4 Solutions to Exercises Ch 4

Exercise 4.1. Prove that no finite set is equinumerous to a proper subset of itself.

Sol. By induction on n the cardinality of a finite set. If X has cardinality 0 then $X = \varnothing$ and it is trivial. Let Z have cardinality n. Then there is $f: n \approx Z$.

Suppose g: $Z \approx Z_0$ where $Z_0 \subseteq Z$. If Z_0 is a proper subset of Z then $f^{-1}``Z_0$ is a proper subset of n. (It omits some k where $f(k) \in Z - Z_0$.) But then $f^{-1} \circ g \circ f$ is a bijection between n and a proper subset of itself, contradicting Lemma 4.4. the Pidgeon-Hole Principle.

Exercise 4.2. Prove (i) Any finite set is equinumerous to a unique natural number.
(ii) ω is infinite.

Sol. (i) This can be proven directly by induction. Or: if $f: X \approx n$ and $g: X \approx m$ with $m \neq n$ we should have (if say $n < m$) that $f \circ g^{-1}: m \approx n$ again contradicting the Pidgeon hole Principle (PHP)

(ii) Suppose for a contradiction that $f: \omega \approx n$. Then ω is equinumerous with a proper subset of itself. By Cor. 4.5 it cannot be finite. Hence it is infinite.

Exercise 4.3. Show that if $A \subsetneqq n \in \omega$ then $A \approx m$ for some $m < n$. Deduce that any subset of a finite set is finite.

Sol: Let $A \subsetneqq n \in \omega$. Let $A \approx m$ (such an m exists by Ex. 4.2(i).) Then $m \neq n$ by PHP and for the same reason $m \not> n$. So $m < n$. For the second part if $A \subseteq B$ with B finite, then $f: B \approx n$ for some f, n then $f``A \subseteq n$. If $A$ is a proper subset of $B$ then $f``A$ is a proper subset of n, and so is bijective with some $m < n$. Hence it, and A, is finite.

Exercise 4.4. Suppose A is finite and $f: A \longrightarrow A$. Show that f is (1-1) iff $\text{ran}(f) = A$.

Sol: ($\Rightarrow$). Suppose $\text{ran}(f) \neq A$. If f were (1-1) this would contradict Ex. 4.1. ($\Leftarrow$) Suppose $\text{ran}(f) = A$. Let $h: A \approx n$. Define $g: A \longrightarrow A$ as follows. Let $X_a = \{b \in A \mid f(b) = a\}$. Let $g(a) =$ that member b of X_a with $h(b)$ least. Then g is welldefined. If f is not (1-1) there is some X_a with more than one element. Hence $\text{ran}(g) \neq A$. But then g maps the finite set A into a proper subset of itself contradicting Ex. 4.1.

Exercise 4.5. Let A, B be finite. Without using any arithmetic, show that $A \cup B$ and $A \times B$ is finite.

Sol: By induction on $m \approx B$. Let $m = 0$. Then $A \cup 0 = A$ is finite for any finite A. Suppose $m = 1$. Let $B = \{b\}$ have size 1. Check that $A \cup \{b\}$ is then also finite. (It has size that of A or one more, depending on whether $b \in A$ or not.) Suppose true for all B of size $m' \leq m$, and consider a set B of size $m + 1$. Let b_0 be chosen from B and then set $B_0 = B - \{b_0\}$. Then B_0 is of size m (check this). Then $A \cup B_0$ is finite. But by the case $m = 1$, so is $(A \cup B_0) \cup \{b_0\}$. Hence it is true for $m + 1$.

A similar argument works for $A \times B$.

Exercise 4.6. Show that if A is finite and $\langle A, R\rangle$ is a strict total order, then it is a wellorder (and note in this case that $\langle A, R^{-1}\rangle \in \text{WO}$ too).

Sol: By induction on $|A|$. If $|A|$=0, it is trivial. Suppose true for all strict total orders $\langle A, R\rangle$ with $|A|$=k. Let $k + 1 = |A| = |A' \cup \{a\}|$ for some $a \in A$. Then $\langle A', R\rangle$ is a wellorder by the inductive hypothesis, as $|A'|$=k. Let $\varnothing \neq X \subseteq A$. If $X \subseteq A'$ then we are done; otherwise $a \in X$. Let b be R-least in $A' \cap X$; if bRa then b is R-least in X. If not a is.

Exercise 4.7. (i) Show that $X \preceq Y$ implies that $\mathcal{P}(X) \preceq \mathcal{P}(Y)$; (ii) Show that if $X \preceq X'$ and $Y \preceq Y'$, then $X \times Y \preceq X' \times Y'$.(ii) Show that if $X \preceq X'$ and $Y \preceq Y'$, then $X \times Y \preceq X' \times Y'$. (iii) Give an example to show that $X \prec X'$ and $Y \preceq Y'$, does not imply that $X \times Y \prec X' \times Y'$.

Sol: (i) Suppose $f: X \preceq Y$. Define $F: \mathcal{P}(X) \preceq \mathcal{P}(Y)$ by $F(A) = \{f``u \mid u \in A\}$. As f is assumed (1-1) so will F be. (ii) If $f: X \preceq X'$ and $g: Y \preceq Y'$ are both (1-1) then trivially $\langle x, y\rangle \rightarrowtail \langle f(x), g(y)\rangle$ is a (1-1) function demonstrating that $X \times Y \preceq X' \times Y'$. (iii) Consider $X = 1, X' = 2$ and $Y = Y' = \omega$. Find a (1-1) map from $2 \times \omega$ into $1 \times \omega$.

Exercise 4.8. Show that (i) $(-1, 1) \approx \mathbb{R}$; (ii) $(0, 1) \approx [0, 1]$ by finding directly suitable bijections, *without* using Cantor-Schröder-Bernstein.

Sol: (i) Consider $f(x) = \tan \frac{\pi}{2} x$. (ii) Let f be the identity function apart from the following points: $f(0) = 1/3$; $f(1) = 1/9$; $f(1/(3^k)) = 1/(3^{k+2})$. Then $f: [0, 1] \approx (0, 1)$.

Exercise 4.9. Show that $\mathcal{P}(\omega) \approx \mathbb{R} \approx {}^{\omega}2$. [Hint: First show that $\mathcal{P}(\omega) \approx (0, 1)$. May be easier to show that $\exists f: \mathcal{P}(\omega) \preceq (0, 1)$ (by using characteristic functions of $X \subseteq \omega$ and mapping them to binary expansions). Then show that $\exists g$: $(0, 1) \preceq \mathcal{P}(\omega)$ using the same device. Then appeal to Cantor-Schröder-Bernstein to obtain the first $\approx$. Now note that $\mathcal{P}(\omega) \approx {}^{\omega}2$ is easy: subsets $X \subseteq \omega$ are in (1-1) correspondence with their characteristic functions χ_X.]

