3 Chapter 3 Exercise Solutions

Exercise 3.1. Let $<$ be the usual ordering on $\mathbb{N}^+ = \{n \in \omega \mid n \neq 0\}$. For $n \in \mathbb{N}^+$ define $f(n)$ to be the number of distinct prime factors of n. Define a binary relation $mRn \Longleftrightarrow f(m) < f(n) \vee (f(m) = f(n) \wedge m < n)$. Show that R is in fact a wellordering of $\mathbb{N}^+$. Draw a picture of it.

Sol: In later terminology this will look like ω^2. The first block of ω consists of all primes together with 1 in their natural ordering, then all products of two primes with their natural ordering, then all products of 3 primes, and so on. To show formally that this is a wellordering: let $X \subseteq \mathbb{N}^+$ be non-empty. Let $X_n \subseteq X$ be those members that are the product of n primes (for $n > 0$). As $X \neq \varnothing$, for some n $X_n \neq \varnothing$; so take n least with this property and let k be least in X_n (in the natural ordering). Then check that k is the least element of X in our new ordering.

Exercise 3.2. Show that $\langle A, \prec\rangle \in \mathrm{WO}$ implies that there is no set $\{x_n \mid n \in \omega\}$ with $(\forall n\, x_{n+1} \prec x_n)$. (Is there a reason one might hesitate to replace the 'implies' by '$\longleftrightarrow$' here?)

Sol: Suppose $Y = \{x_n \mid n \in \omega\}$ with $(\forall n\, x_{n+1} \prec x_n)$ existed as a counterexample. Then clearly Y has no $\prec$-least member, and thus $\prec$ cannot be a wellordering. (For the comment on the reverse implication: this is a subtle point. If $\langle A, \prec\rangle \notin \mathrm{WO}$ then there is $X \subseteq A$ with no $\prec$-least element. One can presumably then pick an element of X, x_0, and as x_0 is not $\prec$-least there is $x_1 \in X$ with $x_1 \prec x_0$. And so forth. However this "and so forth" requires one to make infinitely many choices; one in effect tries to define a function $f\colon \omega \longrightarrow X$ with $\forall n\, f(n+1) \prec f(n)$. Why does such a function exist at all? [We need some form of the Axiom of Choice (to be introduced later) to ensure that we can do an infinite amount of "choosing". We are choosing an infinite chain depending on our choices already made. The technical version of the axiom needed is then called the *Axiom of Dependent Choice*.])

Exercise 3.3. Show that if $\langle X, \prec\rangle$ is a total ordering, then

$$\langle X, \prec\rangle \in \mathrm{WO} \iff \forall u \in X (\forall Z \subseteq X_u =_{\mathrm{df}} \{z \in X \mid z \prec u\}, \text{ if } Z \neq \varnothing, \text{ then } Z \text{ has a } \prec\text{-least element})$$

[Thus it suffices for a total order to be a wellorder, if its restrictions to all its proper initial segments are wellorders.]

Sol: ($\Rightarrow$) follows from the definition of $\langle X, \prec\rangle \in \mathrm{WO}$. So suppose RHS holds. Let $Z \subseteq X$, $Z \neq \varnothing$. We require an $\prec$-least element of Z. Let $t \in Z$ be arbitrary. By supposition X_t is wellordered by $\prec$. If $Z \cap X_t = \varnothing$ then t itself is the $\prec$-least element of Z; otherwise let u be the $\prec$-least element of $Z \cap X_t$. Then u is also the $\prec$-least element of Z.

Exercise 3.4. Let $f\colon \langle X, \prec\rangle \longrightarrow \langle Y, \prec'\rangle$ be an order isomorphism with $\langle X, \prec\rangle, \langle Y, \prec'\rangle \in \mathrm{WO}$ as in the last Lemma 3.6. Show that for any $z \in X$, $f \restriction X_z\colon \langle X_z, \prec\rangle \cong \langle Y_{f(z)}, \prec'\rangle$.

Sol: As f is order preserving and (1-1) into Y, so is $f \restriction X_z$ into $Y_{f(z)}$. (Namely $t \prec z \Rightarrow f(t) \prec' f(z) \Rightarrow f(t) \in Y_{f(z)}$). Similarly $f``X_z \supseteq Y_{f(z)}$ $(u \prec' f(z) \Rightarrow u = f(v)$ (for some $v$, as $f$ is onto) $\Rightarrow v \prec z$(as $f$ is order preserving) $\Rightarrow u \in f``X_z$. So $f \restriction X_z$ is onto $Y_{f(z)}$ and is therefore an isomorphism.

Exercise 3.5. Find an example of two totally ordered sets which are not order isomorphic, although each is order isomorphic to a subset of the other.

Sol: Let $X_1 = \mathbb{Q} \cap (0,1]$ and $X_2 = \mathbb{Q} \cap [0,1)$ both with the usual ordering. Then X_1 and X_2 are not order isomorphic (eg one has a first element, the other does not). But embed X_1 into X_2 via using $r \rightarrowtail r/2$ and X_2 into X_1 *via* $r \rightarrowtail 1/2 + r/2$.

Exercise 3.6. Suppose $\langle X, \prec_1\rangle$ and $\langle Y, \prec_2\rangle$ are wellorderings. Show that $\langle X \times Y, \prec_{\mathrm{lex}}\rangle \in \mathrm{WO}$ where we define $\langle u, v\rangle \prec_{\mathrm{lex}} \langle t, w\rangle$ if $u \prec_1 t \vee (u = t \wedge v \prec_2 w)$.

Sol: It is easy to check that it is a strict total ordering. For wellorder: let $Z \subseteq X \times Y$, $Z \neq \varnothing$. We require $x \in Z$, the $\prec_{\mathrm{lex}}$ least element. First let $u_0 = \min_{\prec_1}\{u \mid \exists v(\langle u, v\rangle \in Z)\}$ and $v_0 = \min_{\prec_2}\{v \mid \langle u_0, v\rangle \in Z\}$. Take $x = \langle u_0, v_0\rangle$.

Exercise 3.7. Show that if $\langle X, \in\rangle$ is an ordinal, then so is $\langle S(X), \in\rangle$ (where $S(X) = X \cup \{X\}$).

Sol: As $\langle X, \in\rangle$ is an ordinal, Trans(X) and $\in$ wellorders it. But easily Trans(X) $\longrightarrow$ Trans($S(X)$) (see Exercise 1.22). Then if $C \subseteq S(X) \wedge C \neq \varnothing$, then either (i) $C \cap X \neq \varnothing$ in which case, as X is an ordinal there is an $\in$-least element of $C \cap X$ which is then the $\in$-least element of $S(X)$ or else (ii) $C \cap X = \varnothing$. In this latter case $C = \{X\}$ and X is the $\in$-least element of $C \cap S(X)$. Hence $S(X)$ is wellordered by $\in$. Hence $S(X)$ is an ordinal.

