5 Solutions to Exercises Chapter 5

Exercise 5.1. Show that AC $\Leftrightarrow$ UP.

Sol: ($\Rightarrow$) Let $R \subseteq X \times Y$ is any relation, let $\mathcal{G} = \{X_u | u \in X\}$ where $X_u =_{\rm df} \{v | \langle u, v\rangle \in R\}$ Let f be a choice function for $\mathcal{G}$. Then f does the job.

($\Leftarrow$) Let G be a family of non-empty sets. Consider the subset $R \subseteq G \times \bigcup G$ where $\langle X, u\rangle \in R$ iff $u \in X$. A function that uniformises R is a choice function for G.

Exercise 5.2. Show that AC $\Rightarrow$ IFP.

Sol: Similar to the above. Let $H: X \longrightarrow Y$ be an onto function. Let $\mathcal{G} = \{X_y | y \in Y\}$ where $X_y =_{\rm df} \{x | H(x) = y\}$. Let G be a choice function for $\mathcal{G}$. Note that $\mathrm{dom}(G) = Y$ as H was assumed onto.

Exercise 5.3. Show that WP $\Leftrightarrow$ Cardinal Comparison.

Sol: ($\Rightarrow$) By WP let $g: X \approx \gamma$ and $f: Y \approx \delta$. If $\delta \leq \gamma$ then $g^{-1} \circ f: Y \preceq X$. If $\delta \geq \gamma$ we get $X \preceq Y$ by similar reasoning.

($\Leftarrow$) Let X be any set. By the Corollary following Hartogs' Theorem, there is an ordinal ν so that $\nu \not\preceq X$. By Comparability then, $f: X \preceq \nu$ for some f. But then define the wellorder R of X by $a R b \leftrightarrow f(a) < f(b)$.

Exercise 5.4. Show that AC $\Leftrightarrow$ Tychonoff Property.

Sol: ($\Rightarrow$) Let X_i *(for $i \in I$)* be any sequence of non-empty sets. Let $\mathcal{G} = \{X_i | i \in I\}$. Then any choice function f for $\mathcal{G}$ is in the direct product $\Pi_{i \in I} X_i$ ($=_{\rm df} \{f \,|\, f: I \longrightarrow \bigcup_{i \in I} X_i \wedge \forall i \in I (f(i) \in X_i)\}$).

($\Leftarrow$) If G is a family of non-empty sets, then any $f \in \Pi_{X \in G} X$ is a choice function for G.

Exercise 5.5. Show that WP $\Rightarrow$ Vector Space property.

Sol: Just follow the Hint.

Exercise 5.6. Show that if C is any proper class and F any (1-1) function, that $F``C$ is a proper class.

Sol: Suppose for a contradiction that $F``C = u$ where $u$ is a set. Then $F^{-1}``u = C$ a proper class, but also is a set by the Axiom of Replacement. A contradiction.

Exercise 5.7. For sets X, Y let $\mathcal{F} = \{h \mid h \subseteq X \times Y \wedge h$ is a (1-1) function$\}$. Assume ZL and show that there is a $g \in \mathcal{F}$ with either $\mathrm{dom}(g) = X$ or $\mathrm{ran}(g) = Y$. Deduce that using ZL we have that for any sets X, Y we have either $X \preceq Y$ or $Y \preceq X$.

Sol: If $C \subseteq \mathcal{F}$ is any chain, check that $\bigcup C \in \mathcal{F}$. So now assume g is a maximal element. If the conclusion fails for this g, pick an element $u \notin \mathrm{dom}(g)$ and an element $v \notin \mathrm{ran}(g)$. Then $g' = g \cup \{\langle u, v\rangle\} \in \mathcal{F}$ contradicting the supposed maximality of g.

Exercise 5.8. Use various equivalents of WP to show that if $f: X \to Y$ is an onto function, that there is $g: Y \to X$ with $\mathrm{id} = f \circ g$.