Sol: $(0, 1) \approx \mathbb{R}$ by a suitable tan function as in Ex.4.8 (i) above: $f(x) = \tan \pi\left(x - \frac{1}{2}\right)$. As remarked $\mathcal{P}(\omega) \approx {}^{\omega}2$ is easy: given $\chi \in {}^{\omega}2$ define X by $\chi(n) = 1 \longleftrightarrow n \in X$. Then check that $X \rightarrowtail \chi_X$ is (1-1) from $\mathcal{P}(\omega)$ onto ${}^{\omega}2$. [(1-1): suppose $X \neq Y$. Let $k \in X \Delta Y$. Then $\chi_X(k) \neq \chi_Y(k)$ and so $\chi_X \neq \chi_Y$. (onto): let $g \in {}^{\omega}2$. Let $X = \{n \mid g(n) = 1\}$; then $\chi_X = g$.] To show $\mathcal{P}(\omega) \approx (0, 1)$: firstly$f$: $\mathcal{P}(\omega) \preceq (0, 1)$ where we let $f(X) = 0.\chi_X(0)\chi_X(1)\chi_X(2)\chi_X(3)\cdots\chi_X(k)\cdots$ where the latter is considered a *decimal* expansion. Secondly consider *binary* expansions of elements of $r \in (0, 1)$: let $r = 0.00101...$ (where by convention we decide that no expansion ends in an infinite sequence of 1's alone, because, *eg*, 0.01000... is the same real as 0.00111111...); then the binary expansion defines an element $X_r \in \mathcal{P}(\omega)$ $X_r = \{n \mid$ the n'th digit in the binary expansion of r is a $1\}$. Then $r \neq q \longrightarrow X_r \neq X_q$. This establishes a map $g: (0, 1) \preceq \mathcal{P}(\omega)$. By CSB this suffices.

Exercise 4.10. Show directly (without using that $\mathcal{P}(X) \approx {}^X2$ or the CSB Theorem) that $X \prec {}^X 2$.

Sol: That $X \preceq {}^X2$ is straightforward: let $g(u) = g_u \in {}^X2$ be the function with $\mathrm{dom}(g_u) = X$ which is zero everywhere, except for u where $g_u(u) = 1$. Then g is (1-1). Suppose for a contradiction that $f: {}^X2 \longrightarrow X$ were an injection. We may define a bijection $h: \mathrm{ran}(f) \approx {}^X2$ by $h(x) = f^{-1}(x)$. Define $v \in {}^X2$ by:

$v(x) = 0$ if $h(x)(x)$ is defined and equals 1; otherwise $v(x) = 1$. Let $f(v) = z$. Now conclude that $v(z) = 0 \leftrightarrow v(z) = 1$ for a contradiction.

Exercise 4.11. (i) Show that $\varnothing \neq X$ is countable iff there is $f: \omega \longrightarrow X$ which is onto. [Hint for ($\Leftarrow$): Construct a (1-1) map from f, demonstrating $X \preceq \omega$.]

(ii) Prove that X is countable and infinite $\Longleftrightarrow X$ is countably infinite,

Sol:(i) ($\Rightarrow$) If g: $X \preceq \omega$ with g (1-1), then we can define f: $\omega \twoheadrightarrow X$ onto, by setting $f(m) = x_0$ for some fixed element of X for any $m \notin \mathrm{ran}(g)$; and $f(m) = g^{-1}(m)$ otherwise.

($\Leftarrow$) For $u \in X$ let $A_u =_{\mathrm{df}} \{n \in \omega \mid f(n) = u\}$. As f is onto, every $A_u \neq \varnothing$ and $u \neq v \longrightarrow A_u \cap A_v = \varnothing$. Define h: $X \preceq \omega$ by: $h(u) = \min A_u$. Then h is a (1-1) function because the A_u are disjoint.

(ii) Let X be countable and infinite. Then $X \preceq \omega$. Hence there exists a (1-1) $g: X \longrightarrow \omega$. Let $Y = \mathrm{ran}(g)$. As X is infinite, for no $n \in \omega$ do we have $Y \subseteq n$ (otherwise, by Ex 4.3 $Y \approx m$ for some $m \leq n$ and hence $X \approx m$). Define $k: \omega \to Y$ by $k(m) = m$'th element of Y in the usual $<$ ordering. Then $k^{-1} \circ g: X \approx \omega$.

Conversely suppose X is countably infinite, *i.e.* $\exists f: X \approx \omega$ a bijection. Clearly X is not finite, and $f: X \preceq \omega$ trivially.

Exercise 4.12. Show that $\omega \approx \omega \times \omega$. [Hint: consider the function $f(m, n) = 2^m(2n+1) - 1$. For future reference we let $(u)_0$ and $(v)_1$ be the (1-1) "unpairing" inverse functions from ω to ω so that $f^{-1}(u) = \langle (u)_0, (u)_1 \rangle$.]

Sol: Nothing left to do: just observe that f is a bijection! We can define the unpairing functions by: $(u)_0 =_{\text{df}}$ the greatest power of 2 dividing $u+1$.

$(u)_1 =_{\text{df}} \frac{u+1}{2^{(u)_0}}$.

Exercise 4.13. Show that $\mathbb{Z}, \mathbb{Q}$ are both countably infinite.

Sol: Clearly both sets are infinite. For $\mathbb{Z}$ define $f\colon \omega \approx \mathbb{Z}$ by $f(2n) = n$ for n even, and $f(2n+1) = -n$ for n odd. For $\mathbb{Q}$, first let $g\colon \omega \approx \omega \times \omega$ from the last exercise, and then h: $\omega \times \omega \longrightarrow \mathbb{Q}$ be defined by $h(\langle p, 0\rangle) = 0; h(\langle 2p, q\rangle) = p/q$ and $h(\langle 2p+1, q\rangle) = -p/q$. Then $h \circ g$ is onto $\mathbb{Q}$. Now appeal to Ex. 4.11.

Exercise 4.14. Prove: let X and Y be countably infinite sets, then $X \cup Y$ is countably infinite.

Sol: Let $h\colon \omega \approx X$ and $g\colon \omega \approx Y$. We shall define $l\colon \omega \approx X \cup Y$. *Case 1* If $Y \backslash X$ is finite, with $k\colon n \approx Y \backslash X$ for some n, k, then define l by: $l(i) = k(i)$ for $i < n$, and $= h(i-n)$ for $i \geq n$. *Case 2* $Y \backslash X$ is infinite. Define $l(2i) = h(i)$. Define $l(2i+1) = g(j)$ where j is least so that $g(j) \in Y \backslash (X \cup \operatorname{ran} l \restriction 2i+1) = Y \backslash (X \cup \{l(0), l(1) \ldots, l(2i)\})$.

Exercise 4.15. Let X, Y, Z be sets. Either by providing suitable bijections, or by establishing injections in each direction and using Cantor-Schröder-Bernstein, in each case show that:

(i) $X \times (Y \times Z) \approx (X \times Y) \times Z)$ and $X \times (Y \cup Z) \approx (X \times Y) \cup (X \times Z)$ (assume $Y \cap Z = \varnothing$) ;
(ii) ${}^{X \cup Y}Z \approx {}^{X}Z \times {}^{Y}Z$; (assume $X \cap Y = \varnothing$)
(iii) ${}^{X}(Y \times Z) \approx {}^{X}Y \times {}^{X}Z$;
(iv) ${}^{X}({}^{Y}Z) \approx {}^{(X \times Y)}Z$.