Exercise 3.8. Show that if $\langle A, R\rangle, \langle B, S\rangle \in \mathrm{WO}$, then the sum $\langle C, T\rangle \in \mathrm{WO}$.

Sol: It is easy to check that $\langle C, T\rangle$ is a strict total order. Let $\varnothing \neq X \subseteq C$. *Case 1* $X \cap A \neq \varnothing$. In that case check that the R-least element of $X \cap A$ is the T-least element of X. *Case 2* Otherwise. Check that the S-least element of X is the T-least element of X.

Exercise 3.9. Show that if $\langle A, R\rangle, \langle B, S\rangle \in \mathrm{WO}$, then the product $\langle C, U\rangle \in \mathrm{WO}$.

Sol: Again strict totality of U is straight forward. (Although a little lengthy: to show, *e.g.* connectivity, suppose $\langle x, y\rangle, \langle u, v\rangle \in C$. Suppose further that $\langle x, y\rangle \neq \langle u, v\rangle$ and $\langle x, y\rangle \not{U} \langle u, v\rangle$. Then Case 1 $v = y$: then $x \neq u$; but then $x \not{R} u$. As R is connected, $u R x$, and so $\langle u, v\rangle U \langle x, y\rangle$. Case 2 $v \neq y$ and transitivity *etc.* are similar.) For wellorderedness: let $\varnothing \neq X \subseteq C = A \times B$. Let $E =_{\mathrm{df}} \{b \mid \exists a \in A\, (\langle a, b\rangle \in X)\}$. Then $E \neq \varnothing$. Let b_0 be the S-least element of E. Now let $F =_{\mathrm{df}} \{a \mid \langle a, b_0\rangle \in X\}$. Check that if a_0 is the R-least element of A then $\langle a_0, b_0\rangle$ is the U-least element of X.

Exercise 3.10. Suppressing the usual ordering $<$ on the following sets of numbers, show that in the product orderings: $\mathbb{Z} \times \mathbb{N} \not\cong \mathbb{Z} \times \mathbb{Z}$. Is $\mathbb{N} \times \mathbb{Z} \cong \mathbb{Z} \times \mathbb{Z}$? Is $\mathbb{Q} \times \mathbb{Z} \cong \mathbb{Q} \times \mathbb{N}$?

Sol: In $\mathbb{Z} \times \mathbb{N}$ there are elements p (in the first 'block' of $\mathbb{Z}$) so that X_p (the initial segment below p of $\mathbb{Z} \times \mathbb{N}$) contains no ascending ω-like sequence. But that is not true of $\mathbb{Z} \times \mathbb{Z}$. So there could never be an isomorphism between the two orders. In $\mathbb{Z} \times \mathbb{Z}$ every element has an immediate predecessor and immediate successor. This fails in $\mathbb{N} \times \mathbb{Z}$, so again no isomorphism between them. $\mathbb{Q} \times \mathbb{Z} \cong \mathbb{Q} \times \mathbb{N}$ - this is Exercise 4.52 - any countable dense linear order without endpoints is isomorphic to $\mathbb{Q}$.

Exercise 3.11. (i) Express $(\omega +' \omega) +' \omega$ using the multiplication symbol $.'$ only.
(ii) Express $\omega .' \omega$ using the addition symbol $+'$ only.

Sol: (i) $\omega \cdot' 3$ (ii) $\omega +' \omega +' \omega +' \cdots$. This latter expression is not formally a part of proper language.

Exercise 3.12. Show that the distributive law $(\alpha +' \beta) \cdot' \gamma = \alpha \cdot' \gamma +' \beta \cdot' \gamma$ is not valid. On the other hand, convince yourself that $\alpha \cdot' (\beta +' \gamma) = \alpha .' \beta +' \alpha \cdot' \gamma$ will be true.

Sol: Take $\alpha = 2, \beta = 3, \gamma = \omega$. Then $(2 +' 3) \cdot' \omega = 5 \cdot' \omega = \omega \neq \omega +' \omega = 2 \cdot' \omega + 3 \cdot' \omega$. We shall prove the valid distributive law later when we give our official definitions of $+$ and $\cdot$.

Exercise 3.13. (i) Compute $\sup(\beta + 1)$ and verify that it equals $\bigcup\, (\beta + 1)$. Suppose that $0 < \lambda \in \mathrm{On}$. Show that λ is a limit ordinal iff $\lambda = \bigcup\, \lambda$. (ii) Prove that if X is a transitive set of ordinals, then X is an ordinal.

Sol: (i) $\sup(\beta + 1) = \beta$ by definition of sup. But $\beta = \bigcup\, S(\beta)$ by Lemma 1.33. For the second sentence: ($\Rightarrow$): $\sup(\lambda) = \lambda$ by definition of sup, and the fact that λ is the set of its $<$-predecessors, so this follows by Lemma 3.31. ($\Leftarrow$): this follows by the first sentence, if $\lambda = \beta + 1$ then $\bigcup\, \lambda = \beta \neq \lambda$.

(ii) We are told Trans(X) so for X to be an ordinal, we only need that $\in$-wellorders X. But Lemma 3.24 (6) says the whole class On is wellordered by $\in$, so certainly $X \subseteq \mathrm{On}$ is too.

Exercise 3.14. Suppose λ, λ' are both limit ordinals, and that $\langle\alpha_\xi|\ \xi<\lambda\rangle$ and $\langle\beta_\zeta|\ \zeta<\lambda'\rangle$ are two increasing sequences of ordinals with the property that $\forall\xi<\lambda\exists\zeta<\lambda'(\alpha_\xi<\beta_\zeta)$ and also that $\forall\zeta<\lambda'\exists\xi<\lambda(\beta_\zeta<\alpha_\xi)$. Show that $\sup\{\alpha_\xi|\ \xi<\lambda\}=\sup\{\beta_\zeta|\ \zeta<\lambda'\}$.

Sol: Let $\gamma<\sup\{\alpha_\xi|\ \xi<\lambda\}$. As $\langle\alpha_\xi|\ \xi<\lambda\rangle$ is increasing there is some $\xi<\lambda$ with $\gamma<\alpha_\xi$. But then for some $\zeta<\lambda'$, $\gamma<\alpha_\xi<\beta_\zeta<\sup\{\beta_\zeta|\ \zeta<\lambda'\}$. Thus $\sup\{\alpha_\xi|\ \xi<\lambda\}\le\sup\{\beta_\zeta|\ \zeta<\lambda'\}$. The argument for the reverse inequality is symmetrical.

