

## HYPERMACHINES

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**Abstract.** The Infinite Time Turing Machine model [8] of Hamkins and Kidder is, in an essential sense, a “ $\Sigma_2$ -machine” in that it uses a  $\Sigma_2$  *Liminf Rule* to determine cell values at limit stages of time. We give a generalisation of these machines with an appropriate  $\Sigma_n$  rule. Such machines either halt or enter an infinite loop by stage  $\zeta(n) =_{\text{df}} \mu\zeta(n)[\exists\Sigma(n) > \zeta(n) L_{\zeta(n)} \prec_{\Sigma_n} L_{\Sigma(n)}]$ , again generalising precisely the ITTM case.

The collection of such machines taken together, computes precisely those reals of the least model of analysis.

**§1. Introduction.** The Infinite Time Turing Machine (ITTM) model described in [8] is an attractive model of *transfinite time computation* based on the standard Turing machine with an infinite one way tape, and a finite transition table or instruction set. The latter specifies how the machine behaves at successor steps as is usual, and one needs really only to specify precisely how such a machine behaves at limit steps in time to give a complete description. The model in [8] resets the read/write head position on the first cell (or rather triplet of first cells on each of three parallel tapes for *input*, *output*, and *scratch work*) and assigns 0/1 values to a cell contents by means of a *limsup* rule: the value at time  $\lambda$  is the limsup of the previous values. The use of limsup as opposed to *liminf* is immaterial in terms of computational functionality, and we tend to use *liminf* (as in this paper). The basic properties of these machines were explored in [8]; in [14] the *halting problem* and correspondingly, the *decidable*, and *semi-decidable* sets of integers were characterised. The companion structure to this notion of computation turned out to be the least level of the Gödel  $L$ -hierarchy,  $L_\zeta$ , which has a proper  $\Sigma_2$ -elementary end extension. The decidable sets of integers are those  $\Delta_1$ -definable (without parameters) over  $\langle L_\zeta, \in \rangle$ . (In the terminology of an earlier age, the ITTM decidable sets of integers are the “*abstract 1-section*” of the admissible set  $\langle L_\zeta, \in \rangle$ , and thus, by a theorem of Sacks [13], form the *1-section* of some *type-2 functional*  $G$ :  $1\text{-sc}(G)$ ). It is therefore possible to view this form of computation as a particular example of higher type recursion in the manner of generalised recursion theory).

With hindsight this is perhaps not unsurprising: either of the rules mentioned exhibits the essential  $\Sigma_2$ -nature of the machines: *e.g.* under *liminf*, the value of a cell at a limit

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time  $\lambda$  is 1 if and only if “ $\exists\beta < \lambda\forall\gamma \in (\beta, \lambda)[\dots]$ ”. It is therefore apparent that if  $L_\zeta \prec_{\Sigma_2} L_\Sigma$ , then thinking of the machines running inside  $L$  (which we may, as their construction and operation is absolute to  $L$ ), we should have that the machine has either halted or is entering an infinite loop at time  $\zeta$ .

This naturally leads to the question: “Is there a notion of computation, presumably based on a  $\Sigma_3$  style rule, that has as corresponding companion structure the level of the  $L$ -hierarchy at the least  $\zeta(3)$  with some  $\Sigma(3) > \zeta(3)$  and  $L_{\zeta(3)} \prec_{\Sigma_3} L_{\Sigma(3)}$ ?” Such a notion would then have its “ $\Sigma_3$ -ITTM-*decidable sets*” as those of the abstract 1-section of  $\langle L_{\zeta(3)}, \in \rangle$ , *i.e.* those that are  $\Delta_1(\langle L_{\zeta(3)}, \in \rangle)$ , and would then presumably either be halted or be entering an infinite loop by time  $\zeta(3)$ .

It is the purpose of this note to provide such a notion. We further indicate how to generalise this to give a positive answer to the question for larger  $n$  with  $\Sigma_n$ -limit rules. We only assume some familiarity with [8], with the basic facts of the constructible hierarchy  $L_\alpha$  and the Jensen version, the  $J_\alpha$ -hierarchy (see [2], [5] or Jensen’s original paper [9]). We let  $On$  denote the class of ordinals. For  $H$  any class of ordinals we let  $H^*$  be the class of limit points of  $H$ . We remark that all of these arguments could be formulated within second-order number theory.

In a concluding section we mention some open problems, but also connections with other notions of quasi-induction in the literature, and with determinacy questions.

**§2. The  $\Sigma_3$ -Machine construction.** The  $\Sigma_3$ -machine has four tapes: besides the three parallel tapes of [8] for *input*, *scratch* and *output*, there is an additional parallel *rule* tape that the Read/Write head surveys, and reads and writes to, just as for the other three. (This is a convenience only: we could imagine the rule tape as being absorbed as a simple infinite recursive subtape of the official scratch tape, and stipulations about the rule tape which we are about to formulate could be made in terms of some priorly fixed recursive function  $F : {}^\omega 2 \longrightarrow {}^\omega 2$  applied to the scratch tape cells  $\langle C_{3i+1} \mid i < \omega \rangle$ .) A *computation* is defined much as before: if  $P_e(x)$  is the  $e$ ’th program acting on input  $x$ , the successor stages are governed by the program instructions just as for a standard Turing machine. If we enumerate the cells’ values (for all the tapes) at time  $\nu$  by  $\langle C_i(\nu) \mid i < \omega \rangle$  then at limit stages  $\lambda$  the value of  $C_i(\lambda)$  is given by a generalised limit rule controlled dynamically by the rule tape. (This is not the only method one can envisage for developing a new limit rule: one could by exterior fiat describe a rule in an absolute manner independent of which computation is being run - as is the case for  $n = 2$ . However this would result in a somewhat arbitrary action for programs not computing *universal computations*. The reader might be forgiven for thinking after a first reading of our limit rule below, that such a prior absolute limit rule that assigns to each (relevant) ordinal a mode for computing the limit values at that step, as being simpler to comprehend. However the only way we have thought of this as possible, is one that would have a rather arbitrary action on computations that were not of the universal nature, that is those that ended up halting before stage  $\Sigma(n)$ . Even for the non-halting computations, there would not be any good relationship between the limit rule modes for those ordinal stages in a computation of  $P_e(0)$  and for those on a real input  $x$ ,  $P_e(x)$  say, where  $\Sigma(n, x)$  could be far greater than  $\Sigma(n, 0)$ . In short we see no way around the difficulty. That is not to say that there are no simpler methods: we have used one that works. In its defense, whereas the usual *liminf* rule of the ITTM’s, is one that uses a very stable principal of continuity, namely, look at *all* ordinals  $\alpha$  less than

some limit  $\lambda$ , we seem to need to step up to some higher order notion of computational stability, which looks at points of ‘computational closure’ in a fashion. After one is used to it, this seems not so unnatural; moreover using it we reach higher ordinals. We therefore prefer the dynamic approach to be given below.)

Before describing this action we make some preliminary definitions.

**DEFINITION 1.** *An ordinal  $\alpha$  is called good if  $\omega\alpha = \alpha$ . We write  $\alpha \in G$  for  $\alpha$  good or  $\alpha = 1$ .*

**REMARK 1.**  *$\alpha$  is good iff it is a multiple of  $\omega^\omega$ ; if  $\gamma \in \text{On}$ , we write  $\gamma^+$  for the least limit of good ordinals greater than  $\gamma$ . This is  $\gamma + \omega^\omega \cdot \omega$ .*

Instead of taking *liminf*’s over all ordinals  $\nu < \lambda$  for  $\text{Lim}(\lambda)$  in order to determine  $C_i(\lambda)$  we do this for a restricted set of ordinals below  $\lambda$  which correctly reflect certain patterns of occurrence on the rule tape. These ordinals we shall call *1-correct in  $\lambda$* . This we shall define in a moment, but we state the limit rule now:

**DEFINITION 2.** *Given a Turing machine program  $P_e(y)$  with input  $y$ ,  $C_i(\nu)$  is defined as follows:*

*If  $\nu = \bar{\nu} + 1$  then  $C_i(\nu)$  is determined by the usual program action from  $\vec{C}_i(\bar{\nu})$ .*

*If  $\nu = \lambda$  a limit, then*

$$C_i(\lambda) =_{\text{df}} \liminf_{\nu \rightarrow \lambda} \{C_i(\nu) \mid \nu \text{ 1-correct in } \lambda\}.$$

*As a shorthand notation we shall write this as:  $C_i(\lambda) = \liminf_{\nu \rightarrow \lambda}^* C_i(\nu)$ .*

To put this limit case in another way:

$$C_i(\lambda) = 1 \iff \exists \nu_0 < \lambda \forall \nu \in (\nu_0, \lambda) [\nu \text{ 1-correct in } \lambda \implies C_i(\nu) = 1].$$

$$C_i(\lambda) = 0 \text{ Otherwise.}$$

To make sense of this we need to define “1-correct in  $\lambda$ ”. We shall give a descriptive indication of this (proper definitions and full definitions will be given below). Very broadly, a computation may work with information that has appeared on the scratch tape (or perhaps on a recursive sub-tape of the scratch tape), thought of as a real  $x$  say, and may also at some stage produce certain further information about  $x$ . Below  $\lambda$  there may be stages, some sort of stable point, at which the computation has said all it is going to say about  $x$  (before stage  $\lambda$ ). We might then define a *stable* point (below  $\lambda$ ) as some  $\alpha$ , where any relevant scratch tape real  $x$ , if it appears before stage  $\alpha$ , also has that all information about  $x$  has also appeared before stage  $\alpha$ : that is, nothing new is said about such  $x$  in the interval  $(\alpha, \lambda)$ . Such points are those then of *stable informational content*. A first approximation to our new  $\liminf^*$  rule is that we could take *liminf*’s along these points of stable content. In fact we do something slightly more refined, and also more generous in terms of points: we shall take *liminf*’s along those points  $\alpha$  where  $\alpha$  sees the same stable points (in the above sense) below it, that  $\lambda$  sees are below  $\alpha$ . Such an  $\alpha$  will be called “1-correct in  $\lambda$ .”