Sol. (i) Define $F(\langle x, \langle y, z\rangle\rangle) = \langle\langle x, y\rangle, z\rangle$ and check that F is (1-1) and onto.

For the second conjunct: $X \times (Y \cup Z) = (X \times Y) \cup (X \times Z)$ so $X \times (Y \cup Z) \approx (X \times Y) \cup (X \times Z)$ trivially! There is literally nothing to do.

(ii) Given $f \in {}^{X \cup Y}Z$ define $F(f) = \langle g, h\rangle \in {}^{X}Z \times {}^{Y}Z$ by setting $g(x) = f(x)$ if $x \in X$, and $h(y) = f(y)$ if $y \in Y$. Clearly $f \neq k \longrightarrow F(f) \neq F(k)$. So F is (1-1). But it is clearly onto: given any $\langle g, h\rangle \in {}^{X}Z \times {}^{Y}Z$ define $f \in {}^{X \cup Y}Z$ in the obvious way: $f(u) = g(u)$ $(h(u))$ if $u \in X$ $(u \in Y)$.

(iii) Let now $(-)_0$ and $(-)_1$ be those functions so that if $x = \langle u, v\rangle$ is any ordered pair, then $(x)_0 = u$ and $(x)_1 = v$. Let $f \in {}^{X}(Y \times Z)$ and then define $F(f) = \langle g, h\rangle \in {}^{X}Y \times {}^{X}Z$ by setting $g(x) = (f(x))_0$ and $h(x) = (f(x))_1$. F is easily seen to be (1-1), but also onto: if $\langle g, h\rangle \in {}^{X}Y \times {}^{X}Z$, then define $f(x) = \langle g(x), h(x)\rangle$ and check that $F(f) = \langle g, h\rangle$.

(iv) Suppose $f \in {}^{X}({}^{Y}Z)$. Then define $F(f) \in {}^{(X \times Y)}Z$ to be the function defined by: $F(f)(x, y) = (f(x))(y)$ (noting that $f(x) \in {}^{Y}(Z)$). Again F is (1-1). If $G \in {}^{(X \times Y)}Z$ then define $f \in {}^{X}({}^{Y}Z)$ by setting $f(x) \in {}^{Y}Z$ to be the function evaluated as $f(x)(y) = G(x, y)$. So F is onto.

Exercise 4.16. Suppose that K, L are sets bijective with ordinals. Show that $K \cup L$, and $K \times L$ are bijective with ordinals.

Sol: The first part is easiest if we assume that K, L are disjoint. Let $f\colon K \approx \kappa, g\colon L \approx \lambda$. Define $h\colon K \cup L \approx \kappa + \lambda$ in the obvious way: $h(u) = f(u)$ if $u \in K$, $= \kappa + g(u)$ otherwise. (If they are not disjoint we can define a recursion that takes some $\mu \leq \kappa + \lambda$ steps, defining an enumeration $h\colon \mu \approx K \cup L$ by: $h(\alpha) = f^{-1}(\alpha)$ for $\alpha < \kappa$, and then for $\kappa \leq \alpha < \mu$ let $h(\alpha)$ be the element v of L not yet enumerated by $h \restriction \alpha$ so far, which has $g^{-1}(v)$ least in λ. After some number of stages all of $K \cup L$ will have been enumerated, and the recursion stops with $\operatorname{dom}(h)$ some ordinal μ.)

Define $k\colon K \times L \approx \lambda \cdot \kappa$ by $k(\langle u, v\rangle) = \lambda \cdot f(u) + g(v)$.

Exercise 4.17. Complete (ii) $X \preceq Y \Longleftrightarrow |X| \leq |Y|$;and (iii) $X \prec Y \Longleftrightarrow |X| < |Y|$ of Lemma 4.21.

Sol: In the notation of the proof of the Lemma, for (ii) ($\Rightarrow$) if $f: X \preceq Y$ and $\lambda < \kappa$ we should have a (1-1) map $h \circ f \circ g^{-1}: \kappa \longrightarrow \lambda$. As the identity is a (1-1) map of λ into κ, by C-S-B we should have $\kappa \approx \lambda$ which is absurd. For ($\Leftarrow$) If $\kappa \leq \lambda$ then $h^{-1} \circ g: X \preceq Y$.

For (iii) this is very similar. ($\Leftarrow$) We already have $h^{-1} \circ g: X \preceq Y$ by part (ii). But $|Y| \not\leq |X|$ implies $X \not\approx Y$ by part (i). Hence by C-S-B $Y \not\preceq X$. Thus $X \prec Y$. For ($\Rightarrow$): We have by (ii) that $|X| \leq |Y|$, but also by (ii) $\lambda \leq \kappa \longrightarrow Y \preceq X$. As we are assuming $Y \not\preceq X$ we conclude $\kappa < \lambda$.

Exercise 4.18. Check: each $n \in \omega$ is a cardinal, ω itself is a cardinal. [Hint: just consult the definition, together with some previous lemmas and corollaries.]

Sol: If $n \in \omega$ were not a cardinal then it would be equinumerous with some $m < n$. This would contradict the Pidgeon-Hole Principle. Similarly if ω were not a cardinal we should have $\omega \approx n$ for some $n < \omega$. This contradicts Corollary 4.8

Exercise 4.19. Suppose $\alpha \geq \omega$. (i) Show $\alpha \approx \alpha + 1$. (ii) Show that $\alpha + n$ is not a cardinal (if $n > 0$), nor is $\alpha + \omega$. [Hint: try it with $\alpha = \omega$ first; find a (1-1) map f from $\alpha + n$ (or $\alpha + \omega$ respectively) into α.]

Sol: (i) Let $f(\beta) = \beta$ if $\beta \in \alpha - \omega$ or $\beta = \omega$ (when $\alpha > \omega$). Let $f(0) = \alpha$ and $f(n+1) = n$.

(ii) By induction: we have just shown that $\alpha + 1 \approx \alpha$ and thus $\alpha + 1$ is not a cardinal. By repeating this argument n times we see that $\alpha + n$ is not a cardinal either. For $\alpha + \omega$ define $f(\beta) = \beta$ if $\beta \in \alpha - \omega$, and $f(n) = 2n + 1$, and $f(\alpha + n) = 2n$. Then $f: \alpha + \omega \approx \alpha$.

Note: Exercise 4.19 shows that infinite cardinals are limit ordinals.

Exercise 4.20. Let S be a set of cardinals without a largest element. Show that $\sup S$ is a cardinal.