Exercise 3.15. Prove the Lemma 3.41. Suppose $\alpha,\beta,\gamma\in\mathrm{On}$ are such that $\beta<\gamma$. (i) If $\alpha>0$ then $\alpha\cdot\beta<\alpha\cdot\gamma$; (ii) If $\alpha>1$ then $\alpha^\beta<\alpha^\gamma$. Hence both M_α, E_α are also (1-1) and strictly increasing.

Sol: We do (i). ((ii) is similar.) Suppose $\alpha>0$ and $\beta<\gamma$. We suppose that $\alpha\cdot\beta<\alpha\cdot\delta$ holds for all $\delta<\gamma$. Suppose then $\beta<\gamma$. If $\gamma=\delta+1$, we have *either* $\beta=\delta$ (and then $\alpha\cdot\delta=\alpha\cdot\delta+0<\alpha\cdot\delta+\alpha=\alpha\cdot(\delta+1)$ (using that $A_{\alpha\cdot\delta}$ is strictly increasing)), *or* we have $\beta<\delta$ in which case $\alpha\cdot\beta<\alpha\cdot\delta<\alpha\cdot(\delta+1)$ using what we have just shown for the second $<$, and the induction hypothesis for the first $<$.

If $\mathrm{Lim}(\gamma)$ then, as $\beta<\beta+1<\gamma$, by the inductive hypothesis for the first $<$, we have: $\alpha\cdot\beta<\alpha\cdot(\beta+1)\le\sup\{\alpha\cdot\delta\,|\,\delta<\gamma\}=\alpha\cdot\gamma$.

Exercise 3.16. By applying the last two lemmas, justify the following cancellation laws (and hence all these implications could be replaced by equivalences).

(a) $\alpha+\beta=\alpha+\gamma\longrightarrow\beta=\gamma$.
(b) $(0<\alpha\wedge\alpha\cdot\beta=\alpha\cdot\gamma)\longrightarrow\beta=\gamma$.
(c) $\alpha^\beta=\alpha^\gamma\longrightarrow\beta=\gamma$.
(d) $\alpha+\beta<\alpha+\gamma\longrightarrow\beta<\gamma$.
(e) $(\alpha\cdot\beta<\alpha\cdot\gamma)\longrightarrow\beta<\gamma$.
(f) $\alpha^\beta<\alpha^\gamma\longrightarrow\beta<\gamma$.

Sol: There is little to do here: the previous lemmata said that the arithmetic operations A_α, M_α and E_α are strictly increasing on their inputs. Eg for (c): If $\beta\ne\gamma$, then say, $\beta<\gamma$ and then by the previous lemma $\alpha^\beta<\alpha^\gamma$ *i.e.* $\alpha^\beta\ne\alpha^\gamma$. So (c) must hold. Again for (f) proving the contrapositive: $\beta\not<\gamma\longrightarrow(\beta=\gamma\vee\gamma<\beta)$. Which implies $\alpha^\beta=\alpha^\gamma\vee\alpha^\gamma<\alpha^\beta$ (the latter as E_α is strictly increasing), *i.e.* $\alpha^\beta\not<\alpha^\gamma$.

Exercise 3.17. Show that for any γ and any $\alpha\le\beta$:

(a) $\alpha+\gamma\le\beta+\gamma$;
(b) $\alpha\cdot\gamma\le\beta\cdot\gamma$;
(c) $\alpha^\gamma\le\beta^\gamma$.

Sol: By induction on γ. We do just (a). For $\gamma=0$ this is trivially true. Suppse true for $\delta<\gamma$. If $\gamma=\delta+1$ then $\alpha+(\delta+1)=(\alpha+\delta)+1\le(\beta+\delta)+1$ (by the inductive hypothesis) $=\alpha+(\delta+1)$. And thus it is true for γ. Lastly suppose $\mathrm{Lim}(\gamma)$. Then $\alpha+\gamma=_{\mathrm{df}}\sup\{\alpha+\delta|\delta<\gamma\}$. But $\sup\{\beta+\delta|\delta<\gamma\}$ is the least ordinal greater than or equal to any ordinal in this set, which in turn will be greater than or equal to any ordinal in the set $\{\alpha+\delta|\delta<\gamma\}$ (using the Ind. Hyp.). That is $\sup\{\alpha+\delta|\delta<\gamma\}\le\sup\{\beta+\delta|\delta<\gamma\}=\beta+\delta$.

Exercise 3.18. Complete the proof of (ii) of the Lemma that $\alpha\cdot\beta=\{\alpha\cdot\xi+\eta\,|\,\xi<\beta\wedge\eta<\alpha\}$

Sol: This was an induction on β. 0 and successor cases had been proven. Suppose it is true for $\beta<\lambda$ and $\mathrm{Lim}(\lambda)$. Then

$$\begin{aligned}\alpha\cdot\lambda &=_{\mathrm{df}}\sup\{\alpha\cdot\beta\,|\,\beta<\lambda\}=\bigcup\{\alpha\cdot\beta\,|\,\beta<\lambda\}\\ &=\bigcup\{\{\alpha\cdot\xi+\eta\,|\,\xi<\beta\wedge\eta<\alpha\}|\beta<\lambda\}\ \text{ by the Ind. Hyp.}\\ &=\bigcup_{\beta<\lambda}\{\alpha\cdot\xi+\eta\,|\,\xi<\beta\wedge\eta<\alpha\}\ \text{(just rewriting by def.)}\\ &=\{\alpha\cdot\xi+\eta\,|\,\xi<\lambda\wedge\eta<\alpha\}\ \text{ just as a union of sets.}\end{aligned}$$

Exercise 3.19. Show that if $\alpha<\omega^3$ then there exist unique $n,k,l\in\omega$ with $\alpha=\omega^2\cdot n+\omega\cdot k+l$.

Sol: First let n be least such that $\alpha<\omega^2\cdot(n+1)$. (Such an n exists, since $\omega^3=\sup\{\omega^2\cdot n|\,n<\omega\}$.) Secondly let k be least with $\alpha<\omega^2\cdot n+\omega\cdot(k+1)$. (Such a k exists, since $\omega^2\cdot(n+1)=\sup\{\omega^2\cdot n+\omega\cdot k|\,k<\omega\}$.) Thirdly let l be least with $\alpha<\omega^2\cdot n+\omega\cdot k+(l+1)$.

Exercise 3.20. Say that γ is an *end segment* of β if there is an α so that $\alpha+\gamma=\beta$. (Note that β is an end segment of itself.) Show that any β has at most finally end segments.