We use the following notation: for  $y \in 2^{\mathbb{N}}$  let  $1 \frown y \in {}^\omega 2$  be defined by:  $1 \frown y(k+1) = y(k)$  and  $1 \frown y(0) = 1$ . Then  $1 \frown y$  is the sequence  $y$  prefixed by a ‘1’. Let  $n * y \in {}^\omega 2$  be similarly  $y$  prefixed by  $n$  zeroes. We assume the rule tape values are listed as:  $\langle R_i(\nu) \mid i < \omega \rangle =_{\text{df}} \langle C_{4i}(\nu) \mid i < \omega \rangle$ . We abbreviate  $\langle R_i(\nu) \mid i < \omega \rangle$  by  $R(\nu)$  etc. In the following  $\lambda$  will always denote a limit ordinal.

DEFINITION 3.  $S_\lambda^1$  (the 1-stable in  $\lambda$  ordinals) Set:

$$\alpha \in S_\lambda^1 \iff \alpha \in G \cap \lambda \wedge \forall x \in {}^\omega 2 \forall \nu < \alpha \forall n \forall \beta < \lambda [(1 \frown x = R(\nu) \wedge R(\beta) = n * 1 \frown x) \longrightarrow \exists \beta' < \alpha (R(\beta) = R(\beta'))].$$

Thus: any pattern of the form  $n * 1 \frown x$  (where  $1 \frown x$  itself occurred before  $\alpha$ ) which occurs before  $\lambda$  must also have occurred before  $\alpha$ . The following may be easily established.

- $S_\lambda^1 \cap \alpha \subseteq S_\alpha^1$ .
- If  $\alpha \in S_\lambda^1$  then  $S_\lambda^1 \cap \alpha = S_\alpha^1$ .
- $S_\lambda^1 \subseteq \lambda$  and is closed below  $\lambda$ .

DEFINITION 4.  $E_\lambda^1$  (the 1-correct in  $\lambda$  ordinals)

If  $\lambda \notin G^*$  then  $E_\lambda^1 = \lambda$ ;

If  $\lambda \in G^*$  then  $E_\lambda^1 =_{\text{df}} \{\alpha < \lambda \mid S_\lambda^1 \cap \alpha = S_\alpha^1\}$ .

We also note:

- $\alpha \in S_\lambda^1 \longrightarrow \alpha \in E_\lambda^1$ ; if  $\lambda \in G^*$  then  $E_\lambda^1 \subseteq G$ ;
- $E_\lambda^1$  is closed and unbounded in  $\lambda$ .

(Proof of this latter remark: closure is clear; assume  $\lambda \in G^*$  but the unboundedness failed. Then  $\alpha_0 = \max S_\lambda^1 < \lambda$ . Thus on a tail of  $\alpha \in \lambda$  there is a least  $\beta(\alpha) \in S_\alpha^1 \setminus S_\lambda^1$ . Such a  $\beta(\alpha)$  is greater than  $\alpha_0$ . However if  $\alpha' > \alpha$  is least with  $\beta(\alpha) \notin S_{\alpha'}^1$  (and such must exist as  $\beta(\alpha) \notin S_\lambda^1$ ), one can see that  $\max(S_{\alpha'}^1) = \alpha_0$ . Thus  $\alpha' \in E_\lambda^1$ .)

Notice that this definition means that for ordinals  $\lambda \notin G^*$ , the modified *liminf*<sup>\*</sup> rule is just the previous standard *liminf* rule. It is only for the limits of good ordinals (the ones we are principally interested in) that the rule may be different. This gives some substance to our previously described motivation that we take *liminf*'s at limits of good ordinals by considering only those earlier stages which are “*correctly reflecting of stable informational content*.” This finishes the description of the general  $\Sigma_3$ -ITTM acting for a general program  $P_e$ . We next perform the more difficult task of demonstrating that there is a program that first commences a loop at stage  $\zeta(3)$ .

**§3. The  $\Sigma_3$ -Theory Program.** We now describe an algorithm programmable as some  $P_e$  on a  $\Sigma_3$ -ITTM-machine, and demonstrate that it first enters an infinite loop at the lexicographically least pair  $(\zeta(3), \Sigma(3))$  where  $\Sigma(3) > \zeta(3) \wedge L_{\zeta(3)} \prec_{\Sigma_3} L_{\Sigma(3)}$ . During its run it will produce codes  $l(\alpha)$  for levels of the hierarchy  $\langle J_\alpha, \in \rangle$  for  $\alpha < \Sigma(3)$  in parallel with their complete theories. The reader may have noticed the similarity between the above definitions of 1-stability to those obtained in the  $L$ -hierarchy. This of course is no accident. We now define the  $L$ -counterparts to the above definitions. We recall also that in the definition of the Jensen  $J$ -hierarchy, that  $\text{On} \cap J_\alpha = \omega \cdot \alpha$  (and thus  $J_1 = \text{HF} = L_\omega$ ), and that  $L_\alpha = J_\alpha$  if and only if  $\alpha$  is good. We let the language of set theory be  $\mathcal{L}_{\dot{\in}}$  and we assume for any  $0 < n \leq \omega$  that we have a recursive enumeration of the following  $\Sigma_n$ -formulae of that language  $\langle \varphi_i^n(v_0, v_1) \mid i < \omega \rangle$  with the two variables displayed. For  $\alpha \leq \beta$   $J_\alpha \prec_{\Sigma_n} J_\beta$  has its usual meaning: given  $\varphi_i^n(v_0, v_1)$  and  $x_1 \in J_\alpha$ , if  $J_\beta \models \exists v_0 \varphi_i^n(v_0, x_1)$  then  $J_\alpha \models \exists v_0 \varphi_i^n(v_0, x_1)$ .

DEFINITION 5.  $\widehat{S}_\lambda^n$  (the  $\Sigma_n$ -stable in  $\lambda$  ordinals) We set:

$$\alpha \in \widehat{S}_\lambda^n \iff \alpha < \lambda \wedge J_\alpha \prec_{\Sigma_n} J_\lambda.$$

- “ $\alpha \in \widehat{S}_\lambda^n$ ” is uniformly  $\Pi_n^{J_\lambda}$ -definable. For any  $\delta$ ,  $\widehat{S}_\delta^1 \subseteq (G^*)^*$ .
- $\widehat{S}_\lambda^n$  is a closed subset of  $\lambda$ .

We remark the following: suppose  $\beta_0$  is the least ordinal so that  $L_{\beta_0} \models \text{ZF}^-$ . Then every ordinal  $\beta < \beta_0$  satisfies  $J_\beta \models \text{“}V = \text{HC”}$  (that is, every set is hereditarily countable). Moreover for such  $\beta$  if  $\gamma < \beta$  is  $\Sigma_1$ -definable in  $J_\beta$  by some parameter-free formula, then since there is a  $\Delta_1^{J_\beta}$  definable map (in the parameter  $\gamma$ ) of  $\omega$  onto  $\gamma$ , every ordinal  $\gamma' \leq \gamma$  is also so definable. This ensures that if  $X \prec_{\Sigma_1} J_\beta$  then  $X \cap \omega \cdot \beta \in \omega \cdot \beta + 1$ . Moreover, if  $\gamma < \beta$  is additionally closed under the Gödel pairing function then standard methods show there is a uniform parameter-free map  $\omega \cdot \gamma \leftrightarrow J_\gamma$  which is also  $\Delta_1^{J_\gamma}$ . In particular any finite tuple  $\vec{x}$  from  $J_\gamma$  is enumerated by some ordinal  $\xi < \gamma$  under this map. It follows that for such  $\gamma$ , an  $X$  as above containing  $\gamma$ , also contains  $J_\gamma$  as a subset.

For  $\beta < \beta_0$  arguments from [6] can be used to establish that for each  $n > 0$   $J_\beta$  has a *uniform parameter-free*  $\Sigma_n$ -Skolem function  $h_\beta^n(v_0, v_1)$ . (In general this fails for levels  $J_\gamma$  and  $n > 1$ .) The function  $h_\beta^n$  is itself  $\Sigma_n^{J_\beta}$ -definable without parameters, with (as an inspection of the argument of [6] shows) the same definition uniformly for any  $\beta < \beta_0$ . The Skolem function *uniformises* the  $\Sigma_n^{J_\beta}$  relations: given an  $x \in J_\beta$  if there is a  $y \in J_\beta$  such that  $J_\beta \models \varphi_i^n[x, y]$ , then  $h_\beta^n[i, x]$  is a witness such that  $J_\beta \models \varphi_i^n[x, h_\beta^n[i, x]]$ . These Skolem functions readily yield  $\Sigma_n$ -Skolem hulls: if  $A \subseteq J_\beta$  then  $h_\beta^n \omega \times [A]^{<\omega}$  is the smallest  $\Sigma_n$ -elementary submodel of  $J_\beta$  containing  $A$  and is thus the  $\Sigma_n$ -Skolem hull of  $A$  in  $J_\beta$ .