Sol: Let $\tau = \sup S$. Suppose $\delta < \tau$ and $f: \tau \longrightarrow \delta$ is a bijection (for a contradiction). As $\tau = \sup S$ there is a cardinal $\gamma \in S \wedge \delta < \gamma < \tau$. Hence $f \restriction \gamma: \gamma \longrightarrow \delta$ is (1-1). Hence $\gamma \preceq \delta$. But trivially $\delta \preceq \gamma$. By CSB then $\gamma \approx \delta$ and so γ is not a cardinal! Contradiction.

Exercise 4.21. Show that an infinite set cannot be split into finitely many sets of strictly smaller cardinality. [Hint: Suppose that Y is an infinite set. First consider two sets. Let $X \subseteq Y$, and suppose that $|X| < |Y|$. Show that $|Y \backslash X| = |Y|$. See if you can show this directly, without using Cor. 4.27 to come.]

Sol: If $|Y \backslash X| < |Y|$ then we should have that Y was the disjoint union of two sets of smaller cardinality than itself. We show this is impossible. Let $\kappa = \max\{|X|, |Y \backslash X|\} < \lambda = |Y|$. Define an ordinal to be *even* if it is of the form $\alpha + 2n$ for some $n \in \omega$ and limit ordinal α; otherwise it is odd. (Notice that limit ordinals are even). Let $g: \kappa \twoheadrightarrow X$, $h: \kappa \twoheadrightarrow Y \backslash X$ be onto, and define $f: \kappa \twoheadrightarrow Y$ by: $f(\lambda + 2n) = g(\lambda + n)$, $f(\lambda + 2n + 1) = h(\lambda + n)$ where $\mathrm{Lim}(\lambda)$ and $n < \omega$. This is a contradiction to $\kappa < \lambda = |Y|$. Now apply induction: if Y cannot be split into k such sets of smaller cardinality, then let $Y = X_0 \cup X_1 \cup \cdots \cup X_k$, with each $|X_i| < |Y|$. But then consider Y as $(X_0 \cup X_1) \cup \cdots \cup X_k$, a union of $k - 1$ sets, and apply the inductive hypothesis. (With Cor 4.27 this becomes easy as $|X_0 \cup X_1 \cup \cdots \cup X_k| = \max\{|X_0|, |X_1|, \ldots, |X_k|\}$.)

Exercise 4.22. Perform the inductions in the last lemma.

Sol: We just do $+$. Let m be arbitrary.

$n = 0$: Then $m + 0 = m$ (def. of ordinal addition) $= m \oplus \varnothing$.

Suppose true for $n = k$: Then

$m + (k+1) = (m+k) + 1$ (recursive def. of ordinal addition)

$= (m \oplus k) + 1$ (I.H) $= m \oplus k \cup \{m \oplus k\}$

$= (m \oplus k) \oplus 1$. (The latter is established *via* a function $f: m \oplus k \cup \{m \oplus k\} \approx (m \oplus k) \times \{0\} \cup 1 \times \{1\}$ defined by: $f(l) = \langle l, 0 \rangle$ for $l \in m \oplus k$ and $f(m \oplus k) = \langle 0, 1 \rangle$.

Exercise 4.23. For any ordinals α, β: $|\alpha +' \beta| = |\alpha| \oplus |\beta|$; $|\alpha \cdot' \beta| = |\alpha| \otimes |\beta|$ (and so the same will hold for ordinal $+$ and $\cdot$).

Sol: Just look at the definitions of $+'$ and $\oplus$. and the $\cdot'$ and $\otimes$.

Exercise 4.24. Suppose $\langle A, R\rangle \in \mathrm{WO}$ and there is a cardinal κ with $|A| \geq \kappa$, but so that for every $b \in A$ the initial segment $\langle A_b, R\rangle \cong \langle \delta, \in\rangle$ for a $\delta < \kappa$. Show that $\mathrm{ot}(\langle A, R\rangle = \kappa$.

Sol: If $\mathrm{ot}(\langle A, R\rangle) > \kappa$ then for some $a \in A$ we should have that b is the "κ'th element": $\mathrm{ot}(\langle A_b, R\rangle) = \kappa$, but this is contrary to what we are told about $\langle A, R\rangle$. Hence $\mathrm{ot}(\langle A, R\rangle \leq \kappa$. However we are also told that $|A| \geq \kappa$, so A cannot be bijective with any $\gamma < \kappa$. Hence $\mathrm{ot}(\langle A, R\rangle \geq \kappa$.

Exercise 4.25. Show that for infinite cardinals $\omega \leq \kappa \leq \lambda$ that $\kappa \oplus \lambda = \lambda$ directly, that is without use of Hessenberg's Theorem.

Sol: Let f: $\kappa \times \{0\} \cup \lambda \times \{1\} \longrightarrow \lambda$ be defined by: $f(\langle \alpha + n, 0\rangle) = \alpha + 2n$; $f(\langle \alpha + n, 1\rangle) = \alpha + 2n + 1$ for $n < \omega$ and $\mathrm{Lim}(\alpha)$ with $\alpha < \kappa$. For any β with $\kappa \leq \beta < \lambda$, let $f(\langle \beta, 1\rangle) = \beta$. Now check f is a bijection.

Exercise 4.26. Let $\lhd$ be the wellorder on $\kappa \times \kappa$ from Hessenberg's Theorem. For $\alpha \geq \omega$ let $o(\alpha) =_{\mathrm{df}} \mathrm{ot}(\alpha \times \alpha, \lhd)$. Show (i) $\{\langle \alpha, \beta\rangle | \langle \alpha, \beta\rangle \lhd \langle 0, \gamma\rangle\} = \gamma \times \gamma$; (ii) $o(\alpha + 1) = o(\alpha) + \alpha + \alpha + 1$; (iii) $o(\omega) = \omega$; $o(\omega \cdot 2) = \omega \cdot \omega$; in general $o(\alpha) \leq \alpha^3$;(added Dec 7th 20). (iv) $o(\alpha) = \alpha$ implies α is additively indecomposable. (v)* (Harder) $o(\alpha) = \alpha$ if and only if α is multiplicatively indecomposable, that is, $\eta, \gamma < \alpha \longrightarrow \eta \cdot \gamma < \alpha$. [See Ex.3.32.]

Sol: (i) is just by definition of $\lhd$. (ii) The difference between $\alpha + 1 \times \alpha + 1$ and $\alpha \times \alpha$ in the diagram is the points added first along the horizontal, in order type α, followed by those in the vertical, of order type α; the final point $\langle \alpha, \alpha\rangle$ being added at the end. Thus making $o(\alpha) + \alpha + \alpha + 1$ altogether.

(iii) Just draw the boxes. For the last part this is by induction on β with the successor step given by: $o(\alpha + 1) = o(\alpha) + \alpha \cdot 2 + 1$ (by (ii)) $\leq \alpha^3 + \alpha \cdot 2 + 1$ (by I.H.)$\leq \alpha^3 + \alpha^2$ (if $\alpha \geq 3$) $= \alpha^2 \cdot (\alpha + 1) \leq (\alpha + 1)^3$.

