2 Chapter 2 Exercise Solutions

Exercise 2.1. Every natural number is transitive. [Hint: Use Princ. of Math. Induction - in other words, show that the set of transitive natural numbers is inductive.]

Sol: $0 = \varnothing$ is transitive. So $0 \in Z =_{\text{df}} \{k \in \omega \mid \text{Trans}(k)\}$. Suppose now $k \in Z$. By Exercise 1.21(ii) $S(k)$ is transitive. Hence Z is inductive, and hence $Z = \omega$.

Exercise 2.2. Show that X is inductive where $X = \{n \mid \forall m \in \omega(m = n \vee m < n \vee n < m)\}$.

Sol: $0 \in X$ as if $m \in \omega$, but $m \neq 0$ then $m \not< 0$ and $\varnothing \in m$, that is $0 < m$. Suppose $S(k)$ is least not in X. Take any $m \in \omega$ with $m \neq S(k)$ and $m \not< S(k)$. The latter implies that $m \notin S(k) = k \cup \{k\}$. As $k \in X$ we have that $m \notin k$ implies $m = k \vee k < m$. But $m \notin \{k\}$ rules out $m = k$. Thus $k < m$. We have to show that $S(k) < m$. But $k < m \longrightarrow S(k) < S(m) = m \cup \{m\}$ (by Lemma 2.11(ii)). Hence $S(k) < m \vee S(k) = m$. The latter conjunct is ruled out by our initial assumption on m. Hence $S(k) < m$ and thus $S(k) \in X$ after all. Hence X is inductive.

Exercise 2.3. Show that $\forall m, n \in \omega\, (n < m \longleftrightarrow n \subsetneqq m)$.

Sol: Let $X = \{m \in \omega \mid \forall n \in \omega\, (n < m \longleftrightarrow n \subsetneqq m)\}$. Trivially $0 \in X$. Suppose $k \in X$. Consider $S(k)$. Let $n < S(k)$. Then $n = k \vee n < k$. As $k \in X$ the latter disjunct implies that $n \subsetneqq k$ and hence $n \subsetneqq S(k)$, and for the former we trivially have $k \subsetneqq S(k)$. Conversely if $n \subsetneqq S(k)$ then it must be that $k \notin n$; hence (Lem. 2.12) $n = k \vee n < k$ and so $n < S(k)$. Thus $S(k) \in X$ and hence X is inductive.

Exercise 2.4. Let $X \neq \varnothing, X \subseteq \omega$. Show that there is $n \in X$, with $n \cap X = \varnothing$.

Sol: Let n be the least element of X by the Least Number Principle. Then note that $n \cap X = \varnothing$ (if $k \in n \cap X$ then $k < n \wedge k \in X$ which contradicts n's supposed leastness).

Exercise 2.5. (Principle of Strong Induction for ω) Let $X \subseteq \omega$ an suppose Φ is a definite welldefined property of natural numbers. Show that

$$\forall n[\forall k < n\Phi(k) \longrightarrow \Phi(n)] \longrightarrow \forall n\Phi(n).$$

[Hint: Suppose for a contradiction $X = \{n \in \omega \mid \neg\Phi(n)\} \neq \varnothing$. Apply the Least Number Principle.]

Sol: Follow the hint. Trivially $0 \notin X$ since $\forall k < 0\Phi(k)$ holds! So X, which is assumed non-empty, has a least element of the form $S(k)$. Thus $\neg\Phi(S(k))$. Then $k \notin X$ and nor are any $<$-predecessors of k in X. In other words $[\forall m < S(k)\Phi(m)$. Hence $\Phi(S(k))$ by the antecedent clause. Contradiction. Hence $X = \varnothing$.

Exercise 2.6. Prove some of the other clauses of the last Proposition 2.17.

Sol: As another sample we do: (f) $m^{n+p} = m^n.m^p$. By induction on p. Let m, n be arbitrary. Then $m^{n+0} = m^n = m^n.1 = m^n.m^0$. Assume (f) proven for p and we prove it for $p+1$. $m^{n+(p+1)} = m^{(n+p)+1}$ (by Definition of Add.) $= m^{(n+p)}.m^1 =$ $(m^n.m^p).m^1$ (by Ind. Hyp.) $m^n.(m^p.m^1) =$ (by Associativity of Multiplication) $= m^n.m^{p+1}$ (by Def. of Exp.) as required. Hence (f) is true for any p.

Exercise 2.7. (i) Let $h: \omega \longrightarrow \omega$ be given by: $h(0) = 4$ and $h(n+1) = 3.h(n)$. Compute $h(4)$.
(ii) Let $h: \omega \longrightarrow \omega$ be given by $h(n) = 5.n + 2$. Express $h(n+1)$ in terms of $h(n)$ as simply as possible.

Sol: (i) 324 (ii) $h(0) = 2$; $h(n+1) = h(n) + 5$.

Exercise 2.8. Assume f_1 and f_2 are functions from ω to A, and that G is a function on sets, so that for every n $f_1 \restriction n$ and $f_2 \restriction n$ are in $\mathrm{dom}(G)$. Suppose also f_1 and f_2 have the property that
$f_1(n) = G(f_1 \restriction n)$ and $f_2(n) = G(f_2 \restriction n)$. Show that $f_1 = f_2$.