For such  $\beta < \beta_0$  this has the ready consequence, for example, that if  $\gamma_0 = \sup \widehat{S}_\beta^n < \beta$  then the  $\Sigma_n$ -Skolem hull of  $\{\gamma_0\}$  in  $J_\beta$  is all of  $J_\beta$ . (For, as remarked above, every  $\gamma < \gamma_0$  must be in the  $\Sigma_1$ -Skolem hull of  $\{\gamma_0\}$ , and hence  $h_\beta^n \omega \times \{\gamma_0\}$  is a transitive  $\Sigma_n$ -elementary submodel of  $J_\beta$  of the form  $J_\delta \prec_{\Sigma_n} J_\beta$ ; as  $\gamma_0 \in J_\delta$  then we conclude  $\delta = \beta$ .) In particular if  $\gamma_0 = 0$  (because  $J_\beta$  has no proper  $\Sigma_n$ -elementary submodels)  $h_\beta^n$  is then a partial  $\Sigma_n^{J_\beta}$ -definable function of  $\omega$  onto  $J_\beta$ , a fact we shall use in the sequel. More particularly still we shall use this if  $\beta < \Sigma(n)$  where  $(\zeta(n), \Sigma(n))$  are the lexicographically least pair  $(\zeta', \Sigma')$  so that  $\zeta' < \Sigma'$  and  $L_{\zeta'} \prec_{\Sigma_n} L_{\Sigma'}$ .

We particularise the discussion now to  $n = 3$ . Our machine will be a “theory machine” writing out now the  $\Sigma_3$ -theory of levels of the  $J_\alpha$ -hierarchy, for  $\alpha < \Sigma(3)$ , just as the “ $\Sigma_2$ -Theory Machine” of [7] did, as a standard ITTM.

We shall assume that our scratch tape is divided recursively into infinite sub-tapes  $D_0, D_1, \dots$ . We set aside the first cell on the  $D_0$  tape as a “flag cell” - and designate it “ $F$ ”.

We shall describe a process that, *inter alia*, allows a code,  $l(\alpha)$  for  $\langle J_\alpha, \in \rangle$ , written out as a characteristic function of a subset of  $\omega$ , to be uniformly obtainable from  $S(\alpha') =_{\text{df}} \langle C_i(\alpha') \mid i < \omega \rangle$ , the *snapshot* at stage  $\alpha'$ , where  $\alpha' =_{\text{df}} \omega^3 \cdot (\alpha + 1)$ .

To do this we need some further nomenclature. We let  $\mathcal{L}_{\dot{\epsilon}, \dot{p}}$  be the language augmented by an extra constant symbol  $\dot{p}$  and again assume a recursive enumeration of the  $\Sigma_n$ -sentences of this language. We use these enumerations in the following definitions.

**DEFINITION 6.** For  $\alpha \in \text{On}$ ,  $n \leq \omega$  (i) let  $T_\alpha^n \subseteq \mathcal{L}_{\dot{\epsilon}}$  be the  $\Sigma_n$ -Th( $\langle J_\alpha, \in \rangle$ );  
(ii) for  $p \in J_\alpha$  we let  $T_\alpha^n(p) \subseteq \mathcal{L}_{\dot{\epsilon}, \dot{p}}$  be the theory  $\Sigma_n$ -Th( $\langle J_\alpha, \in, p \rangle$ ).

As the base case for an induction, we shall assume that for the least  $\alpha_0 \in G$  (i.e.  $\alpha_0 = 1$ ), our program has written  $T_{\alpha_0}^\omega$  on  $D_3$  and a code  $l(\alpha_0)$  for  $\langle J_1, \in \rangle$  on  $D_2$  by stage  $\omega^3 \cdot (\alpha_0 + 1) = \omega^3 \cdot 2$ .

We shall now describe the process that at the point in time  $\omega^3 \cdot (\alpha + 1)$ : (I) has written a code  $l(\alpha)$  for  $J_\alpha$  to  $D_2$  using (II) the complete  $\Sigma_\omega$ -theory  $T_\alpha^\omega$  which has been written on  $D_3$ . The above is the base case for the least good ordinal  $\alpha_0$ . In the sequel we may refer to *stages* of the process. Each single stage may require infinitely many machine *steps* (each of the latter takes a single unit of ‘time’). We do not wish to give all the details of the machine steps, but shall try and describe the stages that are translatable into steps, and shall endeavour to apply these two words in this way. However note that for  $\lambda$  good it will only take  $\lambda$  steps to perform  $\lambda$  stages, so at such points the numerations have caught up.

Suppose inductively that for  $\alpha \in G$  we have derived (I) and (II) on the tape. We use the following notation:

$$\bar{\alpha} = \sup(\widehat{S}_\alpha^1); \quad \alpha' = \alpha'(\alpha) \text{ is the largest } \alpha' \leq \alpha \text{ with } \alpha' \in G^*.$$

• Between  $\alpha$  and  $\alpha^+ (= \alpha + \omega^\omega \cdot \omega)$ , the next limit of good ordinals after  $\alpha$ , the machine behaves as follows:

(A) (i) It computes a code for  $\bar{\alpha}$  and writes this to  $D_1$ . We make the proviso that if at any time the value on  $D_1$  is changed, whilst that value is being written, then before a cell on  $D_1$  is written to, it is first preceded by a ‘wipe clean’ action that resets that cell value first to zero, before overwriting the new value. (Note that this simply has the effect that any cell that contains a ‘1’, which is destined to stay having a ‘1’ after the new value of  $\bar{\alpha}$  is written, is ‘flashed’  $1 \rightarrow 0 \rightarrow 1$ .) The point of this wipe-clean process is only that after an  $\omega$ -sequence of such changes, the  $D_1$  tape will contain only 0’s. We also, for convenience’s sake, represent the ordinal ‘0’ as ‘1000...’ (or some such) to distinguish it from the ‘empty’ tape ‘0000...’.

Given  $T_\alpha^\omega$ , and  $l(\alpha)$ , as  $\widehat{S}_\alpha^1$  is  $\Pi_1^{J_\alpha}$  definable, it takes only finitely many steps to determine whether the value on  $D_1$  needs to be changed, and if so, only  $\omega + \omega$  steps to identify  $\bar{\alpha}$  and write out a code for it.

(ii) From  $T_\alpha^\omega$  it looks for  $\alpha' = \alpha'(\alpha) \in G^*$ , and sets

$$\begin{aligned} F &= 0 \text{ if } \alpha' \in \widehat{S}_\alpha^1; \\ &= 1 \text{ otherwise.} \end{aligned}$$

This takes only  $< \omega$  steps.

Note for (iii) and (iv) to come, that given any set  $x \in J_\alpha$  and any  $n \leq \omega$ ,  $T_\alpha^n(x) \leq_T T_\alpha^\omega$  since  $T_\alpha^n(x)$  is the set of  $\Sigma_n$ -sentences true of  $f_\alpha(k)$  for some  $k$ , where  $f_\alpha : \omega \rightarrow J_\alpha$  is a canonical onto map defined from  $T_\alpha^\omega$  and  $f_\alpha(k) = x$ . As our induction proceeds for all ordinals  $\beta < \Sigma(3)$  we may assume that  $f_\beta$  is always a  $\Sigma_3^{J_\beta}$  parameter-free map (uniformly defined for all  $\beta$ ). We set  $\bar{f}_\alpha = f_\alpha \cap (\omega \times \omega^2)$ .

(iii) From  $\bar{\alpha}$  (which was identified at (i)) and using  $T_\alpha^\omega$ , it computes  $T_{\bar{\alpha}}^\omega$  and writes it to  $D_0$  (once  $\bar{\alpha}$  is identified, in the manner just remarked, we may easily find the recursive function for the reduction  $T_{\bar{\alpha}}^\omega \leq_T T_\alpha^\omega$ ; this takes  $\omega$  many steps).

(iv) We also require that  $T_\alpha^1(\bar{\alpha})$  be written out to  $D_4$ . Again a further  $\omega$  steps, given  $T_\alpha^\omega$  and  $\bar{\alpha}$ .

(v) From  $l(\alpha)$  and  $T_\alpha^\omega$  (on  $D_3$ ) it can assemble a code for  $l(\alpha + 1)$  and replace  $l(\alpha)$  with this on  $D_2$  ( $\omega \cdot 2$  steps).

(vi) Using  $l(\alpha + 1)$  it writes the  $\Sigma_\omega$ -theory  $T_{\alpha+1}^\omega$  to  $D_3$ . ( $\omega^2$  steps).

(vii) On our rule tape, we may successively list

(a)  $\bar{f}_\alpha(0)$  (if defined), is written in  $\omega$  steps to  $R$ . That done, we then successively list all those  $n * 1 \frown \bar{f}_\alpha(0)$ , for which  $J_\alpha \models \varphi_n^1[\bar{f}_\alpha(0)/\bar{p}]$  (where  $\varphi_n^1$  is our prior fixed recursive enumeration of the  $\Sigma_1$ -sentences of  $\mathcal{L}_{\in, \bar{p}}$ ). Note that after this has been done (which takes  $\omega^2$  many steps) each cell  $R_i$  of  $R$  has value 0, as  $R$  fills up with zero's from the left, because  $T_{\bar{f}_\alpha(0)}^1$  is infinite. We return to (a) looking in turn at those of  $\bar{f}_\alpha(1), \bar{f}_\alpha(2), \dots, \bar{f}_\alpha(k), \dots$  etc. which are defined, in a similar fashion.

This process will take care of our rule requirements. Note that it takes  $\omega^3$  steps, and this in fact is the totality of all steps taken at this stage, which is now finished.

We now have  $l(\alpha+1)$  and  $T_{\alpha+1}^\omega$  correctly written out and are at step  $\omega^3 \cdot (\alpha+1) + \omega^3 = \omega^3 \cdot (\alpha+2)$ . So now the machine may return to (A). To help summarise, we produce in the next  $\omega^3$  steps:

$\overline{\alpha+1} \simeq \sup(S_{\alpha+1}^1)$  on  $D_1$  ;  
 $\alpha'(\alpha+1)(= \alpha'(\alpha))$ , setting  $F$  appropriately;  
 $T_{\alpha+1}^\omega$  on  $D_0$  ;  
 $T_{\alpha+1}^1(\overline{\alpha+1})$  on  $D_4$  ;  
 $l(\alpha+2)$  on  $D_2$  ;  
 $T_{\alpha+2}^\omega$  on  $D_3$  ;  
 $1 * x$  and listings of  $T_{\alpha+1}^1(x)$  on  $R$  as above, for all  $x \in {}^\omega 2 \cap J_{\alpha+1}$  in turn.