(iv) If α were not (add.) indecomposable, there would be by definition (see Ex 3.30) η, $\gamma < \alpha$ with $\eta + \gamma \geq \alpha$, in particular if, wlog, η is the larger of the two, $\eta + \eta \geq \alpha$. But clearly $\mathrm{Lim}(\alpha)$ so $\eta + 1 < \alpha$ and then $o(\alpha) > o(\eta + 1) = o(\eta) + \eta + \eta + 1 > \alpha$ by (ii). For (v): Note first that α is mult. indecomposable is equivalent to saying that $\eta < \alpha \longrightarrow \eta^2 < \alpha$. (Check). Or moreover $\eta < \alpha \longrightarrow \eta^3 < \alpha$. So suppose α is such. By (iii): $o(\eta) = \mathrm{ot}(\eta \times \eta, \lhd) \leq \eta^3$. Then $\alpha \leq o(\alpha) = \mathrm{ot}(\alpha \times \alpha, \lhd) = \sup_{\eta < \alpha} \mathrm{ot}(\eta \times \eta, \lhd) \leq \sup_{\eta < \alpha} \eta^3 = \alpha$. Conversely suppose $o(\alpha) = \alpha$. First show by ind. on $\beta \geq \omega$ that $\beta \cdot \beta \leq o(\beta + \beta)$. (Successor step: $o((\beta + 1) + (\beta + 1)) = o((\beta + \beta + 1)) = o((\beta + \beta)) + (\beta + \beta) \cdot 2 + 1 \geq \beta^2 + \beta \cdot 2 + 1 = (\beta + 1)^2$.) We then show that $\eta < \alpha \longrightarrow \eta^2 < \alpha$ and so α is mult. indec. Fix such an $\eta < \alpha$; noting that $\eta + \eta < \alpha$ (as α is add. indecomp. by (iv)), we have $\eta^2 \leq o(\eta + \eta) < o(\alpha) = \alpha$.

Exercise 4.27. Show that if $\kappa \geq \omega$ is an infinite cardinal, then it is a fixed point of any of the ordinal arithmetic operations A_α, M_α or E_a for any $\alpha < \kappa$: $\alpha + \kappa = \kappa$; $\alpha \cdot \kappa = \kappa$ and $\alpha^\kappa = \kappa$.

Sol: For example consider M_α: $\alpha \cdot \kappa = \sup\{\alpha \cdot \beta | \beta < \kappa\}$. On the one hand $|\alpha \cdot \beta| = |\alpha \cdot' \beta| \leq |\alpha \times \beta| < \kappa$ (think of $\alpha \cdot' \beta$ as occupying the box $\alpha \times \beta$, and $|\alpha \times \beta| = |\alpha| \otimes |\beta| < \kappa$). On the other M_α is strictly increasing, so this sup is greater than or equal to κ.

Exercise 4.28. Show that ${}^n A \approx A \times \cdots \times A$ (the n-fold cartesian product of A).

Sol: This is easy: we just "identify" the finite function $f \in {}^n A$ with the n-tuple in $A \times \cdots \times A$: $f = \{\langle 0, f(0)\rangle, \langle 1, f(1)\rangle\rangle, \ldots \langle n - 1, f(n - 1)\rangle\} \leftrightarrow \langle f(0), f(1), \ldots, f(n - 1)\rangle$. This is clearly a bijection between the two sets.

Exercise 4.29. Assume WP. Let $|X_n| = \kappa \geq \omega$ for $n < \omega$. Show that that $|\bigcup_n X_n| = \kappa$. (This is the generalisation of Lemma 4.21 for uncountable sets X_n.) [Hint: Follow closely the format of Lemma 4.21; use the fact that we now know $\omega \times \kappa \approx \kappa$ to replace $\omega \times \omega = \omega$ in that argument.]

Sol: Clearly $|\bigcup_n X_n| \geq \kappa$. Let $Z = \{g \mid \exists i < \omega\ (g\colon \kappa \approx X_i)\}$. Then Z is a set (it is a subset of $\bigcup \{{}^{\omega}X_i \mid i < \omega\}$). Let R be a wellordering of Z. Suppose we have a function f satisfying $f\colon \omega \times \kappa \approx \kappa$. Then we may as before define $g\colon \kappa \longrightarrow \bigcup_{i<\omega} X_i$ by $g(f(i, \nu)) = g_i(\nu)$ and where g_i is the R-least function $g\colon \kappa \approx X_i$. Then *Check* that g is onto. We know such an f exists, since by cardinal arithmetic $|\omega \times \kappa|$=max$\{\omega, \kappa\} = \kappa$. Hence $\omega \times \kappa \approx \kappa$.

Exercise 4.30. Show that $|{}^{X}Y| = |{}^{X}|Y|\,| = |{}^{|X|}Y| = |Y|^{|X|}$. Deduce that the definition of κ^λ is independent of the choices of sets L, K.

Sol: Let X,Y be any sets with $|X| = \lambda$, $|Y| = \kappa$. Let $f\colon X \approx \lambda$ and $g\colon Y \approx \kappa$. Define $F\colon {}^{X}Y \longrightarrow {}^{\lambda}\kappa$ by $F(h) = g \circ h \circ f^{-1}$. Note that F is injective:

$F(h_1) = F(h_2) \rightarrow g \circ h_1 \circ f^{-1} = g \circ h_2 \circ f^{-1} \rightarrow g^{-1} \circ g \circ h_1 \circ f^{-1} \circ f = g^{-1} \circ g \circ h_2 \circ f^{-1} \circ f \rightarrow h_1 = h_2$.

But F is also onto: let $k \in {}^{\lambda}\kappa$. Then $k = F(h)$ where $h = g^{-1} \circ k \circ f$.
Thus $|{}^{X}Y| = |{}^{|X|}|Y|\,|$. And thence $|{}^{\lambda}\kappa| = \kappa^\lambda$, by def. of κ^λ, which is $|Y|^{|X|}$. By the same considerations $|{}^{X}|Y|\,| = |{}^{|X|}Y| = |{}^{|X|}|Y| = 6\,|Y|^{|X|}$.

Exercise 4.31. (Without WP, that is without assuming there is γ with $\gamma \approx x$.) Prove the last Corollary 4.23, that for any set x there is an ordinal ν so that $\nu \not\preccurlyeq x$. [Hint: this is really Hartogs' theorem, with the set x substituted for α throughout.]

Sol: As the hint suggests, let $S =_{\rm df} \{R \mid \langle x, R\rangle \in {\rm WO}\}$. Let $\tilde{S} = \{{\rm ot}(\langle x, R\rangle) \mid R \in S\}$. Then take $\nu = \sup \tilde{S}$.

Exercise 4.32. Are there ordinals α so that $\alpha = \omega_\alpha$? If so find one. (Such would be a *fixed point* of the cardinal enumeration function F, as we should have $F(\alpha) = \alpha$.)