Sol: For each $y \in Y$ let $X_y = \{u \in X | \; f(u) = y\}$. As f is onto, each $X_y \neq \varnothing$. Let $\mathcal{F} = \{X_y | \; y \in Y\}$ and consider a choice function G for $\mathcal{F}$. Define $g(y) = G(X_y)$.

Exercise 5.9. * ZL is often stated in an apparently stronger form, ZL^+: *Let $\mathcal{F}$ be a set so that for every chain $\mathcal{G} \subseteq \mathcal{F}$ then $\mathcal{G}$ has an upper bound in $\mathcal{F}$. Then $\mathcal{F}$ contains a maximal element Y.* Show that this increase in strength is indeed only apparent: $\mathrm{ZL} \Leftrightarrow \mathrm{ZL}^+$.

Sol: The direction ($\Leftarrow$) is immediate since $\bigcup \mathcal{G}$ is an upper bound for any chain $\mathcal{G}$ which is in $\mathcal{F}$. For the other direction we take the easy route through WP which we have seen is equivalent to ZL, and utilise the argument that (WP $\Rightarrow$ ZL). At the final part there we had built a chain $\mathcal{H}$ and claimed $\bigcup \mathcal{H}$ is a maximal element of $\mathcal{F}$. We do the same here. Now our ZL$^+$ only states that there is an upper bound $Y \in \mathcal{F}$ for $\mathcal{H}$. But again for that upper bound Y, we must have that $Y \supseteq \bigcup \mathcal{H}$. The argument finishes in exactly the same way, as $Y = k(\nu)$ for some ν, and so can only be equal to $\bigcup \mathcal{H}$.

Exercise 5.10. Use ZL to show that for any partial order $\langle A, \preceq\rangle$ there is an extension $\preceq' \supseteq \preceq$, so that $\langle A, \preceq'\rangle$ is a total order. [Hint: (i) If $\langle A, \preceq\rangle$ is not total pick $u, v \in A$ that are $\preceq$-incomparable; let $\preceq_0 = \preceq \cup \{\langle x, y\rangle \mid x \preceq u \wedge v \preceq y\}$; check that $\preceq \subset \preceq_0$ is still a partial order; (ii) apply ZL to the set of partial orders on A. This is known as the *Order Extension Principle*.] Deduce that that there is a total order $\preceq$ extending the partial order $\subseteq$ on $\mathcal{P}(\mathbb{N})$.

Sol: Let $\mathcal{F}$ be the set of partial ordering relations $\preceq_1 \subseteq A \times A$ with $\preceq_1 \supseteq \preceq$. Then note that if $\mathcal{G} \subseteq \mathcal{F}$ is any chain of such partial orders, that $\bigcup \mathcal{G}$ is also such a partial order, and is thus in $\mathcal{F}$. But now we can apply ZL and conclude there is a maximal $\preceq'$ in $\mathcal{F}$. By (i) of the hint $\preceq'$ must be total for otherwise we could extend it: it would thus not be maximal.

Exercise 5.11. Show that AC is equivalent to: every family of sets contains a maximal subfamily of disjoint sets. Formally: let $\mathrm{DF}(y) \longleftrightarrow_{\mathrm{df}} \forall u, v \in y(u \neq v \longrightarrow u \cap v = \varnothing)$. Show that

$$\mathrm{AC} \longleftrightarrow \forall y \exists x \subseteq y(\mathrm{DF}(x) \wedge \forall z \subseteq y(\mathrm{DF}(z) \longrightarrow y \not\subset z)).$$

Sol: ($\Rightarrow$) Use ZL. Let $\mathcal{C}$ be the collection of subsets x of y with $\mathrm{DF}(x)$. Note that if $\mathcal{H}$ is any chain of such subsets $x \subseteq x'$ with the disjointness property that

Exercise 5.12. Let Φ be the statement: *for any two non-empty sets X, Y, either there exists an onto map $f: X \longrightarrow Y$ or there exists an onto map $g: Y \longrightarrow X$.*

(i) Show that WP $\Rightarrow \Phi$.