This fully describes the ‘‘successor’’ stage action relating the transitions from  $J_\alpha$  to  $J_{\alpha+1}$  to  $J_{\alpha+2}$ . Note that after the last stage the time clock is at  $\omega^3 \cdot (\alpha+3)$ . Moreover that every cell on the  $R$  tape has unboundedly often in this time contained a 0 (and thus actually contains only 0's at this time by the  $\liminf^*$  rule, which at this simple stage devolves into taking simple  $\liminf$ 's).

At limit stages  $\lambda \in (\alpha, \alpha^+)$  the  $\liminf^*$  rule is to be used. As  $\lambda \notin G^*$ , the rule is again merely the former ITTM rule of straight  $\liminf$ . This thus has the following effects at this stage:

- (1) On  $D_1$ : as  $\beta \longrightarrow \lambda$   $\bar{\beta}$  is eventually constant, with eventual value  $\bar{\lambda} < \lambda$  say, with  $\bar{\lambda} = \sup \widehat{S}_\lambda^1$ . This implies that  $F$  is eventually constant, but moreover:
- (2) On  $D_0$  : we have  $T_{\bar{\lambda}}^\omega$ ;
- (3) On  $D_4$ : we have  $T_{\bar{\lambda}}^1(\bar{\lambda})$  (as both these are the simple union of earlier  $\Sigma_1$ -theories.)
- (4) On  $D_3, D_2$ : we have the  $\liminf_{\beta \rightarrow \lambda}^* T_\beta^\omega$  ; and  $\liminf_{\beta \rightarrow \lambda}^* l(\beta)$ .
- (5) On  $R$ : every cell is again zero, due to the unbounded setting of all cells to zero during listings and the  $\liminf$  rule.

• If  $D_1$  is not empty, this is because  $\bar{\lambda} < \lambda$ . (This will always happen if  $\lambda \notin G^*$ .) In this case there is a  $\Sigma_1^{J_\lambda}(\{\bar{\lambda}\})$  map uniform in  $\lambda$  and the parameter  $\bar{\lambda}, g_\lambda : \omega \rightarrow J_\lambda$  (meaning uniform for those  $\lambda$  with  $\bar{\lambda} < \lambda$  - note  $\bar{\lambda} = 0$  is included as a possibility). Using  $D_4$  this allows us to compute a code  $l(\lambda)$  for  $\langle J_\lambda, \in \rangle$  to be written to  $D_2$  in  $\omega$  many steps. From  $l(\lambda)$  we may compute  $T_\lambda^\omega$  and write this to  $D_3$  (this takes  $\omega^2$  time). Rather superfluously at this stage we run the Rule tape writing algorithm solely in order to keep the enumeration of steps in line.

We are now at time  $\omega^3 \cdot (\lambda + 1)$ . The machine then returns to (A) and continues as before. We now consider what to do when we arrive at a  $\lambda \in G^*$ .

DEFINITION 7. For  $\lambda \in G^*$  let  $\widehat{E}_\lambda^1 =_{\text{df}} \{\alpha < \lambda \mid \widehat{S}_\alpha^1 = \widehat{S}_\lambda^1 \cap \alpha\}$ .

The point of listing  $\Sigma_1$ -theories of the form  $\langle J_\gamma, \in, x \rangle$  on the Rule tape  $R$  is precisely to establish (6) below. This will have the consequence that if  $\gamma \in G \cap \lambda$  then  $\gamma$  is 1-correct in  $\lambda$  in the sense of computation if and only if  $\gamma$  is similarly ‘ $\Sigma_1$ -correct’ about which ordinals are  $\Sigma_1$ -stables in the set-theoretic context; this implies:

$$\bar{\gamma} < \gamma \in G \cap E_\lambda^1 \longrightarrow (J_{\bar{\gamma}} \prec_{\Sigma_1} J_\gamma \leftrightarrow J_{\bar{\gamma}} \prec_{\Sigma_1} J_\lambda).$$

We thus show:

$$(6) \quad \lambda \in G \longrightarrow \widehat{S}_\lambda^1 = S_\lambda^1. \text{ Hence } \lambda \in G^* \longrightarrow \widehat{E}_\lambda^1 \cap G = E_\lambda^1 \cap G.$$

Proof: We show by induction on  $\alpha < \lambda$  that  $\widehat{S}_\alpha^1 \cap \alpha = S_\alpha^1 \cap \alpha$ . Suppose this shown for  $\alpha$ . Let  $\gamma = \min((\widehat{S}_\alpha^1 \cup S_\alpha^1) \setminus \alpha)$ . Suppose  $\gamma \in \widehat{S}_\alpha^1$ . Then  $J_\gamma \prec_{\Sigma_1} J_\lambda$  and  $\gamma$  is highly closed, indeed admissible. However by our rule tape construction we have continually for any  $\beta < \lambda$ , listed  $T_\beta^1(p)$  for any  $p \in J_\beta$  of the form  $p = x \in \omega^2$ . But any  $\Sigma_1$  sentence  $\varphi_n[x]$  with  $x \in \omega^2 \cap J_\gamma$  in this theory, is also in such a theory  $T_\delta^1(x)$  for a  $\delta \in G \cap \gamma$ , as  $J_\gamma \prec_{\Sigma_1} J_\lambda$ . Thus  $n * 1 \frown x$  appears on  $R$  before time  $\gamma$ . Hence  $\gamma \in S_\alpha^1$ .

Now suppose  $\gamma \in S_\alpha^1$ . Then  $\gamma \in G$ . We should like to have  $J_\gamma \prec_{\Sigma_1} J_\lambda$ . Note that it suffices to require  $\Sigma_1$  elementarity for formulae  $\varphi_n(\xi/v_0)$  for ordinals  $\xi < \gamma$ , and moreover as there is a real  $x_\xi \in \omega^2 \cap J_{\xi+1}$  coding  $\xi$ , it thus suffices to consider this for formulae  $\varphi_n(x/v_0)$  with  $x \in \omega^2 \cap J_\gamma$ . By our construction for any  $1 * x$  listed on the rule tape by stage  $\gamma$ , as  $\gamma \in S_\alpha^1$  then  $T_\lambda^1(x)$  is recursive in  $\langle R(\nu) \mid \nu < \gamma \rangle$ . (If  $1 * x = R(\nu_0 + \omega)$  then it is  $\langle R(\nu_0 + \omega \cdot k) \mid k < \omega \rangle$  that lists  $T_\lambda^1(x)$ ). Of course if  $x \in J_\gamma$  so is any  $n * x$  for any  $n$ , and so the latter will also be listed at some point: but that is why we restrict our initial listing of any such  $x$  as prefixed by a ‘1’, and their theories by 0’s.) Running the machine inside  $J_\gamma$  this would make  $T_\lambda^1(x)$   $\Sigma_1$ -definable over  $J_\gamma$  for any  $x \in \omega^2 \cap J_\gamma$ . Hence  $T_\lambda^1(x) = T_\gamma^1(x)$  for all such  $x$ . This suffices to imply that  $J_\gamma \prec_{\Sigma_1} J_\lambda$ . QED (6)

Suppose now  $\lambda$  equals  $\alpha^+$ . Note that  $E_\lambda^1$  even in this case is simply a tail of  $G \cap \lambda$ . As  $\beta \longrightarrow \lambda$   $\bar{\lambda}$  will settle down in value below  $\lambda$ , from some point on. (Recall that  $\bar{\lambda} = \lambda$  would imply that  $\lambda$  is in fact strongly admissible, and thus cannot be of the form  $\alpha^+$ .) The  $\lim \inf^*$  rule now comes into play in full, but we still have (1)-(5) holding just as for a limit which is a non-limit of good ordinals, as above, and the actions are the same. One point to note is:

- If  $\beta = \alpha^+ + 1$  then  $F$  must then by the end of the operations at stage  $\beta$  be set to 1: this is because  $\alpha^+ \notin \widehat{S}_\beta^1$  (an element of any  $\widehat{S}_\beta^1$  is always in  $(G^*)^*$ ).

However for a general  $\lambda \in G^*$   $\bar{\lambda}$  may equal  $\lambda$ . Our Flag  $F$  is designed to alert the machine when we are at such a point:

$$(7) \quad (\lambda \in (G^*)^* \wedge F(\lambda) = 0) \longleftrightarrow \bar{\lambda} = \lambda.$$

Proof: ( $\leftarrow$ ) Suppose  $\bar{\lambda} = \sup \widehat{S}_\lambda^1 = \lambda$ . Then for unboundedly many  $\beta < \lambda$  we have  $\beta \in \widehat{S}_\lambda^1$  and thus  $\lambda \in (G^*)^*$ . Further for  $\beta \in \widehat{S}_\lambda^1$ , we have that the Flag  $F$  at stage  $\beta + 1$



is set to 0, as  $\beta = (\beta + 1)' \wedge \beta \in \widehat{S}_{\beta+1}^1$ . This value of 0 persists for any of the steps  $\eta \in (\beta + \omega^3, \beta + \omega^\omega]$ . Thus at the next good ordinal  $\beta + \omega^\omega$  the Flag is zero. Moreover  $\beta + \omega^\omega \in E_\lambda^1$ .

However by step  $\beta^+ + \omega^3$   $F$  has been set to 1 (see the last bullet point above). But  $\beta^+ \notin \widehat{S}_\delta^1$  for any  $\delta \in (\beta^+, \beta^+ + \omega^\omega]$ . Hence the Flag stays set at 1 in this interval (and beyond). But both  $\beta + \omega^\omega$  and  $\beta^+ + \omega^\omega$  are in  $G$  and are 1-correct in  $\lambda$ . This happens for unboundedly many  $\beta < \lambda$ . Hence because of these alternating values,  $\liminf_{\nu \rightarrow \lambda}^* F(\nu) = 0$ .