Sol: Define pairs α_i, β_i with $\alpha_i+\beta_i=\beta$ as follows. The β_i will enumerate the end segments of β. Let $\alpha_0=0$, $\beta_0=\beta$. Let $\alpha_1>\alpha_0$ be least (if defined) so that there is $\beta_1\neq\beta_0$ with $\alpha_1+\beta_1=\beta$. Clearly $\beta_1<\beta_0$. As addition is strictly increasing β_1 is uniquely defined for this α_1. Note also there can be no other end segment $\bar\beta$ in (β_1,β_0) as that would require some $\bar\alpha<\alpha_1$ in order that $\bar\alpha+\bar\beta=\beta$ - by the monotonicity of addition again. But this contradicts the leastness in the choice of α_1. Continuing in this fashion let $\alpha_{i+1}>\alpha_\iota$ be least (if defined) so that there is $\beta_{i+1}\neq\beta_i$ with $\alpha_{i+1}+\beta_{i+1}=\beta$. Again, clearly $\beta_{i+1}<\beta_i$ and is uniquely defined if α_{i+1} is; further it is the next smallest end segment. However now the β_i form a strictly descending sequence of ordinals, and thus by wellorderedness it must be finite, and for some $j<\omega$ α_j will become undefined.

Exercise 3.21. Let X be a set of ordinals without a largest element. Show
(i) $\alpha+\sup X=\sup\{\alpha+\tau\,|\,\tau\in X\}$;
(ii) $\alpha\cdot\sup X=\sup\{\alpha\cdot\tau\,|\,\tau\in X\}$;
(iii) $\alpha^{\sup X}=\sup\{\alpha^\tau\,|\,\tau\in X\}$.

Sol: Let $\lambda=\sup X$. Note first that for any sets of ordinals $X\subseteq Y$ that $\sup\{\alpha+\tau\,|\,\tau\in X\}\leq\sup\{\alpha+\tau\,|\,\tau\in Y\}$. (And similarly for $\cdot$ and exponentiation.)

(i) $\sup\{\alpha+\tau\,|\,\tau\in X\}=\sup\{\alpha+\tau\,|\,\tau\in\lambda\}$. [Because every $\tau\in X$ is less than some $\tau'\in\lambda$, (and also conversely). And for such τ, τ' as A_α is strictly increasing we deduce $\alpha+\tau<\alpha+\tau'$. Hence $\sup\{\alpha+\tau\,|\,\tau\in X\}\leq\sup\{\alpha+\tau\,|\,\tau\in\lambda\}$ (And conversely, $\geq$).] But $\sup\{\alpha+\tau\,|\,\tau<\lambda\}$ is $\alpha+\lambda$ by definition of A_α.

(ii) Similarly $\sup\{\alpha\cdot\tau\,|\,\tau\in X\}=\sup\{\alpha\cdot\tau\,|\,\tau<\lambda\}$ ["$\leq$" since $X\subseteq\lambda$ and "$\geq$" because every $\tau'<\lambda$ is less than some $\tau\in X$, and by Ex 3.15 (i) $\alpha\cdot\tau'<\alpha\cdot\tau$.] But $\sup\{\alpha\cdot\tau\,|\,\tau<\lambda\}$ is $\alpha\cdot\lambda$ by definition of M_α.

(iii) Follow the format of (ii).

Exercise 3.22. Find subsets of $\mathbb{Q}$ with order types ω^2,ω^ω, and $\omega^\omega+\omega^3+17$ under the natural $<$ ordering.

Sol. Let F: $[0,\infty)\to[0,1)$ be the piecewise linear map that sends $[0,1)\to[0,\frac{1}{2})$, $[1,2)\to[\frac{1}{2},\frac{2}{3}),\cdots,[k-1,k)\to[1-\frac{1}{k},1-\frac{1}{k+1}),\cdots$. Let $A^k=\{k-\frac{1}{n}\,|\,n>1\}$ for $k\geq1$, $k\in\mathbb{N}$. Then the order type of each A^k is ω. Let $B_2=\bigcup_{k\geq1}F``A^k$. Then $B_2\subseteq[0,1)$ and has order type $\omega\cdot\omega=\omega^2$. Now let $C^k=\{k+l|\,l\in B\}$ for $k\in\mathbb{N}$. Then $C$ has order type $\omega^2\cdot\omega=\omega^3$. Let $B_3=\bigcup_{k\geq1}F``C^k$. Repeated applications of this idea leads to subsets B_n of $[0,1)$ of order type ω^n. To get ω^ω put a copy of B_n in $[n,n+1)$ for all $n\geq1$. The union of these sets in these intervals is a set D, say, with order type ω^ω. To get $\omega^\omega+\omega^3+17$: apply F to D, to obtain $F``D$ in $[0,1)$, put a copy of ω^3 in $[1,2)$ and then let the rest of the points be any 17 larger rationals.

Exercise 3.23. Prove that if $0<\alpha,\beta$ then: (i) $\alpha+\beta=\beta\leftrightarrow\alpha\cdot\omega\leq\beta$.
(ii) $\alpha+\beta=\beta+\alpha\leftrightarrow\exists\gamma\exists m,n\in\omega\ (\alpha=\omega^\gamma\cdot m\wedge\beta=\omega^\gamma\cdot n)$.

(i) ($\rightarrow$) Assume $\alpha+\beta=\beta$. Then $\alpha<\beta$ and by induction on $k<\omega$ we always have $\alpha\cdot k+\beta=\beta$, *i.e.* $\alpha\cdot k\leq\beta$. But then $\alpha\cdot\omega=\sup\{\alpha\cdot k|\, k<\omega\}\leq\beta$. ($\leftarrow$) Assume $\alpha\cdot\omega\leq\beta$. Then $\alpha<\beta$ *a fortiori* so there exist (Cor.3.43) χ, ρ with $\beta=\alpha\cdot\chi+\rho$ and $\rho<\alpha$. By assumption $\chi\geq\omega$. But then $\alpha+\beta=\alpha+(\alpha\cdot\chi+\rho)=\alpha\cdot(1+\chi)+\rho=\alpha\cdot\chi+\rho$ (as $\omega\leq\chi\longrightarrow 1+\chi=\chi$) $=\beta$.

(ii) We prove this by induction on β. Suppose the statement holds for all $\beta'<\beta$.