(ii) ($*$) Show that $\Phi \Rightarrow$ WP. [Hint: Consider the family of maps of a set X onto an ordinal. Use a Hartogs' like argument to show that the supremum of such ordinals exists.]

Sol: ($\Rightarrow$) Let X, Y be arbitrary but non-empty. We use Cardinal Comparison. Then either $X \preceq Y$ or $Y \preceq X$. WLOG suppose the former. Let $f: X \preceq Y$ be (1-1). Let $u \in X$. Define an onto map $g: Y \longrightarrow X$ by $g(v) = f^{-1}(v)$ if $v \in \mathrm{ran}(f)$, $g(v) = u$ otherwise.

($\Leftarrow$) Let X be any set. Define a *pre-wellordering* of a set X to be a wellordering R, but where the requirement that $uRv \wedge vRu \longrightarrow u = v$ is dropped. Here the ordering $\preceq$ now divides up X into equivalence classes: $u \sim_R v$ iff $uRv \wedge vRu$. (Note that if $f_\alpha: X \twoheadrightarrow \alpha$ is onto we may define such a PWO of X by $u \preceq_\alpha v \longleftrightarrow f_\alpha(u) \leq f_\alpha(v)$, and here an equivalence class is of the form $f_\alpha^{-1}(\xi)$ for some $\xi < \alpha$.)

If $\sim_R$ is this equivalence relation, then $X/\sim_R$ is a WO and so isomorphic to a unique ordinal. (With R arising from an f_α as above, then, by design, $\mathrm{ot}(X/\sim_R) = \alpha$.)

Let $Y_0 = \{R \subseteq X \times X \mid \mathrm{dom}(R) = X \wedge R \text{ is a PWO}\}$. Y_0 is a set by Axiom of Power Set and the Axiom of Subsets.

Let $Y = \{\alpha \mid \exists R \in Y_0(\mathrm{ot}(X/\sim_R) \cong \alpha)\}$. Y is a set of ordinals, by the Axiom of Replacement. Hence $\theta = \sup(Y)$ exists. It can be easily seen that Y has no largest element. Hence θ is a strict upper bound on the ordinals onto which X can be mapped. (One may show that θ is a cardinal as in Hartogs' Theorem but that is not relevant here.) As there is no map of X onto θ by our supposed principle there is a map θ onto X. However then we can find a wellorder of X and so WP holds for X. [If g is the given map, define for u, $v \in X$, uRv iff (the least α with $g(\alpha) = u$ is less than the least β such that $g(\beta) = v$).]

6 Solutions to Exercises Ch 6

Exercise 6.1. Compute (i) $\rho(S(x))$ in terms of $\rho(x)$. (ii) Show that $\rho(\bigcup x) \leq \rho(x)$, and give examples of sets x_1, x_2 with $\rho(\bigcup x_1) < \rho(x_1)$ but $\rho(\bigcup x_2) = \rho(x_2)$; can you characterise those sets z for which $\rho(\bigcup z) < \rho(z)$? (iii) Suppose $\rho(x) = \rho(y) = \alpha$ and $f: x \to y$. Compute $\rho(\langle x, y, x\rangle)$; $\rho(f)$; $\rho({}^x y)$; $\rho({}^\alpha x)$.