( $\rightarrow$ ) Suppose  $\bar{\lambda} < \lambda$ . Let  $\gamma$  be the least element of  $G^*$  greater than  $\bar{\lambda}$ . Then  $F$  is at stage  $\gamma + 1$  set to value 1. It will only be 0 at any stage  $\delta > \gamma$  if  $\bar{\lambda} < \delta' \in \widehat{S}_\delta^1$ . But such a  $\delta$  is not in  $E_\lambda^1$ ! Hence  $\{\nu \in (\bar{\lambda}, \lambda) \mid F(\nu) = 0\} \cap E_\lambda^1 = \emptyset$ . Hence  $F(\lambda) = 1$  by the  $\liminf^*$  rule. QED (7)

(8) If  $\lambda \in G$  and  $\bar{\lambda} < \lambda$  then  $T_\lambda^\omega$  is on  $D_0(\lambda)$ .

Proof: By induction on  $\bar{\lambda} < \beta \leq \lambda$ , for  $\beta \in E_\lambda^1$  show that  $T_\lambda^\omega$  is on  $D_0(\beta)$ , noting that  $\beta = \bar{\lambda}$  for such  $\beta$ . QED(8)

(9)  $\forall \lambda < \Sigma(3)$  (a code for  $l(\lambda)$  can be extracted from  $S(\lambda)$ )

Proof: We've seen how to do this using  $T_\lambda^1(\bar{\lambda})$  which is on  $D_4$ , when  $\bar{\lambda} < \lambda$  (by using the method of an onto map explained just after (5)). So assume  $\bar{\lambda} = \lambda$  (and *a fortiori*  $\lambda \in (G^*)^*$ ). Then the Flag  $F(\lambda) = 0$  (see (7)) and hence the machine knows this fact. We show first how to determine  $T_\lambda^3$  from  $D_3(\lambda) = \liminf_{\alpha \rightarrow \lambda}^* T_\alpha^\omega$  which is written on  $D_3$ .

Suppose  $\varphi \equiv \exists u \psi(u)$  is  $\Sigma_3$  with  $\psi \in \Pi_2$  in  $\mathcal{L}_\in$ . We use the following equivalence.

(10) Assume  $\bar{\lambda} = \lambda$ .  $\varphi \in T_\lambda^3 \iff \exists n \in \omega [\exists \gamma_0 \forall \gamma > \gamma_0 (\gamma \in \widehat{S}_\lambda^1 \longrightarrow J_\gamma \models \sigma_n)]$   
 where  $\sigma_n$  is the following sentence: “ $\exists \beta [(\beta \in \widehat{S}_\gamma^2 \cup \{0\}) \wedge \exists k (k = h_\gamma^2(n, \beta) \wedge \psi(k))]$ ”.

Proof of (10): ( $\rightarrow$ ). Suppose  $\varphi \in T_\lambda^3$  and is of the form illustrated, with some  $x \in J_\lambda$  witnessing  $J_\lambda \models \psi[x]$ . Then for some  $\beta \in \widehat{S}_\lambda^2 \cup \{0\}$ ,  $x = h_\lambda^2(n, \beta)$ . Choose  $\gamma \in \widehat{S}_\lambda^1$  sufficiently large with  $x \in J_\gamma \models x = h_\gamma^2(n, \beta)$ . This is possible by our assumption on  $\lambda$ . However for any  $\gamma' \geq \gamma$  with  $\gamma' \in S_\lambda^1$  we also have that  $x = h_{\gamma'}^2(n, \beta)$  and  $\beta \in \widehat{S}_{\gamma'}^2$ . Moreover  $(\psi[x])_{J_\lambda}$  is  $\Pi_2$  and so goes down to such  $\gamma'$ :  $(\psi[x])_{J_{\gamma'}}$ . Hence the right hand side holds.

( $\leftarrow$ ): Suppose the RHS holds for some  $n$ , but the LHS failed for a contradiction. Then we first claim that  $\widehat{S}_\lambda^2$  must be bounded in  $\lambda$ : for if it were unbounded then we could always take a  $\gamma$  on the RHS to be from  $\widehat{S}_\lambda^2$ . However then the  $\Pi_2$  statement  $\psi[k]$  about  $k$ , included in  $\sigma_n$ , would go up to  $\lambda$  and the LHS would hold. Hence  $\sup(\widehat{S}_\lambda^2) = \beta_0 < \lambda$ . Now there are unboundedly many  $\gamma < \lambda$  with  $\gamma \in \widehat{S}_\lambda^1$ ,  $\gamma > \beta_0$ ,  $\widehat{S}_\gamma^2 = \widehat{S}_\lambda^2 \cap \gamma$ , and with  $J_\gamma \models \sigma_n$ . For such a  $\gamma$ , let  $\beta = \beta_\gamma$  be the least ordinal witnessing the existential quantifier of the quoted statement  $\sigma_n$ . By what we may call the “2-correctness” of  $\gamma$ ,  $\beta \in \widehat{S}_\lambda^2 \cup \{0\}$ . If for some such  $\gamma$  satisfying these clauses, we had  $\beta_\gamma < \beta_0$ , we should have that “ $k = h_\gamma^2(n, \beta_\gamma) \wedge \psi(k)$ ” which is true in  $J_\gamma$ , would be absolute to  $J_{\beta_0}$ . But

$\beta_0 \in \widehat{S}_\lambda^2 \cup \{0\}$  so it is also true in  $J_\lambda$  thus verifying  $\psi$ . On the other hand if for all such  $\gamma$  satisfying the clauses we had  $\beta_\gamma = \beta_0$ , then  $h_\gamma^2(n, \beta_\gamma)$  has constant value some  $k$  for all such  $\gamma$  on a tail of  $\widehat{S}_\lambda^1$ . Moreover as  $(\psi[k])_{J_\gamma}$  in all sufficiently large  $\gamma \in \widehat{S}_\lambda^1$  and as  $\psi \in \Pi_2$ , we conclude  $(\psi[k])_{J_\lambda}$ , another contradiction. QED(10)

(11) *There is a (1-1) recursive  $F : \mathbb{N} \rightarrow \mathbb{N}$  so that for any  $\lambda < \Sigma(3)$ , if  $\bar{\lambda} = \lambda$  then*

$$\varphi \in T_\lambda^3 \iff \exists n \in \omega F(\prec n, \ulcorner \varphi \urcorner \succ) \in D_0(\lambda).$$

Hence for such  $\lambda$ ,  $T_\lambda^3$  is uniformly r.e. in  $S(\lambda) = \overrightarrow{C}_i(\lambda)$ .

Proof: Assume  $\bar{\lambda} = \lambda$ , then we recast (10) as:

$$\varphi \in T_\lambda^3 \iff \exists n \in \omega [\exists \gamma_0 \forall \gamma > \gamma_0 (\gamma \in E_\lambda^1 \implies \sigma_n \in D_0(\gamma))]$$

( $\rightarrow$ ) This follows from (6) and (10).

( $\leftarrow$ ) As  $S_\lambda^1$  is unbounded in  $\lambda$ , and is contained in  $E_\lambda^1$  this follows from the argument of (10).

From this our  $\text{liminf}^*$  rule then shows  $D_0(\lambda)$  has the correct information. QED (11)

For  $\lambda < \Sigma(3)$  we shall thus be able, by the comments on Skolem hulls *etc.* after Definition 5, to use a  $\Sigma_3^{J_\lambda}$  map  $h$  of  $\omega$  onto  $J_\lambda$  (which by the above is at worst uniformly r.e. in  $S(\lambda)$  in the case that  $\bar{\lambda} = \lambda$ ) to define a code  $l(\lambda)$  for  $J_\lambda$ : if  $h(k)$  is defined, we may form the equivalence class of  $k'$  such that  $h(k) = h(k')$ , and define the binary relation  $kEm \iff h(k) \in h(m)$ . This yields a code  $l(\lambda)$  for  $\langle J_\lambda, \in \rangle$  which may be written to  $D_2$  in, say a further  $\omega$  steps. QED(9)

The above process lasts for as long as differing  $\Sigma_3$  theories are produced for the different levels  $\lambda < \Sigma(3)$ . However,  $(\zeta(3), \Sigma(3))$  is also the lexicographic least pair  $(\pi, \chi)$  of ordinals with  $T_\pi^3 = T_\chi^3$ . (Assume for a contradiction that  $(\pi, \chi) <_{\text{lex}} (\zeta(3), \Sigma(3))$  have the same  $\Sigma_3$ -theories. Firstly if  $\chi < \Sigma(3)$ , using the onto  $\Sigma_3$  function  $h$ , we have that the  $\Sigma_3$  sentences  $\sigma_{n,m} \equiv "h(n) < h(m) \in \text{On}"$  are in  $T_\chi^3$  and yield a wellorder of type  $\chi$  (as  $h_\chi$  is onto  $\chi$ ). But it is impossible that such  $\sigma_{n,m}$  are all in  $T_\pi^3$  as the latter there yield a wellorder only of type  $\pi$ ! Hence we must have  $\chi = \Sigma(3) \wedge \pi < \zeta(3)$ . But clearly  $T_{\Sigma(3)}^3 = T_{\zeta(3)}^3$ , and the same argument shows that the sentences  $\sigma_{n,m}$  in  $T_{\zeta(3)}^3$  are in  $T_\pi^3$  for the same contradiction.)

Hence the machine before stage  $\Sigma(3)$  produces different theories, and at stage  $\Sigma(3)$  produces only  $T_{\zeta(3)}^3$  and then  $l(\zeta(3))$  on  $D_2$ , and so commences to cycle.