Sol: Let $\alpha_0 = \omega_0$. Let $\alpha_{n+1} = \omega_{\alpha_n}$. Then note $\alpha_n < \alpha_{n+1}$. Let $\tilde{\alpha} =_{\rm df} \sup\{\alpha_n \mid n < \omega\}$. Then check $\gamma < \tilde{\alpha} \leftrightarrow \exists n \gamma < \alpha_n \leftrightarrow \exists n \gamma < \omega_{\alpha_n} \leftrightarrow \gamma < \omega_{\sup\{\alpha_n\}} \leftrightarrow \gamma < \omega_{\tilde{\alpha}}$.

Exercise 4.33. Show that CH is equivalent to the statement that every ordinal less than $2^{\aleph_0}$ is countable.

Sol: Obvious from the definition.

Exercise 4.34. Show that (i) the set of countable subsets of $\mathbb{R}$ has cardinality $2^{\aleph_0}$;
(ii) the set of countable subsets of $\mathbb{R}$ which contain $\mathbb{Q}$ also has cardinality $2^{\aleph_0}$;
(iii) the set of open intervals of $\mathbb{R}$ has cardinality $2^{\aleph_0}$.

Sol: X is countable if and only if it is the range of a map with domain ω: $f\colon \omega \longrightarrow X$ (Ex 4.11). Thus X is a countable subset of $\mathbb{R}$ iff $\exists f \in {}^{\omega}\mathbb{R}({\rm ran}(f) = X)$. But $|{}^{\omega}\mathbb{R}| = |\mathbb{R}|^\omega = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \otimes \aleph_0} = 2^{\aleph_0}$ which shows (i). (ii) By: (i) the set in question has cardinality $\leq 2^{\aleph_0}$. But for any countable set $X \subseteq \mathbb{R}$. $X \cup \mathbb{Q} \subseteq \mathbb{R}$ is a countable subset, hence the cardinality of such is $\geq 2^{\aleph_0}$. For (iii): such an open interval is of the form (r_0, r_1) for a pair of reals $r_0, r_1 \in \mathbb{R}$. But $|{}^{2}\mathbb{R}| = |\mathbb{R}|^2 = |\mathbb{R}| \otimes |\mathbb{R}| = |\mathbb{R}| = 2^{\aleph_0}$.

Exercise 4.35. Prove that there is λ with $\lambda = \beth_\lambda$.

Sol: As in Ex 4.32, replacing ω_{α_n} by $\beth_{\alpha_n}$ throughout.

Exercise 4.36. Show that the union of $\kappa \geq \omega$ many sets of cardinality κ is a set of cardinality κ. [Hint: If $\langle A_\iota | \iota < \kappa \rangle$ are the sets with $|A_\iota| = \kappa$ then consider a (1-1) map into $\kappa \oplus \kappa$.]

Sol: One way: simply follow the proof of Lemma 18 replacing "countable" by "of cardinality κ". [In detail: Let $g\colon \kappa \longrightarrow \bigcup A_\iota$ be defined by $g(f(\iota, \beta)) = g_\iota(\beta)$, where now g_ι is the R-least function from $Z = \{g | \exists \iota < \kappa(g\colon \kappa \approx X_\iota)\}$ where $\langle Z, R \rangle \in$ WO. Then $\mathrm{dom}(g) = \mathrm{ran}(f) = \kappa$ and is onto. Thus $|\bigcup A_\iota| \leq \kappa$. But also as any $|A_\iota| = \kappa$, $|\bigcup A_\iota| \geq \kappa$.]

Exercise 4.37. Place in correct order the following cardinals using $=, <, \leq$:

$\aleph_{13}$, $\aleph_{\omega^2}$, $\varnothing$, $\aleph_{\omega_1}^{\aleph_{\omega_1}}$, $\sup\{\aleph_n | n < \omega\}$, $\aleph_{\omega_1} \oplus \aleph_\omega$, $\aleph_\omega$, $\aleph_{\omega_1} \otimes \aleph_{\omega_1}$, $\aleph_\omega \oplus \aleph_{\omega_1}$, $2^\varnothing$, $\aleph_{\omega_1}$.

You should give your reasons; apart from the 'ω^2' in the second cardinal, the arithmetic is all cardinal arithmetic.

Sol: $\varnothing < 1 = 2^\varnothing < \aleph_{13} < \sup\{\aleph_n | n < \omega\} = \aleph_\omega < \aleph_{\omega^2} < \aleph_{\omega_1} \otimes \aleph_{\omega_1} = \aleph_\omega \oplus \aleph_{\omega_1} = \aleph_{\omega_1} = \aleph_{\omega_1} \oplus \aleph_\omega < \aleph_{\omega_1}^{\aleph_{\omega_1}}$. The first three $<$ are clear as $2^\varnothing = 1$ is finite; and $\aleph_{13} < \aleph_n$ for $n > 13$. $\aleph_\omega$ is the first limit cardinal, and is equal to $\sup\{\aleph_n | n < \omega\} = \aleph_\omega$. As $\omega < \omega^2$ and the cardinal enumeration function is strictly increasing we have $\aleph_\omega < \aleph_{\omega^2} < \aleph_{\omega_1}$. By cardinal arithmetic $\aleph_{\omega_1} \otimes \aleph_{\omega_1} = \aleph_\omega \oplus \aleph_{\omega_1} = \aleph_{\omega_1} \oplus \aleph_\omega = \max\{\aleph_\omega, \aleph_{\omega_1}\} = \aleph_{\omega_1}$. By Cantor's theorem $\aleph_{\omega_1} < 2^{\aleph_{\omega_1}} \leq \aleph_{\omega_1}^{\aleph_{\omega_1}}$.

Exercise 4.38. Simplify where possible: $2^{\aleph_0}$; $\aleph_\omega \oplus \aleph_{\omega_1}$; $(2^{\aleph_0})^{\aleph_1}$; $\aleph_\omega^3 \oplus (\aleph_5)^2$.

You should do this twice: the first time without assuming the Generalised Continuum Hypothesis, and the second time assuming it. (The operations are all cardinal arithmetic.)

Sol: Without GCH: $2^{\aleph_0}, \aleph_{\omega_1}, (2^{\aleph_0})^{\aleph_1} = 2^{\aleph_0 \otimes \aleph_1} = 2^{\aleph_1}, \aleph_\omega^3 \oplus (\aleph_5)^2 = \aleph_\omega \oplus \aleph_5 = \aleph_\omega$.
With GCH: $\aleph_1, \aleph_{\omega_1}, \aleph_2, \aleph_\omega$.

Exercise 4.39. Show directly (without using Hessenberg's Theorem) that for $n < \omega$ $(\beth_n)^2 = \beth_n$. [Hint: use induction on n.]

Sol: We already know that $\omega = \omega_0 = \omega_0 \otimes \omega_0 = \beth_0 \otimes \beth_0$ since we showed directly that $\omega \approx \omega \times \omega$. Assume that $\beth_n \otimes \beth_n = \beth_n$ is proven. The $\beth_{n+1} \otimes \beth_{n+1} = 2^{\beth_n} \otimes 2^{\beth_n} = 2^{\beth_n \oplus \beth_n}$. But Ex. 4.25 shows directly that $\kappa \oplus \kappa = \kappa$ for any cardinal $\kappa \geq \omega$. So the latter equals $2^{\beth_n} = \beth_{n+1}$.