($\rightarrow$) So suppose, w.l.o.g., $0<\alpha<\beta$ and $\alpha+\beta=\beta+\alpha$. As $\alpha<\beta$, for some χ and some $\rho<\alpha$, we have $\beta=\alpha\cdot\chi+\rho$. Then $\alpha+\beta=\alpha+(\alpha\cdot\chi+\rho)=\alpha(1+\chi)+\rho$. Note that $\chi=k<\omega$ (for some k, since otherwise $1+\chi=\chi$, and then $\alpha+\beta=\beta\neq\beta+\alpha$). Now, if $\rho=0$ we are done, as then $\alpha+\beta=\alpha+\alpha\cdot k=\alpha\cdot 2+\alpha\cdot(k-1)=\alpha\cdot(k-1)+\alpha\cdot 2$. Applying the I.H. with $\alpha\cdot(k-1)<\beta$ we have $\exists\gamma\exists n,m<\omega(\alpha\cdot 2=\omega^\gamma\cdot m\cdot 2\wedge\alpha\cdot(k-1)=\omega^\gamma\cdot n)$. Then $\alpha=\omega^\gamma\cdot m$, $\beta=\omega^\gamma\cdot(n+m)$. If $\rho>0$, then $\alpha+\beta=\alpha+(\alpha\cdot k+\rho)=\alpha\cdot k+(\alpha+\rho)$, But $\beta+\alpha=(\alpha\cdot k+\rho)+\alpha=\alpha\cdot k+(\rho+\alpha)$. Hence (as $A_{\alpha\cdot k}$ is (1-1)) $\alpha+\rho=\rho+\alpha$. As $\rho<\alpha<\beta$ by the IH we have that for some ω^γ, some $p,q<\omega$, $\alpha=\omega^\gamma\cdot p\wedge\rho=\omega^\gamma\cdot q$, and then it is easily checked that $\beta=\omega^\gamma\cdot(pk+q)$ and we are done.

($\leftarrow$) This is easy: $\omega^\gamma\cdot m+\omega^\gamma\cdot n=\omega^\gamma\cdot(m+n)=\omega^\gamma\cdot(n+m)=\omega^\gamma\cdot n+\omega^\gamma\cdot m$.

Exercise 3.24. In each of (i)-(iii) find α and X a set of ordinals without a largest element with the properties

(i) $\sup X+\alpha\neq\sup\{\tau+\alpha\,|\,\tau\in X\}$;
(ii) $\sup X\cdot\alpha\neq\sup\{\tau\cdot\alpha\,|\,\tau\in X\}$;
(iii) $(\sup X)^\alpha\neq\sup\{\tau^\alpha\,|\,\tau\in X\}$.
[Hint: in each case X can be found with $X=\{\beta_n|\, n<\omega\}$ with $\beta_n<\beta_{n+1}$.]

Sol: For all three parts take $\beta_n=n$ and $\alpha=\omega$. Then compute: (i) $\sup\{\beta_n\}+\alpha=\omega+\omega\neq\sup\{\beta_n+\omega|\, n<\omega\}=\omega$; (ii) $\sup\{\beta_n\}\cdot\alpha=\omega\cdot\omega\neq\sup\{\beta_n\cdot\omega|\, n<\omega\}=\omega$; (iii) $\sup\{\beta_n\}^\alpha=\omega^\omega\neq\sup\{\beta_n^\omega|\, n<\omega\}=\omega$.

Exercise 3.25. (i) Prove that if $\beta<\gamma$ then $\omega^\beta+\omega^\gamma=\omega^\gamma$. (ii) Prove that if $\alpha<\beta\leq\omega^\gamma$ then
$$\alpha+\beta=\omega^\gamma\leftrightarrow\ \beta=\omega^\gamma.$$

Sol: (i) Let $\tau>0$ be the unique ordinal such that $\beta+\tau=\gamma$ (by Cor.3.43). Then, using the laws of arithmetic, $\omega^\beta+\omega^{\beta+\tau}=\omega^\beta+\omega^\beta\cdot\omega^\tau=\omega^\beta\cdot(1+\omega^\tau)=\omega^\beta\cdot\omega^\tau=\omega^{\beta+\tau}=\omega^\gamma$.
(ii) Assume $\alpha<\beta\leq\omega^\gamma$. ($\rightarrow$) For a contradiction assume $\alpha+\beta=\omega^\gamma$ but $\beta<\omega^\gamma$. Then as E_ω is increasing, we can find $\gamma'<\gamma$ and $0<k<\omega$ with $\beta<\omega^{\gamma'}\cdot k$. But then $\alpha+\beta\leq\omega^{\gamma'}\cdot k+\omega^{\gamma'}\cdot k<\omega^{\gamma'+1}\leq\omega^\gamma$. Contradiction.

($\leftarrow$) This is the argument of (iii) $\Rightarrow$(ii) of Exercise 3.30.

Exercise 3.26. Prove that if $\alpha\geq 2$ then $\forall\beta(\alpha\cdot\beta\leq\alpha^\beta)$.

Sol: Let $\alpha\geq 2$. Then by induction on β: $\alpha\cdot 0=0<1=\alpha^0$. If $\alpha\cdot\beta'\leq\alpha^{\beta'}$ holds for $\beta'<\beta$ and: if $\beta=\beta'+1$ then $\alpha\cdot\beta=\alpha\cdot(\beta'+1)=\alpha\cdot\beta'+\alpha\leq\alpha^{\beta'}+\alpha$ (by I.H.) $\leq\alpha^{\beta'}\cdot\alpha=\alpha^{\beta'+1}$ (implicitly using $\gamma+\alpha\leq\gamma\cdot\alpha$ here for $\gamma>0$); or if β a limit, then $\alpha\cdot\beta=\sup\{\alpha\cdot\gamma|\,\gamma<\beta\}\leq\sup\{\alpha^\gamma|\,\gamma<\beta\}$ (I.H.) $=\alpha^\beta$.

Exercise 3.27. If $\sigma=\omega^\tau$ for some $\tau>0$, and $\alpha<\sigma$, then show that there are $\delta<\tau, k<\omega$, and $\gamma<\omega^\delta$ with $\alpha=\omega^\delta\cdot k+\gamma$.

Sol: Suppose $\sigma=\omega^\tau$. As E_ω is increasing, let δ be least such that $\omega^\delta\leq\alpha<\omega^{\delta+1}$. As $\alpha<\sigma$, $\delta+1\leq\tau$. Then 'by division' there are $\eta<\omega^\delta$, and ξ with $\alpha=\omega^\delta\cdot\xi+\eta$. But $\xi<\omega$ since otherwise if $\xi=\omega+\zeta$, then $\alpha=\omega^\delta\cdot(\omega^1+\zeta)+\eta\geq\omega^{\delta+1}$.

Exercise 3.28. Cantor Normal Form Exercise: Omitted.