REMARK 2. Since it is (reasonably) clear that any machine with this limit rule will either halt before  $\zeta(3)$  or enter a loop at this point (by considering such machines as running inside  $L_{\Sigma(3)}$ ) we thus have a complete description of the *semi-decidable, decidable* predicates, *halting problem* and so on, just as the calculation determining the role of the ordinals  $(\zeta(2), \Sigma(2))$  did for the original ITTM model. (See [15] for a somewhat cleaner development of this  $\Sigma_2$ -theory.) For example, the halting problem set will be recursively isomorphic to the  $\Sigma_1$  truth set of  $\langle L_{\lambda(3)}, \in \rangle$  where  $\lambda(3)$  is least such that  $L_{\lambda(3)} \prec_{\Sigma_1} L_{\zeta(3)}$ . The other assertions concerning abstract 1-sections *etc.* from the Introduction then also follow.

§4.  $\Sigma_n$ -Machines. We now consider machines related to levels of the  $L$ -hierarchy at the least  $\zeta(n)$  with some  $\Sigma(n) > \zeta(n)$  and  $L_{\zeta(n)} \prec_{\Sigma(n)} L_{\Sigma(n)}$  for larger  $n < \omega$ .

In developing the next level,  $\Sigma_4$ -machines, which halt or loop by  $\zeta(4)$ , we accordingly make use of the appropriate notions  $E_\lambda^2, \widehat{E}_\lambda^2, S_\lambda^2, \widehat{S}_\lambda^3, \widehat{S}_\lambda^4$ , etc. We assume now the rule tape  $R$  is recursively split into two infinite pieces,  $Q, R$ . (We keep the letter  $R$  as we intend to expand on what the machine of the last Section does.) There is no difference between what is written to the  $R$  tape in the current machine and that of the last section; we use  $R$  to define  $S_\lambda^1$  and  $E_\lambda^1$  just as before. The  $Q$ -part will be used to define  $S_\lambda^2$  etc. We further recursively split  $Q$  into  $2 \times \omega^2$  many infinite pieces  $Q(i, k, m)$  with  $i < 2, k, m \in \omega$ . As before  $Q(\beta, i, k, m)$  will denote that piece viewed at time  $\beta$ .

We adopt the notation that  $\alpha_i =_{\text{df}} \sup S_\alpha^i$  for  $i = 1, 2$ .

DEFINITION 8.  $\alpha \in S_\lambda^2 \iff \alpha = \alpha_1 \wedge \alpha \in S_\lambda^1 \wedge$   
 $\forall x \in {}^\omega 2 \forall \nu < \alpha \forall n [(1 \frown x = R(\nu) \wedge \exists \beta' \in E_\lambda^1 Q(\beta', i, k, m) = 1 \frown x) \longrightarrow$   
 $\longrightarrow \exists \beta' \in E_\alpha^1 (Q(\beta', i, k, m) = 1 \frown x)].$

DEFINITION 9.  $E_\lambda^2$  (the 2-correct in  $\lambda$  ordinals)

$$E_\lambda^2 = \{\alpha \in E_\lambda^1 \mid \alpha_{1,2} (=_{\text{df}} \sup S_{\alpha_1}^2) \in S_\lambda^2\} \quad \text{if } \lambda_1 = \lambda;$$

$$= E_\lambda^1 \quad \text{otherwise.}$$

Our definitions imply:

REMARK 3.  $S_\beta^2 \subseteq E_\beta^2$ ;  $\beta = \beta_1 \longrightarrow S_\beta^1 \cap E_\beta^2$  is closed and cofinal in  $\beta$ .

We now adopt as our Limit Rule:

$$C_i(\lambda) = 1 \iff \exists \nu_0 < \lambda \forall \nu \in (\nu_0, \lambda) [\nu \text{ 2-correct at } \lambda \longrightarrow C_i(\nu) = 1];$$

$$= 0 \quad \text{otherwise.}$$

The above then completes the description of the  $\Sigma_4$ -machine architecture. We turn now to a program that computes theories and codes for all levels of the constructible hierarchy below  $\Sigma(4)$ .

$\Sigma_4$ -theory machine description. Let  $\langle \varphi_n(v_0) \rangle$  effectively enumerate all formulae and  $\langle \psi_n(v_0) \rangle$  all  $\Sigma_1$  formulae in the free variable  $v_0$ . We continue to use the notation:  $\bar{\alpha} =_{\text{df}} \sup \widehat{S}_\alpha^1$ ; and further adopt  $\alpha^2 =_{\text{df}} \sup \widehat{S}_\alpha^2$ . (Later we shall see that for the machine description to be specified we shall have  $\alpha_1 = \bar{\alpha}$  and  $\alpha_2 = \alpha^2$  i.e. with the notions of computable stability coinciding with set theoretical  $\Sigma_n$ -stability.) As in the previous section we have set  $f_\alpha : \omega \twoheadrightarrow J_\alpha$  to be a canonical onto map defined from  $T_\alpha^\omega$ . As our induction proceeds for all ordinals  $\beta < \Sigma(4)$  we may assume that  $f_\beta$  is always a  $\Sigma_4^{J_\beta}$  parameter-free map (uniformly defined for all  $\beta$ ). Again set  $\bar{f}_\alpha = f_\alpha \cap (\omega \times \omega^2)$ . To specify the operation we simply augment the previous  $\Sigma_3$ -theory machine with an extra task at (6).

Let  $\alpha \in G$  and let  $\beta > \alpha$  be least with  $\beta \in G$ . Then at stage  $\beta$  we shall now additionally require:

(6) Let  $\bar{\beta}$  be as above and as already defined, with  $\bar{\beta}^2 = \sup \widehat{S}_{\bar{\beta}}^2$ . Then the complete theories of  $J_{\bar{\beta}}$  and  $J_{\bar{\beta}^2}$  are written in the following way at stage  $\beta$ : if  $y = \bar{f}_{\bar{\beta}}(k)$  is the  $k$ 'th real in  $J_{\bar{\beta}}$  (resp.  $J_{\bar{\beta}^2}$  where in this case  $y = \bar{f}_{\bar{\beta}^2}(k)$ ) and  $J_{\bar{\beta}} \models \varphi_n[y]$  (resp.  $J_{\bar{\beta}^2} \models \varphi_n[y]$ ) then it is required that  $1 \frown y$  is on the the second rule

$Q(\beta, 0, k, n)$  ( $Q(\beta, 1, k, n)$  resp.) tape segment at stage  $\beta$ ; otherwise  $Q(\beta, 0, k, n)(0) = 0$  ( $Q(\beta, 1, k, n)(0) = 0$  resp.).

Starting from a code  $l(\alpha)$  for  $J_\alpha$  the machine writes codes for  $J_\gamma$  for  $\gamma \in [\alpha, \beta)$  writing, just as in the last section, (i) - (vii). We further ensure:

(viii) Using  $T_{\bar{\gamma}}^\omega$  which is written on  $D_0$  and,  $T_{\bar{\gamma}^2}^\omega$  (which is recursive  $T_{\bar{\gamma}}^\omega$ ), we may compute values of  $\bar{f}_{\bar{\gamma}}(k)$  and  $\bar{f}_{\bar{\gamma}^2}(k)$  and write out the theories in the real parameters  $\bar{f}_{\bar{\gamma}}(k) = y$  and  $\bar{f}_{\bar{\gamma}^2}(k) = y$ . Thus we set  $Q(\gamma, 0, k, m) = 1 \wedge \bar{f}_{\bar{\gamma}}(k)$  iff  $J_{\bar{\gamma}} \models \varphi_m[\bar{f}_{\bar{\gamma}}(k)]$  and similarly  $Q(\gamma, 1, k, m) = 1 \wedge \bar{f}_{\bar{\gamma}^2}(k)$  iff  $J_{\bar{\gamma}^2} \models \varphi_m[\bar{f}_{\bar{\gamma}^2}(k)]$ . Mirroring the writing of  $T_{\bar{\beta}}^1(\bar{\beta})$  before on  $D_1$ , we use an additional piece of scratch tape  $D_5$  and record here  $T_{\bar{\gamma}}^2(\gamma^2)$ . This can also be obtained from  $T_{\bar{\gamma}}^\omega$  on  $D_0$ . All this can be done in  $< \omega^2$  many steps. Recall that the previous  $R$ -writing process (vii) required  $\omega^3$  many steps. So by dovetailing this process with that of (vii) we can still stick to the same ordinal arithmetic and have both  $R$ - and  $Q$ - writing done in  $\omega^3$  steps.

Recall that (ii) ensures that the machine may start with a 0 on  $F$  at stage  $\alpha$  but will change it to a 1 as soon as it sees  $\gamma' = \alpha' \notin \widehat{S}_\gamma^1$ .

$$(6') \lambda \in G \longrightarrow \widehat{S}_\lambda^1 = S_\lambda^1. \text{ (Thus } \bar{\lambda} = \lambda_1.) \text{ Hence } \lambda \in G^* \longrightarrow E_\lambda^1 \cap G = \widehat{E}_\lambda^1 \cap G.$$

$$(7') (\lambda \in (G^*)^* \wedge F(\lambda) = 0) \longleftrightarrow \sup \widehat{S}_\lambda^1 = \lambda.$$

We additionally have here the new:

$$(6'') \bar{\beta} = \beta \longrightarrow (\alpha \in S_\beta^2 \longleftrightarrow \alpha \in \widehat{S}_\beta^2). \text{ Thus } \beta^2 = \beta_2.$$

For (6') note that the  $\Sigma_4$ -theory machine extends the action of the  $\Sigma_3$ -theory machine, with  $\widehat{S}_\lambda^1, S_\lambda^1$ , defined from before, so there is nothing to prove.

For (7'): ( $\leftarrow$ ) The argument is the same but here the only difference is that we are taking a  $\liminf^*$  along  $E_\lambda^2$ : there are unboundedly many  $\tau \in E_\lambda^2$  taking each of the values 0 and 1. For ( $\rightarrow$ ) if we assume, using (6'),  $\bar{\lambda} < \lambda$  then in this case  $E_\lambda^2 = E_\lambda^1$  so the previous reasoning holds.