Sol: Consider $Z = \{n \in \omega \mid f_1(n) \neq f_2(n)\}$. We claim that $Z = \varnothing$. By the least number principle if $Z \neq \varnothing$ then there is a least $k \in Z$. Note that $0 \neq k$: this is because $f_1(0) = G(f_1 \restriction 0)$. But $f_1 \restriction 0 = f_2 \restriction 0 = \varnothing$. Hence $f_1(0) = G(\varnothing) = f_2(0)$. Hence $k = S(m)$ for some $m \notin Z$. Thus $f_1(m) = f_2(m)$. But then $f_1 \restriction S(m) = f_2 \restriction S(m)$, and so $f_1(S(m)) = G(f_1 \restriction S(m)) = G(f_2 \restriction S(m)) = f_2(S(m))$. Hence $S(m) \notin Z$ - a contradiction.

Exercise 2.9. Let $h{:}\,\omega \longrightarrow \omega$ be given by: $h(k) = k - 10$ if $k > 100$; and $h(k) = h(h(k+1))$ if $k \leq 100$.
Give a definition of h if possible, using the standard formulation of a definition by recursion, which involves only computing values $h(k)$ from smaller values, or constants. If this is impossible show it so.

Sol: $h(k) = k - 10$ if $k > 100$; $h\ (k) = 91$ if $k \leq 100$.

Exercise 2.10. Find (i) infinitely many functions $h{:}\,\omega \longrightarrow \omega$ satisfying: $h(k) = h(k+1)$; (ii) the unique function $h{:}\,\omega \longrightarrow \omega$ satisfying: (a) $h(0) = 2; h(k) = h(k+1)(h(k+1)+1)$ if $k > 0$.

Sol: (i) Any constant function $h(k) = c$; (ii) the function $h(0) = 2$; $h(k) = 0$ if $k > 0$.

Exercise 2.11. Prove that for any $n, m \in \omega$ that $n + m = 0 \leftrightarrow (n = 0 \wedge m = 0)$.

Sol: Two ways: by definition of the function A_n we have that $A_n(m)$ is always the successor of something, and so non-zero, unless $n = m = 0$. Alternatively one can give a direct proof using mathematical induction on m using basic facts about $+$.

Exercise 2.12. Prove that for any n, m, $k \in \omega$ (i) $n < m \longrightarrow n + k < m + k$; (ii) $k > 0 \wedge n < m \longrightarrow n.k < m.k$.

Sol: By induction on k: for $k = 0$ we have $n + k = n$ etc. so this is trivial. Assume proven for k'. Then fix any $n < m$. By the I.H. $n + k' < m + k'$. Then $(n + k') + 1 \leq m + k' < (m + k') + 1$. By associativity then, the result holds for $k' + 1$.

Exercise 2.13. Prove that for any $n, m \in \omega$ that if $n \leq m$ then there is a unique $k \in \omega$ with $n + k = m$.

Sol: By induction on m. If $m = 0$ then it is trivial. So suppose $m = m' + 1$ and the result is proven for m'. Let $n \leq m$. Let $X = \{k \mid n + k = m\}$. *Claim $X \neq \varnothing$ and has a unique element:* If $n = m$ take $k = 0$. If $n < m$ by the inductive hypothesis there is k' with $n + k' = m'$. So $n + (k' + 1) = (n + k') + 1 = m$ and $k' + 1 \in X$. Suppose X has more than one element: say $k_1 < k_2$. then by the last exercise part (i) (and the commutativity of $+$) we have $n + k_1 < n + k_2$ a contradiction. So the *Claim* holds and the proof for $m' + 1$ is finished.

Exercise 2.14. * (The Ackermann function) Define using the equations below the *Ackermann function:*
$A(0, x, y) = x.y$
$A(k+1, x, 0) = 1$
$A(k+1, x, y+1) = A(k, A(k+1, x, y), x)$
Show that $A(k, x, y)$ is defined for all x, y, k. [Hint: Use a *double induction:* first on k assume that for all x, y $A(k, x, y)$ is defined; then assume for all $y' < y$ $A(k+1, x, y')$ is defined.] What is $A(1, x, y)$?

Sol: For $k = 0$ there is nothing to do, as $A(0, x, y)$ is given to us in this case. Make the assumption of the Hint. If $y = 0$ again there is nothing to do as we are given that $A(k+1, x, 0) = 1$. Assume $y = y' + 1$ and $A(k+1, x, y')$ is then defined by the second (or "inner") inductive hypothesis. But then $A(k, A(k+1, x, y'), x)$ is defined by the first or "outer" inductive hypothesis. Hence for all x, y $A(k+1, x, y)$ is defined. $A(1, x, y) = x^y$. [The Ackermann function is an example of an extremely fast growing function that is recognised by logicians as a recursive function which is not "primitive recursive": it cannot be given by building up functions using composition and the recursion scheme of Recursion Theorem on ω (Thm 2.14). One can show that it grows faster than any function of the latter type. Nevertheless it is a recursive function (a concept which we have not defined) which these were shown by Alan Turing to be 'computable'.