Sol: (i) $\rho(S(x)) = \rho(x \cup \{x\}) = \rho(x) + 1$. (ii) Take $x_2 =$ {Evens}. Then $\rho(x_2) = \omega = \rho(\bigcup x_2)$ (as $\bigcup x_2 = \omega$). In general if $\mathrm{Lim}(\rho(x_2))$ then $\rho(\bigcup x_2) = \rho(x_2)$; (Why? because if $\eta = \rho(\bigcup x_2) < \rho(x_2) = \lambda$ with $\mathrm{Lim}(\lambda)$, then every element $y \in x_2$ has satisfies $y \subseteq V_{\eta+1}$, and thus $x \subseteq V_{\eta+2}$, and thus $\rho(x) \leq \eta + 2 < \lambda$ - contradiction!). If $x_1 = \{\omega\}$, $\rho(x_1) = \omega + 1$, but $\omega = \bigcup \{\omega\}$ and so $\rho(\bigcup x_1) = \omega < \rho(x_1)$. Again in general if $\rho(z)$ is a successor ordinal, $\delta + 1$ say, then $\rho(\bigcup z) = \delta < \rho(z)$.

(iii) $\rho(\langle x, y, x\rangle) = \rho(\langle\langle x, y\rangle, x\rangle) = \rho(\{\{\langle x, y\rangle\}\{\langle x, y\rangle, x\}\}) = \rho(\langle x, y\rangle) + 2 = \alpha + 4$. If $f \in {}^x y$ then $f \subseteq x \times y$. As $x, y \subseteq V_\alpha$ then $x \times y \subseteq V_\alpha \times V_\alpha \subseteq V_{\alpha+2}$ (by Examples 1). Hence $f \subseteq V_{\alpha+2}$ and then ${}^x y \subseteq V_{\alpha+3}$. Thus $\rho({}^x y) = \alpha + 3$. As $\alpha = \rho(\alpha)$ we also have $\rho({}^\alpha x) = \alpha + 3$.

Exercise 6.2. What if α in the above example is a limit ordinal? Can we improve the bounds on ranks? If $\langle\omega, R\rangle$ is an ordering, what is $\rho(R)$? [Hint: compute $\rho(\omega \times \omega)$, and $\rho((\omega+1) \times (\omega+1))$.]

Sol: If α is a limit, then note that for $x, y \in V_\alpha$ so are $S(x), \langle x, y\rangle \in V_\alpha$, and also $\langle x, y, x\rangle$. In particular $V_\alpha \times V_\alpha \subseteq V_\alpha$. Then any relation (and so any ordering or function) R on $V_\alpha \times V_\alpha$ is also a subset of V_α and so $\rho(R) = \alpha$, and $R \in V_{\alpha+1}$. Ths means that ${}^x y \subseteq V_{\alpha+1}$ and so $\rho({}^x y) = \rho({}^\alpha x) = \alpha + 1$.

In the case that $\alpha = \omega$ then by the above, any ordering on ω is an element of $V_{\omega+1}$ and so has rank ω. $\rho(\omega \times \omega) = \omega$, but $\rho((\omega+1) \times (\omega+1)) = \omega + 3$ (note $\langle\omega, \omega\rangle \in (\omega+1) \times (\omega+1)$).

Exercise 6.3. Show that the Axiom of Foundation implies the apparently stronger statement that for any *class* $(A \neq \varnothing \longrightarrow \exists y \in A(y \cap x = \varnothing))$.

Sol: As $A \neq \varnothing$ let a be a set in A. Suppose $\rho(a) = \alpha$. Apply the axiom of foundation to the set $x = A \cap V_{\alpha+1}$.

Exercise 6.4. Let $\mathbb{G} = \langle G, \circ, e, {}^{-1}\rangle$ be a group. Assume WP, but not the Axiom of Foundation. Show that there is a group $\tilde{\mathbb{G}} \in \mathrm{WF}$ with $\mathbb{G} \cong \tilde{\mathbb{G}}$. [Hint: By WP find R so that $\langle G, R\rangle \in \mathrm{WO}$. Then "copy" $\mathbb{G}$ onto $\alpha = \mathrm{ot}(\langle G, R\rangle)$.]