Proof of (6''): suppose  $\alpha \in S_\beta^2$ . Suppose  $J_\beta \models \varphi_n[y]$  with  $\varphi$  expressing a  $\Sigma_2$  property about  $y$  where  $y \in J_\alpha$ , then we want  $J_\alpha \models \varphi_n[y]$ . However ' $J_\beta \models \varphi_n[y]$ ' is equivalent to  $J_{\bar{\delta}} \models \varphi_n[y]$  for some sufficiently large  $\delta \in E_\beta^1$ . Further this is recorded on the  $Q(\delta, 0, k, n)$  part of the second rule tape at a sufficiently large stage  $\delta \in E_\beta^1$  with  $y \in J_{\bar{\delta}}$  and  $y = \bar{f}_{\bar{\delta}}(k)$ . (Note that  $\bar{\delta} \in S_\beta^1$ ). Then  $\alpha \in S_\beta^2$  guarantees that this  $\Sigma_2$  property about  $y$  has been recorded as holding for some sufficiently large  $\delta_0 \in E_\alpha^1$  by  $Q(\delta_0, 0, k, n)$ , and thus  $\varphi_n[y]$  holds in  $J_{\bar{\delta}_0}$  with  $\bar{\delta}_0 \in S_\alpha^1$ . By upwards  $\Pi_1$  elementarity then,  $\varphi_n[y]$  holds in  $J_\alpha$ . We conclude that  $\alpha \in \widehat{S}_\beta^2$ . For the converse one may imagine the machine running in  $J_\beta$  and then apply  $\Sigma_2$ -elementarity. QED

We have the following addition to (8). Here we have harmlessly assumed that  $k_0$  has been chosen so that  $\bar{f}_\beta(k_0) = 0$  for any  $\beta$ .

$$(8') \text{ For any } \lambda, \text{ if } \bar{\lambda} = \lambda \text{ and } \lambda^2 < \lambda \text{ then } T_{\lambda^2}^\omega \text{ is recursive in } Q(\lambda, 1, k_0, n).$$

Proof (8'): Recall that on the  $\langle 1, k, n \rangle$  parts of the  $Q$ -tape were recorded the complete theories  $T_{\bar{\beta}^2}^\omega(y)$  in real parameters  $y = \bar{f}_{\bar{\beta}^2}(k)$ .  $S(\lambda)$  is given by the  $\text{liminf}^*$  taken over 2-correct ordinals  $\beta < \lambda$ . Suppose  $\lambda^2 < \lambda$ . Then for  $\beta \in E_\lambda^2 \setminus \lambda^2$  by definition we have that  $\bar{\beta}^2 = \sup S_\beta^2 = \lambda^2$ , and in particular as  $\beta$  tends to  $\lambda$  through  $E_\lambda^2$ , eventually  $\bar{\beta}^2 = \lambda^2$ . Hence the pure part of the theory  $T_{\lambda^2}^\omega$  is eventually constant on the tape.

(9')  $\forall \lambda < \Sigma(4)$  (a code for  $l(\lambda)$  can be extracted from  $S(\lambda)$ ).

Proof: Using (7') the Flag  $F$  tells the machine if  $\bar{\lambda} = \lambda$ . If  $\bar{\lambda} < \lambda$  the construction of a code  $l(\lambda)$  is just as for the  $\Sigma_3$ -theory machine. So assume  $\bar{\lambda} = \lambda$ . Suppose  $\lambda^2 = \sup \hat{S}_\lambda^2 < \lambda$ . Just as in the proof of (8') as  $\beta$  tends to  $\lambda$  through  $E_\lambda^2$ , eventually  $\bar{\beta}^2 = \lambda^2$ . Consequently at stages for a tail of  $\beta \in E_\lambda^2$ , on  $D_5$  we have either the theory  $T_{\bar{\beta}^2}^2(\lambda^2)$  itself, or a  $\text{liminf}$  of such theories  $T_{\bar{\beta}^2}^2(\lambda^2)$  for an increasing chain of  $\bar{\beta}$ . However such  $\bar{\beta}$  are in  $\hat{S}_\lambda^1$ . By the upwards persistence of  $\Sigma_2$  theories, these  $\text{liminf}$ 's are simple unions and hence at stage  $\lambda$   $D_5$  contains  $T_\lambda^2(\lambda^2)$ .

Our assumption is that there is a  $\Sigma_2^{J_\lambda}(\{\lambda^2\})$  onto function  $f : \omega \rightarrow J_\lambda$ . Moreover the machine does know that  $\lambda^2 < \lambda$ :  $\lambda^2 = \lambda$  if and only if for arbitrarily large  $\delta \in E_\lambda^2$  we have both that  $J_\delta \models \text{"max } S_\delta^2 = \text{max } S_\delta^1\text{"}$  and also its negation. Consequently neither this sentence nor its negation is in  $\text{lim inf}_{\beta \rightarrow \lambda}^* T_\beta^\omega$  on  $D_0$ . Hence the machine knows to construct a code for  $l(\lambda)$  with that theory  $T_\lambda^2(\lambda^2)$ .

If  $\lambda^2 = \lambda$  then the direct generalisation of the argument of (10) one level up yields:

(10') Assume  $\lambda^2 = \lambda$ .

$$\varphi \equiv \exists v_0 \psi(v_0) \in T_\lambda^4 \iff \exists n \in \omega [\exists \gamma_0 \forall \gamma > \gamma_0 (\gamma \in \hat{S}_\lambda^2 \rightarrow J_\gamma \models \sigma_n)]$$

where  $\sigma_n$  is the following sentence: " $\exists \beta [(\beta \in \hat{S}_\gamma^3 \cup \{0\}) \wedge \exists k (k = h_\gamma^3(n, \beta) \wedge \psi(k))]$ ".

And in turn now with some minor adjustments to some  $F_0(\prec n, \ulcorner \varphi \urcorner \succ) := \ulcorner \sigma_n \urcorner$  for the latter  $\sigma_n$ :

(11') There is a (1-1) recursive  $F : \mathbb{N} \rightarrow \mathbb{N}$  so that for any  $\lambda < \Sigma(4)$ , if  $\lambda^2 = \lambda$  then

$$\varphi \in T_\lambda^4 \iff \exists n \in \omega Q(\lambda, 1, k_0, F_0(\prec n, \ulcorner \varphi \urcorner \succ))(0) = 1).$$

Hence for such  $\lambda$ ,  $T_\lambda^4$  is uniformly r.e. in  $S(\lambda) = \vec{C}_i(\lambda)$ .

Consequently in this case we may obtain the  $\Sigma_4$  theory of  $J_\lambda$  in a r.e. manner from  $S(\lambda)$ . Just as in the previous section this allows the machine to construct a code  $l(\lambda)$ . Hence as long as we are below  $\Sigma(4)$ , we obtain new theories, and so codes  $l(\alpha)$ . QED

We hope that the reader, having seen how to obtain  $\Sigma_3$ - and  $\Sigma_4$ -machines will be convinced as to how one can generalise the constructions to any  $\Sigma_{n+2}$  by extending the above. The definitions of the higher  $S_\beta^n$  and  $E_\beta^n$  should reflect those of  $E_\beta^2$  and

$S_\beta^2$ ; the  $Q$ -rule tape should be further subdivided to write down the corresponding theories obtained from the last case, merely by raising complexities by 1. The specifications of the  $\Sigma_{n+2}$ -machine follow the same template. Additional slices of the work-tape  $D_6, D_7, \dots, D_{n+3}$  will record theories up to those of the form  $T_{\beta^{n-2}}^n(\beta^n)$  with  $\beta^{n-2} =_{\text{df}} \sup S_\beta^{n-2}$  etc. The proof of (10') is generalisable over  $n$  *verbatim*. This leads to appropriate statements and proofs of (9') and (11').

### §5. Conclusions. Uniformization for ITTM semi-decidable sets

We make some observations concerning Uniformisation for the pointclasses definable from both the original ITTM's and the machines defined here. As a starting point:

DEFINITION 10. For  $n \geq 2$  and  $x \in 2^{\mathbb{N}}$  define  $(\zeta(n, x), \Sigma(n, x))$  as the lexicographically least pair  $\bar{\zeta}, \bar{\Sigma}$  with  $L_{\bar{\zeta}}[x] \prec_{\Sigma_n} L_{\bar{\Sigma}}[x]$ .

DEFINITION 11. Let  $\Gamma_n$  be the pointclass of  $\Sigma_1$ -semidecidable sets of reals defined by  $\Sigma_n$ -machines: that is, the class of those  $A \subseteq 2^{\mathbb{N}}$  so that there is some  $\varphi_e$ , a  $\Sigma_n$ -machine definable partial function, with

$$A(x) \leftrightarrow \varphi_e(x) \text{ halts with a 1 on the output tape.}$$

This is equivalent to saying that there is some  $\Sigma_1$ - $\varphi$  so that

$$A(x) \leftrightarrow L_{\Sigma(n, x)}[x] \models \varphi[x].$$

These can be seen to be *Spector pointclasses* in the sense of [12].

THEOREM 1. Both  $\text{Unif}(\Gamma_n)$  and  $\text{Scale}(\Gamma_n)$  hold.