Exercise 4.40. This exercise asks you to show that various classes of sequences $\{a_n\}_{n<\omega}$ with each $a_n \in \mathbb{N}$ are countable.

(i) The *eventually constant* sequences: $\exists k_0 \forall k \geq k_0\ a_k = a_{k_0}$;
(ii) The *arithmetic progressions*: $\exists p \forall n\ a_{n+1} = a_n + p$;
(iii) The *geometric progressions*: $\exists p \forall n\ a_{n+1} = a_n \cdot p$.

Exercise 4.41. A real number is said to be *algebraic* if it is a root of a polynomial $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$ where each $a_i \in \mathbb{Q}$. Show that there are only countably many algebraic numbers. A real number that is not algebraic is called *transcendental*. Deduce that almost all real numbers are transcendental, in that the set of such is equinumerous with $\mathbb{R}$.

Sol: Every such polynomial is determined by precisely one finite sequence of coefficients from $\mathbb{Q}$. Hence the set of algebraic real numbers has cardinality $|{}^{<\omega}\mathbb{Q}| = |{}^{<\omega}\omega| = \omega$ (See Cor.4.29). By Ex.4.21 then the set of transcendental numbers has cardinality that of $\mathbb{R}$.

Exercise 4.42. A *word* in an alphabet Σ is a string of symbols from Σ of finite length. Show that the number of possible words made up from the roman alphabet is countable. If we enlarge the alphabet to be now countably infinite, is the answer different?

Sol: A word is then any finite sequence from Σ, and the cardinality of such is $|{}^{<\omega}\Sigma| = |\bigcup_n {}^n\Sigma|$. If Σ is finite then this union has size ω; if infinite, with $\omega \leq \kappa = |\Sigma|$, then each $|{}^n\Sigma| = \kappa$, and the union of countably many κ-sized sets has cardinality κ. Thus the answer is only different if we allow uncountable alphabets Σ.

Exercise 4.43. What is the cardinality of (i) the set of all order isomorphisms $f\colon\mathbb{Q}\to\mathbb{Q}$; (ii) the set of all continuous functions $f\colon\mathbb{R}\longrightarrow\mathbb{R}$? ; (iii) the set of all convergent series $\Sigma_{n=0}^{\infty} a_n$ of real numbers?

Sol: (i) The set of all such order isomorphisms is a subset of ${}^{\mathbb{Q}}\mathbb{Q}$ which has cardinality $\aleph_0^{\aleph_0} = 2^{\aleph_0}$. So $2^{\aleph_0}$ is an upper bound. We show that $2^{\aleph_0} = |\mathcal{P}(\omega)|$ is a lower bound too. Let $X\subseteq\omega$. Define an order iso. (which is also continuous for part(ii)) $f = f_X$ for $p\in\mathbb{Q}$, by $f\restriction(\infty,0) = \mathrm{id}$; if $k\in X$ then $f\restriction[k,k+1)\cap\mathbb{Q}$ is given by $f(p) = k+(p-k)/2$ for $p\in\left[k,k+\frac{1}{2}\right)$; $f(p)=\frac{3}{2}\left(p-\left(k+\frac{1}{2}\right)\right)+\left(k+\frac{1}{4}\right)$ for for $p\in\left[k+\frac{1}{2},k+1\right)\cap\mathbb{Q}$. If $k\notin X$ then $f\restriction[k,k+1)$ =id. Clearly if $X\neq Y$ then $f_X\neq f_Y$. (ii) $|{}^{\mathbb{R}}\mathbb{R}| = (2^{\aleph_0})^{2^{\aleph_0}} = 2^{\aleph_0\otimes 2^{\aleph_0}} = 2^{2^{\aleph_0}} > 2^{\aleph_0}$ so *prima facie* there could be more such functions. But notice that for *continuous* functions their action is entirely fixed by their action on the rationals (as by continuity $f(r)=\lim_n f(p_n)$ where p_n is any sequence of rationals converging to r). So there are no more such continuous functions than there are maps $\mathbb{Q}\to\mathbb{Q}$, which is $2^{\aleph_0}$. By part (i) this is then a lower bound too. (iii) The set of all ω-sequences $(a_i)_{i<\omega}$ from $\mathbb{R}$ is of cardinality $|{}^{\omega}\mathbb{R}| = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$. So this is an upper bound. But for every $r\in\mathbb{R}$ there is easily a power series convergent with sum r. So $|\mathbb{R}| = 2^{\aleph_0}$ is a lower bound too.

Exercise 4.44. (i) The *Cantor set* C is the set of all real numbers of the form $\Sigma_{n=0}^{\infty} a_n\cdot 3^{-n}$ with $a_n\in\{0,2\}$. Show that $C\approx\mathbb{R}$. (ii) The *Hilbert cube* is the set $\mathcal{H} = {}^{\mathbb{N}}[0,1]$. What is $|\mathcal{H}|$?

Sol: (i) Trivially $C\preceq\mathbb{R}$. By the Solution to Ex.4.9 above we have $\mathbb{R}\approx(0,1)$. For any $x\in(0,1)$ consider its binary expansion as $\Sigma_{n=0}^{\infty} b_n\cdot 2^{-n}$ with $b_n\in\{0,1\}$. Now just double every digit. This yields a number with ternary expansion $\Sigma_{n=0}^{\infty} a_n\cdot 3^{-n}$ with $a_n\in\{0,2\}$ where $a_n = 2b_n$. Clearly this is (1-1). Hence $\mathbb{R}\preceq C$. For (ii): we know that $\mathbb{R}\approx[0,1]$, hence $|[0,1]| = 2^{\aleph_0}$. As $\mathbb{N}$ is countably infinite we have then $|\mathcal{H}| = |{}^{\aleph_0}(2^{\aleph_0})| = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0\otimes\aleph_0} = 2^{\aleph_0}$.

Exercise 4.45. Let $\mathcal{V}$ be a vector space, with a basis B. We suppose B to be infinite, in which case we have that $\mathcal{V}$ is an infinite dimensional vector space. How many finite dimensional subspaces does $\mathcal{V}$ have?

Sol: Any finite dimensional subspace has a finite basis with elements which can be taken uniquely from B. If $|B| = \kappa$ then the set of all such finite element sets from B has cardinality $|{}^{<\omega}B| = |B|$, and this is the cardinality of the set of such subspaces.

Exercise 4.46. Show that the set of all permutations of $\mathbb{N}$ has cardinality $2^{\aleph_0}$.

Exercise 4.47. Show that the set of all Riemann integrable functions on $\mathbb{R}$ has cardinality $(2^{\aleph_0})^{2^{\aleph_0}}$.

Exercise 4.48. Let $(\mathbb{N},\prec)$ be any strict total order. Show that there is a (1-1) order preserving embedding of $(\mathbb{N},\prec)$ into $(\mathbb{Q},<)$.