Sol: Following the hint, let $f: \langle G, R\rangle \cong \langle\alpha, <\rangle$. Define $\beta \circ' \gamma = \delta \longleftrightarrow f^{-1}(\beta) \circ f^{-1}(\gamma) = f^{-1}(\delta)$; $\varepsilon = f(e)$ etc. Note that $\circ'$ can be coded as a subset of $\alpha \times \alpha \times \alpha$ (we can define $\langle\beta, \gamma, \delta\rangle \in Y$ iff $\beta \circ' \gamma = \delta$) and thus Y is in $V_{\alpha+4}$. Then $\tilde{\mathbb{G}} = \langle\alpha, \circ', \varepsilon, {}^{-1'}\rangle \cong \mathbb{G}$, and is in V.

Exercise 6.5. Let Φ be the proposition "*There is no sequence of sets x_i for $i \in \omega$, with $x_{i+1} \in x_i$*". a) Show that the Axiom of Foundation implies Φ; b) WP together with Φ implies the Axiom of Foundation.

Sol: a) Suppose Φ false and let $X = \{x_i | i \in \omega\}$ be an example of such a sequence. Then for no $x \in X$ do we have $x \cap X = \varnothing$. This contradicts the AxFoundation. b) Assume Φ and the WP. As WP $\longleftrightarrow$ AC, we may use AC. Suppose there was a failure of AxFoundation. Then for some Y there is no $x_0 \in Y$ which is $\in$-minimal amongst elements of Y, that is $x_0 \cap Y \neq \varnothing$. Pick (by AC) some $x_1 \in x_0 \cap Y$. As again, x_1 is not $\in$-minimal in this sense, we can find $x_2 \in x_1 \cap Y$ (again using AC). Continuing in this way we get an infinite sequence $x_{i+1} \in x_i$.

Exercise 6.6. Show that if $\pi: \langle V, \in\rangle \longrightarrow \langle V, \in\rangle$ is an isomorphism, then $\pi = \mathrm{id}$. There are thus no non-trivial isomorphisms of V with itself. [Hint: Suppose there was an x with $\pi(x) \neq x$. Choose one such of least rank with this property. Then $y \in x \longrightarrow \pi(y) = y$.] (This both generalises Cor. 3.7 and is a special case of: If $f: \langle M, R\rangle \longrightarrow \langle M, R\rangle$ is an isomorphism, where $\langle M, R\rangle$ is a wellfounded relation, then $f = \mathrm{id}$.)

Sol: Choose such an x as in the Hint. Then $x \subseteq \pi(x)$ since $y \in x \longleftrightarrow \pi(y) \in \pi(x)$, as π is an isomorphism. But $\pi(y) = y$ by choice of x. Now, as π is onto V, any $t \in \pi(x)$ is of the form $\pi(u)$ for some u. But $\pi(u) \in \pi(x) \longrightarrow u \in x$. But then $\pi(u) = u$ as $\rho(u) < \rho(x)$. So $\pi(x) \subseteq x$ also.

Exercise 6.7. Show for any x that $\rho(x) = \rho(\mathrm{TC}(x))$.

Exercise 6.8. Let X be any set. Show that $\mathrm{Trans}(X) \to \rho\,``X \in \mathrm{On}$.

Sol: Suppose there is X a transitive set, so that $\rho\,``X \notin \mathrm{On}$; by $\in$-induction we may choose such an X with $\rho(X)$ least. Then there is some least $\gamma < \rho(X)$ with no $z \in X$ with $\rho(z) = \gamma$, whilst at the same time there are $u \in X$ with $\gamma < \rho(u)$. By Exercise 6.7 $\rho(u) = \rho(\mathrm{TC}(u)) = \sup^+\{\rho(z) | z \in \mathrm{TC}(u)\} > \gamma$. By the inductive hypothesis, as $\rho(u) < \rho(X)$, $\rho\,``\,\mathrm{TC}(u) \in \mathrm{On}$. So there is $z \in \mathrm{TC}(u)$ with $\rho(z) = \gamma$. But $\mathrm{TC}(u) \subseteq X$, so $z \in X$. This is a contradiction.