Exercise 3.29. For $\alpha>0$ show that $\omega\cdot\alpha=\alpha$ iff α is a multiple of ω^ω, that is for some δ, $\alpha=\omega^\omega\cdot\delta$.

Sol: $(\leftarrow)$ If $\alpha=\omega^\omega\cdot\delta$ then $\omega\cdot\alpha=\omega\cdot\omega^\omega\cdot\delta=\omega^{(1+\omega)}\cdot\delta=\omega^\omega\cdot\delta=\alpha$. $(\rightarrow)$ Suppose α not of the form $\omega^\omega\cdot\delta$. Then $\alpha=\omega^\omega\cdot\delta+\eta$ for some largest δ and some $0<\eta<\omega^\omega$; but if $\omega^k\leq\eta<\omega^{k+1}$ then $\omega^{k+1}\leq\omega\cdot\eta<\omega^{k+2}$, and so $\eta<\omega\cdot\eta$. But now $\omega\cdot\alpha=\omega\cdot(\omega^\omega\cdot\delta+\eta)=\omega^{(1+\omega)}\cdot\delta+\omega\cdot\eta=\omega^\omega\cdot\delta+\omega\cdot\eta>\omega^\omega\cdot\delta+\omega\cdot\eta>\omega^\omega\cdot\delta+\eta$, the last inequality by the strict monotonicity of addition $(A_{\omega^\omega\cdot\delta})$; we then have $\alpha<\omega\cdot\alpha$.

Exercise 3.30. An ordinal σ is called *(additively) indecomposable* if $\alpha,\beta<\sigma\rightarrow\alpha+\beta<\sigma$. Show that the following are equivalent:
(i) σ is indecomposable;
(ii) $\forall\alpha<\sigma\,(\alpha+\sigma=\sigma)$, *i.e.* σ is a fixed point of A_α for any $\alpha<\sigma$;
(iii) $\sigma=\omega^\delta$ for some ordinal δ.

Sol: (ii)$\Rightarrow$(i): Assume (ii). If $\alpha,\beta<\sigma$ then $\alpha+\beta<\alpha+\sigma=\sigma$ as A_α is strictly increasing.

(i)$\Rightarrow$(iii): Suppose $\neg$(iii). Then for some δ, $\omega^\delta<\sigma<\omega^{\delta+1}=\omega^\delta\cdot\omega=\sup\{\omega^\delta\cdot k|\,k<\omega\}$. Then for some $k<\omega$ we have $\omega^\delta\leq\omega^\delta\cdot k<\sigma\leq\omega^\delta\cdot(k+1)=\omega^\delta\cdot k+\omega^\delta$. But both ω^δ and $\omega^\delta\cdot k$ are $<\sigma$! hence $\neg$(i) .

(iii)$\Rightarrow$(ii). Clearly $\alpha+\sigma\geq\sigma$ for any α,σ. Let $\sigma=\omega^\delta$. If $\alpha<\sigma$ then for some $0\leq\eta<\delta$, some $0<k<\omega$, $\alpha<\omega^\eta\cdot k$. Let $\eta+\mu=\delta$. Then
$$\alpha+\sigma\leq\omega^\eta\cdot k+\omega^\delta=\omega^\eta\cdot(k+\omega^\mu)=\omega^\eta\cdot\omega^\mu=\omega^\delta=\sigma.$$

Exercise 3.31. Show that the least indecomposable ordinal greater than α is $\alpha\cdot\omega$ for $\alpha>0$.

Sol: $\alpha\cdot\omega$ is indecomposable: let $\gamma,\delta<\alpha\cdot\omega=\sup\{\alpha\cdot k|\,k<\omega\}$. Thence for some $k<\omega$ γ, $\delta<\alpha\cdot k$. But then $\gamma+\delta<(\alpha\cdot k)\cdot 2=\alpha\cdot 2k<\alpha.\omega$. Then $\alpha\cdot\omega$ is least such: if $\alpha<\delta<\alpha\cdot\omega$, then for some $0<k<\omega$, $\alpha\cdot k<\delta<\alpha\cdot(k+1)$. However then $\alpha\cdot k+\alpha\cdot k\geq\alpha\cdot(k+1)>\delta$.

Exercise 3.32. An ordinal σ is called *multiplicatively indecomposable* if $\alpha,\beta<\sigma\rightarrow\alpha\cdot\beta<\sigma$. Show that the following are equivalent:
(i) $1<\sigma$ is multiplicatively indecomposable;
(ii) $\forall\alpha(0<\alpha<\sigma\longrightarrow\alpha\cdot\sigma=\sigma)$, *i.e.* σ is a fixed point of M_α for any $0<\alpha<\sigma$;
(iii) $\sigma=\omega^{(\omega^\delta)}$ for some ordinal δ.

Sol: (ii)$\Rightarrow$(i): Assume (ii). If $\alpha,\beta<\sigma$ then $\alpha\cdot\beta<\alpha\cdot\sigma=\sigma$ as M_α is strictly increasing.

(i)$\Rightarrow$(iii): Suppose $\neg$(iii). Then for some δ $\omega^{(\omega^\delta)}<\sigma<\omega^{(\omega^{\delta+1})}$. Then for some least $1\leq k<\omega$ we have $\omega^{(\omega^\delta)}\leq\omega^{(\omega^\delta\cdot k)}<\sigma\leq\omega^{(\omega^\delta\cdot(k+1))}$. But $\omega^{(\omega^\delta\cdot k)}\cdot\omega^{(\omega^\delta\cdot k)}=\omega^{(\omega^\delta\cdot k+\omega^\delta\cdot k)}=\omega^{(\omega^\delta\cdot 2k)}\geq\omega^{(\omega^\delta\cdot(k+1))}\geq\sigma$. So σ is not multiplicatively closed. Hence $\neg$(i) .

(iii)$\Rightarrow$(ii). Let δ,σ be as in (iii). If $\alpha<\sigma=\omega^{(\omega^\delta)}$ then for some $\eta<\omega^\delta$, $\alpha<\omega^\eta$ (as ω^δ is a limit). Then $\alpha\cdot\sigma\leq\omega^\eta\cdot\omega^{(\omega^\delta)}=\omega^{(\eta+\omega^\delta)}=\omega^{(\omega^\delta)}$ (the last equality holds since ω^δ is indecomposable).

Exercise 3.33. Formulate a definition for an ordinal $\sigma>2$ to be *exponentially indecomposable* and demonstrate two equivalences by analogy with the two previous exercises.