Sol: We build an embedding f in pieces. Let $f(0)$ be any element of $\mathbb{Q}$. If $f\restriction k$ has been defined, then define $f(k)$ to respect the ordering $\prec$: if $k\prec m$ or $m\prec k$ for all $m<k$ then pick any $q<f(m)$ or $f(m)<q$ for all $m<k$ and set $f(k)=q$. Otherwise, pick a single q with $f(m_0)<q<f(m_1)$ for all pairs m_0, m_1 with $m_0\prec k\prec m_1$.

Exercise 4.49. Let $(\mathbb{N},\prec)$ be any strict total order; show that there is a (1-1) order preserving map of $(\mathbb{N},<)$ either into $(\mathbb{N},\prec)$ or into $(\mathbb{N},\succ)$.

Sol: If $(\mathbb{N},\prec)\in\mathrm{WO}$ then it is isomorphic to an (infinite) ordinal $\langle\alpha,<\rangle$ and then trivially $\langle\mathbb{N},<\rangle$ is a subordering of α. Otherwise we show the latter alternative: let $\varnothing\neq X\subseteq\mathbb{N}$ be without a $\prec$-least element. Then X is infinite. Let $f(0)$ be any element of X. If $f(k-1)$ has been defined, let $f(k)=n$ where n is the $<$-least element of X, with $n\prec f(k-1)$. This will succeed as no element of X is $\prec$-minimal.

Exercise 4.50. Let $X \subseteq \mathbb{R}$ and suppose that $(X, <) \in \mathrm{WO}$ where $<$ is the usual order on $\mathbb{R}$. Show that X is countable.

Sol: Suppose for a contradiction $\omega_1 \leq |X|$. Then $\mathrm{ot}(\langle X, <\rangle) \geq \omega_1$. Then, for some $\alpha \geq \omega_1$, some f, $f: \langle \alpha, \in\rangle \cong \langle X, <\rangle$. But for every $\beta < \alpha$ $I_\beta =_{\mathrm{df}} (f(\beta), f(\beta+1))$ is an open interval in $\mathbb{R}$ - and these are all disjoint. Pick a rational $q_\beta \in I_\beta$. But then $\{q_\beta | \beta < \alpha\}$ is an uncountable set of rationals - contradicting the countability of $\mathbb{Q}$.

Exercise 4.51. Show that any countable ordinal $(\alpha, \in)$ can be (1-1) order-preserving embedded into $(\mathbb{R}, <)$. Show that no uncountable ordinal can be so embedded.

Sol: Let $f: \alpha \approx \mathbb{N}$. Extend f to $f: \langle \alpha, \in\rangle \longrightarrow \langle \mathbb{N}, \prec\rangle$ an order isomorphism, by defining the strict total order on $\mathbb{N}$ by $n \prec m \longleftrightarrow f^{-1}(n) \in f^{-1}(m)$. The result follows from Ex. 4.48. However if α were uncountable but $g: \langle \alpha, \in\rangle \longrightarrow \langle \mathbb{R}, <\rangle$ were order preserving and (1-1), then $X = \mathrm{ran}(g)$ would contradict Ex.4.50.

Exercise 4.52. * Let $(\mathbb{N}, \prec)$ be any strict total order which is (a) *dense*, that is, for any $n, m \in \mathbb{N}$ there is $q \in \mathbb{N}$ with $n \prec q \prec m$; (b) has no endpoints, *i.e.* no maximum nor minimum elements. Show that $(\mathbb{N}, \prec) \cong (\mathbb{Q}, <)$. Deduce that any two countable dense total orders without endpoints are isomorphic.

Sol: Let $\{q_n\}_{n<\omega}$ enumerate in a (1-1) way the rationals. First make the observation that if $f_0: (D, \prec) \longrightarrow (P, <)$ is an isomorphism of *finite* sets $D \subseteq \mathbb{N}$ and $P \subseteq \mathbb{Q}$, then for any $k \in \mathbb{N}$ and $q \in \mathbb{Q}$ we can extend f_0 to an isomorphism f where now $k \in \mathrm{dom}(f)$ and $q \in \mathrm{ran}(f)$. We deal first with q: suppose that $q \notin \mathrm{ran}(f)$ (or else we have nothing to do) and, as a first case, that for all $q' \in \mathrm{ran}(f_0)$ $q' < q$. Then let n be least (in the sense of the usual $<$ ordering on $\mathbb{N}$), with $n' \prec n$ for all $n' \in \mathrm{dom}(f_0)$; if for all $q' \in \mathrm{ran}(f_0)$ $q' > q$, then instead let n be $<$-least, with $n' \succ n$ for all $n' \in \mathrm{dom}(f_0)$. If neither of these cases occur then for some pair $\langle m, l\rangle$ $f_0(m)$ is $<$-largest in $\mathrm{ran}(f_0)$ below q, and $f_0(l)$ is $<$-least in $\mathrm{ran}(f_0)$ above q. Now let n be $<$-least, with $n \notin \mathrm{dom}(f_0)$ and with $m \prec n \prec l$ (which exists by the denseness of $\prec$). In each of these cases add $\langle n, q\rangle$ to f_0 to get f_0'. We enlarge, if need be, f_0' by an entirely symmetric argument to place $k \in \mathrm{dom}(f)$. We just illustrate with the third case: that is for some pair $\langle m', l'\rangle$, m' is $\prec$-largest in $\mathrm{dom}(f_0')$ $\prec$-below k, and l' is $\prec$-least in $\mathrm{dom}(f_0')$ which is $\prec$ -above k. Now let p be $<$-least, with $q_p \notin \mathrm{ran}(f_0')$ and with $f_0'(m') < q_p < f_0'(l')$. Now add $\langle k, q_p\rangle$ to f_0' to obtain f, with $f: (D', \prec) \longrightarrow (P', <)$ and with $D' = D \cup \{k, n\}$, $P' = P \cup \{q, q_p\}$. We use this observation to make the inductive steps in an argument that starts to construct an increasing sequence of finite isomorphisms $\varnothing = f_0 \subseteq \cdots \subseteq f_n \subseteq f_{n+1} \subseteq \cdots$ where at stage n we ensure that $n \in \mathrm{dom}(f_n)$ and $q_n \in \mathrm{ran}(f_n)$. As can be easily seen, our desired isomorphism is $f = \bigcup_n f_n$.

Exercise 4.53.

Exercise 4.54. Suppose that $(P, <)$ is a dense suborder of $(\mathbb{R}, <)$. Show that there is a a countable $S \subseteq P$ with $(S, <)$ a dense suborder of $(P, <)$.

Sol: Let (p_n, q_n) for $n < \omega$ enumerate all pairs of rationals with $p_n < q_n$. As P is dense in $\mathbb{R}$, for every such pair there is an $s_n \in P \wedge p_n < s_n < q_n$. Let $S = \{s_n | n < \omega\}$. Then S is dense in $\mathbb{R}$, since for any $r, s \in \mathbb{R}$ there is some n so that $r < p_n < q_n < s$. But then $r < s_n < s$. In particular S is dense in P.