Exercise 6.9. Does $\mathrm{Trans}(X) \wedge X \neq \varnothing$ imply that $\varnothing \in X$?

Sol: Yes. Suppose there is a non-empty transitive set X, so that $\varnothing \notin X$; by $\in$-induction we may choose such an X with $\rho(X) = \alpha$ say least. Then check $\alpha > 1$. So let $y \in X$. Then $\rho(y) < \alpha$. But $\rho(y) = \rho(\mathrm{TC}(y))$ (by Exercise 6.7), and by the inductive hypothesis, $\varnothing \in \mathrm{TC}(y)$. But $\mathrm{TC}(y) \subseteq \mathrm{TC}(X) = X$. Contradiction.

Quick way: By Ex. 6.8 $\rho\,``X \in \mathrm{On}$. But then some $t \in X$ has $\rho(t) = 0$. t can only be $\varnothing$.

Exercise 6.10. Show that for all α $|V_{\omega+\alpha}| = \beth_\alpha$.

Sol: By induction on α. Trivial for $\alpha = 0$. Suppose true for $\beta < \alpha$. If $\alpha = \beta + 1$ then $V_{\omega+(\beta+1)} = V_{(\omega+\beta)+1} = \mathcal{P}(V_{\omega+\beta})$, and so $|V_{(\omega+\beta)+1}| = 2^{|V_{\omega+\beta}|} = \beth_{\beta+1}$ as required. If $\mathrm{Lim}(\alpha)$ then $|V_{\omega+\alpha}| = |\bigcup_{\beta<\alpha} V_{\omega+\beta}| = \bigcup_{\beta<\alpha} \beth_\beta = \beth_\alpha$ by definition.

Exercise 6.11. * We say that a function $j: V \longrightarrow V$ is an *elementary embedding* if it preserves the truth about objects. In other words if $\varphi(v_0, ..., v_n)$ is a formula and $x_0, ..., x_n$ are sets; then

$$\varphi(x_0, ..., x_n) \longleftrightarrow \varphi(j(x_0), ..., j(x_n)).$$

If we assume the axioms of set theory (but not AC) our current state of knowledge allows the possibility that such a function j could exist which is not the identity (so $j(x) \neq x$ for some $x \in V$). Show if there is such a j then for some ordinal α, $j(\alpha) \neq \alpha$. [Hint: Consider the formula "$u = \rho(v)$".] (It is known that AC rules out the existence of such a j.)

(Remember the $*$ is supposed to mean this is a harder/more sophisticated exercise, this exercise is **not** examinable.)

Sol: Suppose there was such a j with $j(x) \neq x$ for some x. Then some x is so moved with least rank α say. Then as $y \in x \longrightarrow \rho(y) < \rho(x)$ we can conclude that for any $y \in x$ that $j(y) = y$. As "$y \in x$" can be expressed by a formula, we have by the "elementary embedding" property of j that $j(y) \in j(x)$. We conclude $x \subseteq j(x)$. As $j(x) \neq x$ we must have that there is some $z \in j(x) - x$. Again $\rho(z) \not< \rho(x)$ for otherwise $j(z) = z \in j(x)$ implies that $z \in x$ by "elementarity"" again! So, $\rho(j(x)) > \rho(z) \geq \rho(x)$. Hence, considering the formula "$\alpha = \rho(x)$" we see that $j(\alpha) = \rho(j(x))$ and we thus have $j(\alpha) > \alpha$.

Sol to Ex. 6.7 : As $x \subseteq \mathrm{TC}(x)$, $\rho(x) \leq \rho(\mathrm{TC}(x))$ is trivial. But note that if $x \subseteq V_\alpha$ then $\bigcup^k x \subseteq V_\alpha$ for any $k \in \omega$ also. Hence $\mathrm{TC}(x) \subseteq V_\alpha$ too, and so $\rho(x) \geq \rho(\mathrm{TC}(x))$.