Sol: An ordinal $\sigma>2$ is called *exponentially indecomposable* if $\alpha,\beta<\sigma\rightarrow\alpha^\beta<\sigma$. The following are then equivalent:

(i) σ is exponentially indecomposable;
(ii) $\forall\alpha(1<\alpha<\sigma\longrightarrow\alpha^\sigma=\sigma)$, *i.e.* σ is a fixed point of E_α for any $1<\alpha<\sigma$;
(iii) $\sigma=\omega^\sigma$.

Pf: (ii)$\Rightarrow$(i): Assume (ii). If $\alpha,\beta<\sigma$ then $\alpha^\beta<\alpha^\sigma=\sigma$ as E_α is strictly increasing.

(i)$\Rightarrow$(iii): Suppose $\neg$(iii). If σ is not a limit then trivially (i) fails. Otherwise, then for some $\delta<\sigma$ $\omega^\delta<\sigma\leq\omega^{\delta+1}$. But $\sigma\leq(\omega^\delta)^\sigma=\omega^{\delta\cdot\sigma}$. If $\delta\cdot\sigma$ equalled σ this would contradict our assumption. So $\delta\cdot\sigma>\sigma$ and so σ is not multiplicatively decomposable, and there is some $\eta<\eta+1<\sigma$ with $\delta\cdot\eta<\sigma\leq\delta\cdot(\eta+1)\leq(\omega^\delta)^{\eta+1}$ (using $\delta\leq\omega^\delta$ and Exercise 3.26). But both ω^δ and $\eta+1$ are less than σ. So (i) fails.

(iii)⇒(ii). Let δ, σ be as in (iii). Note then σ is certainly a limit, so if $\alpha < \sigma = \omega^\sigma = \sup\{\omega^\eta | \eta < \sigma\}$, then for some $\eta < \sigma$, $\alpha < \omega^\eta$. Then $\alpha^\sigma \leq (\omega^\eta)^\sigma = \omega^{\eta \cdot \sigma} = \omega^\sigma$ (the last equality holds as $\sigma = \omega^\sigma = \omega^{(\omega^\sigma)}$, so by the last exercise σ is multiplicatively indecomposable).

Exercise 3.34. (i) Consider the set S_0 of all finite strings of Roman letters with the dictionary or lexicographic ordering. (Thus $a <_{\text{lex}} aa <_{\text{lex}} <aaa <_{\text{lex}} \cdots <_{\text{lex}} ab <_{\text{lex}} aba <_{\text{lex}} abd$ etc.) Is $\langle S_0, <_{\text{lex}}\rangle$ a wellordering?

(ii) Now consider the set S_1 of all finite strings of natural numbers (this will be denoted ${}^{<\omega}\omega$). Again consider the lexicographic ordering, where we consider also '$2 <_{\text{lex}} 3$' *i.e.*, so that $<_{\text{lex}}$ also extends the natural $<$ ordering on ω. Is $\langle S_1, <_{\text{lex}}\rangle$ a wellordering?

Sol: (i) That $<_{\text{lex}}$ is a strict total ordering is clear. But it is not a wellorder: consider the descending sequence of words $b > ab > aab > aaab > \cdots$ (It would be a wellorder if we restricted the wellorder to words of a fixed bounded length because with the alphabet finite there are only finitely many words, and every finite total order is a wellorder.) (ii) This is no different, it is still not wellfounded. (But now if we restrict to sequences of finite length it is still a wellorder, even though there are infinitely many sequences, of length 1 say.)

Exercise 3.35. Mephistopheles has coins in a currency with k denominations. He offers Faust the following bargain: Every day Faust must give M. a coin c, and in return receives as many coins as he, Faust, demands, but only in coins of a lower denomination (except when the coin c is already of the lowest denomination, in which case F. will receive nothing in return). Should Faust accept the bargain?

Sol: No, he will end up with nothing. (What wellordering would represent this here?)

Exercise 3.36. Consider the set $\mathcal{P}$ of polynomials in the variable x with coefficients from $\mathbb{N}$. For P, $Q \in \mathcal{P}$ define $P \prec Q \leftrightarrow$ for all sufficiently large $x \in \mathbb{R}$ $P(x) < Q(x)$. Prove $\langle \mathcal{P}, \prec\rangle \in \text{WO}$.

Sol: Let $P(x)$ be: $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ where $n = n(P)$ is the degree of P. Let the code of P be the finite string $c(P) = (n(P), a_{n(P)}, a_{n(P)-1} \ldots, a_0)$. Because the P are polynomials with positive integer coefficients, it is easy to see that $P \prec Q$ iff $c(P) <_{\text{lex}} c(Q)$ where $<_{\text{lex}}$ is the lexicographic ordering on finite strings of natural numbers, ${}^{<\omega}\omega$, from the Ex.3.34. As $\langle {}^{<\omega}\omega, <_{\text{lex}}\rangle \in \text{WO}$, so is $\langle \mathcal{P}, \prec\rangle$, as $P \hookrightarrow c(P)$ is an order preserving map of the latter into the former.

Exercise 3.37. Let ${}^{<\omega}\omega = \{f | \text{Fun}(f) \wedge \exists k (f : k \longrightarrow \omega)\}$ be the set of all functions into ω with domain some $n \in \omega$. The Kleene-Brouwer ordering on ${}^{<\omega}\omega$ is defined by:

$$f <_{\text{KB}} g \longleftrightarrow \exists n [f \upharpoonright n = g \upharpoonright n \ \wedge n \in \text{dom}(f) \wedge (n \notin \text{dom}(g) \vee f(n) < g(n))]$$

Is it a total ordering? A wellordering?

Sol: It is a total ordering: the tricky part is to show transitivity. In the above definition let us say that $f <_{\text{KB}} g$ 'as witnessed by n' (and note that n is unique). Suppose $f <_{\text{KB}} g$ and $g <_{\text{KB}} h$ as witnessed by n, m respectively.

Case 1 $n \in \text{dom}(f) \wedge n \notin \text{dom}(g)$. Then $m < n$ and as $f(m) = g(m)$ we have $f <_{\text{KB}} h$ as witnessed by m.

Case 2 Otherwise. If $n \leq m$ then $f(n) < h(n)$ or $(n = m \wedge n \notin \text{dom}(h))$. Or else if $m < n$ then either $m \notin \text{dom}(h)$ - and so $f <_{\text{KB}} h$ as witnessed by m - or else $f(m) = g(m) < h(m)$ with the same conclusion.

However it is clearly not a wellorder: the constant 0 function c_0: $\omega \longrightarrow 1$ satisfies $c_0 \upharpoonright k+1 <_{\text{KB}} c_0 \upharpoonright k$ for all k.