Proof: Let  $A \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$  be so  $(\Sigma_1)$ -semidecidable. We wish to show that there is  $A^* \subseteq A$  which is  $\Sigma_n$ -machine semidecidable, and uniformizes  $A$ . Note that  $\Sigma(n, (w, z))$  to is a limit of  $(w, z)$ -admissibles, and let  $M_{w, z} =_{\text{df}} L_{\Sigma(n, w, z)}[w, z]$ . Moreover for any  $z \in M_{x, y}$  we have:

$$(1) M_{x, y} \models \varphi(x, z) \leftrightarrow L_{\Sigma(n, (x, y))}[x, y] \models \varphi[x, z].$$

For the discussion of the Shoenfield tree we follow the exposition and notation of [10], Ch.3.13. The  $\Sigma_1$  formula  $\varphi$  has a canonical  $\Sigma_2^1$  'translation'  $\tilde{\varphi}$  which by Shoenfield's argument has a tree representation. Since  $\tilde{\varphi} \in \Sigma_2^1$  let us suppose  $\tilde{\varphi}(x, y) \leftrightarrow \exists z P(x, \langle y, z \rangle)$  with  $P \in \Pi_1^1$ . There is a tree  $T$  on  $(\omega \times \omega^2) \times \omega$  so that  $\langle x, y \rangle \in A \leftrightarrow \exists z T_{x, (y, z)}$  is wellfounded. There is then the corresponding Shoenfield tree  $\hat{T}$  on  $\omega^3 \times \text{On}$  such that

$$(2) \langle x, y \rangle \in A \leftrightarrow \tilde{\varphi}(x, y) \leftrightarrow \exists z \exists g \in {}^\omega \text{On} (\langle x, y, z, g \rangle \in [\hat{T}]).$$

For  $T$  a tree on  $\mathcal{X} \times \text{On}$  we let  $T \upharpoonright \gamma$  be the restriction of the tree to ordinals in the final coordinate less than  $\gamma$ . The definition of the tree  $\hat{T}$  above can be effected in any transitive set model which is admissible, or the union of such: the definition of  $\hat{T}$  is uniformly  $\Delta_1^{\text{KP}}$  or  $\Delta_1^{\text{KPI}_0}$  (where  $\text{KPI}_0$  is the theory asserting that the universe is a limit of admissible sets). For  $\alpha < \alpha'$  we have that  $\hat{T} \upharpoonright \alpha$  is a subtree of  $\hat{T} \upharpoonright \alpha'$ .

The following are equivalent (and in fact are provably equivalent in  $\text{KPI}_0$ ):

- (3)  $\varphi(x, y)$
- (4)  $\tilde{\varphi}(x, y)$
- (5)  $\exists z \hat{T}_{x,y,z}$  is illfounded;
- (6)  $\exists z \hat{T}_{x,y,z} \upharpoonright \alpha$  is illfounded for some ordinal  $\alpha$ .

For  $x$  so that there is some  $y$  with  $\langle x, y \rangle \in A$  we wish to choose a canonical  $\langle y_x, z_x \rangle$  uniformising  $P$ , with  $\langle x, y_x \rangle \in A$ , and do this in a manner so that  $A^*(x, y) \equiv \langle \langle x, y \rangle \in A \wedge y = y_x \rangle$  is semidecidable. With  $y$  chosen so that  $\langle x, y \rangle \in A$ , then in  $M_{x,y}$ ,  $T_{x,y}$  is illfounded. Considering the projected tree  $\hat{T}_x$  we choose the least  $\alpha$  so that  $\hat{T}_x \upharpoonright \alpha$  is illfounded, and take as  $(\langle y_x, z_x \rangle, g_x)$  the leftmost branch through  $\hat{T}_x \upharpoonright \alpha$ . This branch is chosen to be such that if  $(\langle y, z \rangle, g)$  is any other branch, then for  $i$  least so that  $(\langle y(i), z(i) \rangle, g(i)) \neq (\langle y_x(i), z_x(i) \rangle, g_x(i))$ , then  $(\langle y_x(i), z_x(i) \rangle, g_x(i)) <^{\text{lex}} (\langle y(i), z(i) \rangle, g(i))$  with  $<^{\text{lex}}$  the lexicographic ordering. Moreover knowing that  $\hat{T}_x \upharpoonright \alpha$  is illfounded with  $\alpha < On \cap M_{x,y}$ , this leftmost branch can be found by a  $\Sigma_1$ -recursive search within the next admissible set over  $L_\alpha[x, y]$ . It is thus  $\Sigma_1$ -definable within  $M_{x,y}$ . Thus  $\langle x, \langle y_x, z_x \rangle \rangle \in P$ , i.e.  $\langle x, y_x \rangle \in A$  as required.

We leave the reader to establish that the Scale Property can also be shown by using this leftmost branch analysis of the Shoenfield tree for  $\Sigma_2^1$  sets, see for example [11] or [12]. QED

Remark: It can be shown, for example, that uniformisation fails for the pointclass of eventually semidecidable relations of reals defined on the original ITTM's (these are those relations  $A(x, y)$  that are defined by asking that for some machine on input  $(x, y)$  the machine eventually, that is from some point on, has a 1 on the output tape. Such a relation is defined by a  $\Sigma_2$  formula over  $L_{\Sigma(2, (x,y))}[x, y]$ . Indeed pointclasses defined over  $L_{\Sigma(2, (x,y))}[x, y]$  by  $\Pi_1$  formulae, cannot be uniformised by  $\Sigma_2$  defined functions.

This can be seen as follows: Let  $A$  be the class of  $x$  that are Cohen-generic over  $L_{\Sigma(2,x)}$ . (Generic in the sense that they meet all dense sets in the model.) Let  $\varphi(v_0)$  be a  $\Sigma_2$  formula such that  $M(x) = L_{\Sigma(2,x)}[x] \models \varphi[x]$ . Then also  $L_{\zeta(2,x)}[x] \models \varphi[x]$ , but  $x$  is Cohen-generic (for all definable dense sets) over  $L_{\zeta(2,x)}$ . Let  $p$  be a Cohen condition contained in  $x$  which forces that  $L_{\zeta(2,x)}[x] \models \varphi[x]$  and let  $x'$  be  $x$  with exactly one bit changed beyond the length of  $p$ . Then  $(\Sigma(2, x'), \zeta(2, x')) = (\Sigma(2, x), \zeta(2, x))$  so since  $L_{\zeta(2,x)}[x'] \models \varphi[x']$ , so does  $L_{\Sigma(2,x)}[x'] = M(x')$ . This proves that  $A$  contains no  $\Gamma$ -singleton, where  $\Gamma$  is the pointclass defined using  $\Sigma_2$  formulae over  $M(x)$ , and therefore uniformisation fails for  $\Gamma$ .

A similar argument shows that uniformisation also fails for the dual pointclass, defined using  $\Pi_2$  formulae. More generally, if we use  $\Sigma_n$  machines then the pointclass defined using  $\Pi_1$  formulae over  $L_{\Sigma(n,z)}[z]$  cannot be uniformised by the pointclasses using  $\Sigma_n$  or  $\Pi_n$  formulae. This leaves open whether this pointclass could be uniformised by  $\Sigma_{n+1}$  formulae over this structure. In short, we seem to have only reasonable uniformisation for the classes  $\Gamma_n$  defined above.

The above shows that the notion of ITTM machine can be generalised in a satisfactory way to allow for limit rules that correspond to  $\Sigma_n$  descriptions, and the resulting notions of *decidable*, *semidecidable*, *semirecursive* and so on correspond to the appropriate levels of the constructible hierarchy, namely the first  $\Sigma_n$ -extendible ordinals  $\zeta(n)$ . The contents of the output tape of the original ITTM running a standard program is an

example of what Burgess ([1]) would call an *arithmetical quasi-inductive* (AQI) set. (In fact the contents of an eventually stable output tape of a Hamkins-Kidder ITTM is an example of a *recursive quasi-inductive* set - and indeed any AQI process can be simulated on an ITTM, thus showing that AQI processes can all be reduced to recursive ones). A number of questions could now be formulated:

**Question 1** *Can there be a perspicuous or sensible notion of  $n$ -quasi-inductive (where  $n = 2$  corresponds to AQI)?*

In stating this we are of course conscious of the fact that the limit rules for hypermachines we have proposed are explicitly designed to tie up with the  $L$ -hierarchy. Whether there are pleasing notions at levels above  $n = 2$  remains to be seen.

**Question 2** *Are there sensible notions of “machine” that transcend those of this paper?*

In the above one could think very speculatively of machines that go beyond the first model of second-order number theory, or even have some form of in-built extra function allowing them to go outside of  $L$  into say  $K$  the core model?

More concretely, arguments from [17] can be used to show that eventually stable output tapes of ITTM's are all  $\mathcal{O}$ - $\Sigma_3^0$  (but not  $\mathcal{O}$ - $\Sigma_2^0$ ) definable. Recent work of Montalban and Shore implies that those of the  $\Sigma_{n+1}$ -machines are in  $\mathcal{O}$ - $(n\text{-}\Sigma_3^0)$  (where  $(n\text{-}\Sigma_3^0)$  represents the  $n$ -th level of the difference hierarchy on  $\Sigma_3^0$ ). Let  $\Gamma_n$  denote those classes of sets  $A \subseteq \mathbb{R}$  so that for some  $\Sigma_n$ -machine program  $P$ , some real parameter  $y$ , that  $x \in A$  iff  $P(\langle x, y \rangle)$  has a 1 on the first cell of its output tape at stage  $\zeta(n, (x, y))$ . (Thus  $\Gamma_2$  is a boldface pointclass corresponding to AQI sets of reals.) One can ask:

**Question 3** *How strong is  $\text{Det}(\Gamma_n)$ ?*

The second author has shown ([16]) that  $\text{Det}(\Gamma_2)$  implies the existence of inner models with a proper class of strong cardinals; and assuming  $\text{Det}(\Gamma_3)$  that there are, in the nomenclature of Feng and Jensen [3], *type-2* premice.

Lastly we mention a direct application of the limit rules here to a define a class of models generalising one constructed by H. Field in his attempt to define a theory of truth with a conditional operator. The latter's *G-model* in [4] is defined using an  $n = 2$  quasi-induction (with an operator that is  $\Pi_1^1$ -complete rather than arithmetic). By using limit rules at higher levels one can obtain all the desired effects of his *G*-solutions he requires, but with longer hierarchies of his 'determinateness operators'. This yields a family of *G*-models. We leave the interested reader to consult Field's book.

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