Exercise 3.38. Let $\langle X, \prec\rangle \in \mathrm{WO}$. Let Q_X be ${}^{<\omega}X$. Consider the following order $\prec_1$ on Q_X:

$$f \prec_1 g \leftrightarrow \mathrm{dom}(f) < \mathrm{dom}(g) \vee (\mathrm{dom}(f) = \mathrm{dom}(g) \wedge \exists k < \mathrm{dom}(f)(\forall n < k\, f(n) = g(n) \wedge f(k) \prec g(k))).$$

Show that $\langle Q_X, \prec_1\rangle \in \mathrm{WO}$.

Sol: Strict total order: let $f, g \in Q_X$. Connectivity: if $\mathrm{dom}(f) < \mathrm{dom}(g)$ or vice versa, then $f \prec_1 g$ or v.v. So suppose $\mathrm{dom}(f) = \mathrm{dom}(g)$. Let k be the least point of difference in their domains, and check whether $f(k) \prec g(k)$ or v.v. Transitivity: let $f \prec_1 g$ and $g \prec_1 h$. Then argue by cases depending on whether $\mathrm{dom}(f) \le \mathrm{dom}(g)$ and $\mathrm{dom}(g) \le \mathrm{dom}(h)$ *etc.* Wellorder: Let $\varnothing \neq Y \subseteq Q_X$. Let $Y_0 \subseteq Y$ be those elements of Y with least domain, k_0, say. (Then the least element of Y_0 is the least element of Y.) Let $f_0 \in Y_0$ be arbitrary. If f_0 is not the least element of Y_0, let $k_1 < k_0$ be least so that for some $f_1 \in Y_0$ $f_1(k_1) \prec_1 f_0(k_1)$. Pick such an f_1. If f_1 is not the least element of Y_0, then repeat the process finding some $k_2 < k_1$ and an f_2 *etc.* After finitely many steps the k_i's have to stop decreasing; suppose that k_n and $f_n \in Y_0$ are the number and function at this stage. If f_n is not the least element of Y_0, then all the $g \in Y_0$ with $g \prec_1 f_n$ have $g(k_n) \prec f_n(k_n)$. So pick such a g with $g(k_n)$ least. Then g is the $\prec_1$-least element of Y_0.

Exercise 3.39. Show that the following is a wellorder of ${}^n\mathrm{On}$: for $\vec{\alpha} = \langle \alpha_0, \ldots, \alpha_{n-1}\rangle$, $\vec{\beta} = \langle \beta_0, \ldots, \beta_{n-1}\rangle$ set $\vec{\alpha} <^n \vec{\beta}$ iff $\max(\vec{\alpha}) < \max(\vec{\beta})$ or $(\max(\vec{\alpha}) = \max(\vec{\beta})) \wedge (\text{if } i \text{ is least so that } \alpha_i \neq \beta_i \text{ then } \alpha_i < \beta_i)$.

Exercise 3.40. * Let FOn be the class of all finite sets of ordinals. Consider the following ordering $<^*$ on FOn where as usual $p \triangle q = \{\alpha \mid \alpha \in p \backslash q \cup q \backslash p\}$ is the *symmetric difference* of p, q:

$$p <^* q \longleftrightarrow \max(p \triangle q) \in q.$$

(Or to put it another way: $\exists \beta \in q \backslash p\, (p \backslash (\beta+1) = q \backslash (\beta+1))$.) Show that $<^*$ is a wellorder of FOn.

Sol: first note that $<^*$ is the lexicographic ordering when restricted to *descending* finite sequences of ordinals p: $p_0 > p_1 > \cdots > p_k$, q: $q_0 > q_1 > \cdots > q_l$. So, given two sets p, q as above note that $p <^* q$ or $q <^* p$ depending on which is lexicographically first when written out in this descending fashion. So the ordering is a strict total ordering. Let $\varnothing \neq X$ be a set of finite sets of ordinals. Assume each $p \in X$ is enumerated in descending order. Let

$d_0 = \text{least}\, \{\delta \mid \delta = p_0 \text{ for some } p_0 \in p \in X\}$;

$d_1 = \text{least}\, \{\delta \mid \delta = p_1 \text{ for some } p \in X \text{ with } d_0 > p_1 \text{ and } d_0, p_1 \in X\}$. Then $d_1 < d_0$. Proceeding in this way, after a finite number of steps (as the ordinals are wellordered) we arrive at some $d_k < d_{k-1} \cdots < d_1 < d_0$ the minimal element of X.

Exercise 3.41. * Use the Cantor Normal Form to devise a pairing function on ordinals: that is to define a bijection p: $On \times On \longleftrightarrow On$, but with the additional property that $p \restriction \alpha \times \alpha$: $\alpha \times \alpha \longleftrightarrow \alpha$ is a bijection if and only if α is indecomposable (See Ex. 3.30). [Hint: Let $\beta_1 = \omega^{\gamma_0} \cdot d_0 + \omega^{\gamma_1} \cdot d_1 + \cdots + \omega^{\gamma_{k-1}} \cdot d_{k-1}$ and $\beta_2 = \omega^{\gamma_0} \cdot e_0 + \omega^{\gamma_1} \cdot e_1 + \cdots + \omega^{\gamma_{k-1}} \cdot e_{k-1}$ where, in order to match up, some of the d_i's or e_i's may have to be zero (but not both $e_i = d_i = 0$ for any i unless $\beta_1 = \beta_2 = 0$). Let p_0: $\omega \times \omega \longleftrightarrow \omega$ be any pairing function on ω - with the property that $p_0(0,0) = 0$. Then consider $\omega^{\gamma_0} \cdot p_0(d_0, e_0) + \omega^{\gamma_1} \cdot p_0(d_1, e_1) + \cdots + \omega^{\gamma_{k-1}} \cdot p_0(d_{k-1}, e_{k-1})$.]

Sol: With $p(\beta_1, \beta_2) = \omega^{\gamma_0} \cdot p_0(d_0, e_0) + \omega^{\gamma_1} \cdot p(d_1, e_1) + \cdots + \omega^{\gamma_{k-1}} \cdot p_0(d_{k-1}, e_{k-1})$ just note that p is (1-1): we have expanded into their unique CNF's β_1, β_2 and combined the lists of the powers of ω needed; but for no i do we have both $e_i = d_i = 0$. Then because p_0 is (1-1), so is p. As we have specified $p_0(0,0) = 0$, we set also $p(0,0) = 0$ also. Moreover p is onto bcause p_0 is. From Ex. 3.30 an ordinal is indecomposable iff it is of the form ω^γ. Note then that if $\beta_1, \beta_2 < \omega^\gamma$ then the largest powers of ω in their CNF are of the kind ω^{γ_0} for a $\gamma_0 < \gamma$. Check then that $p(\beta_1, \beta_2) < \omega^\gamma$.