Side conditions, adding few reals, and trees

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The problem of building models of consequences, at the level of $H(\omega_2)$, of classical forcing axioms together with CH has a long history, starting with Jensen's landmark result that Suslin's Hypothesis is compatible with GCH.

Most of the work in the area done so far proceeds by showing that some suitable countable support iteration whose iterands are proper forcing notions not adding new reals fails to add new reals at limit stages.

There are (nontrivial) limitations to what can be achieved in this area. In fact, there cannot be any 'master' iteration lemma:

A.–Larson–Moore: Modulo a mild large cardinal assumption, there are two Π_2 statements over $H(\omega_2)$, each of which can be forced, using proper forcing, to hold together with CH, and whose conjunction implies $2^{\aleph_0} = 2^{\aleph_1}$.

Above result closely tied to the following concrete well–known obstacle to not adding reals: Given a ladder system $\vec{C} = (C_\delta \,:\, \delta \in Lim(\omega_1))$, let $\text{Unif}(\vec{C})$ denote the statement that for every colouring $F : \text{Lim}(\omega_1) \longrightarrow \{0, 1\}$ there is $G : \omega_1 \longrightarrow \{0, 1\}$ such that that for every $\delta \in \text{Lim}(\omega_1)$ there is some $\alpha < \delta$ such that $G(\xi) = F(\delta)$ for all $\xi \in C_\delta \setminus \alpha$. We say that G uniformizes F on $\vec{C}$.

Given $\vec{C}$ and F as above there is a natural forcing notion, $\mathcal{Q}_{\vec{C},F}$, for adding a uniformizing function for F on $\vec{C}$ by initial segments. Easy to see that $\mathcal{Q}_{\vec{C},F}$ is proper, adds the intended uniformizing function, and does not add reals. However, any long enough iteration of forcings of the form $\mathcal{Q}_{\vec{C},F}$, even with a fixed $\vec{C}$, will necessarily add new reals. In fact, the existence of a ladder system $\vec{C}$ for which $\text{Unif}(\vec{C})$ holds cannot be forced together with CH in any way whatsoever, as this statement actually implies $2^{\aleph_0} = 2^{\aleph_1}$.

Proof: Fix a bijection $h : \omega \longrightarrow \omega \times \omega$ such that $i \leq n$ if $h(n+1) = (i, j)$. For each $g : \omega_1 \longrightarrow 2$ construct $f_n : \omega_1 \longrightarrow 2$ $(n < \omega)$ such that

$$f_0 = g$$

and

$$f_{n+1} \restriction C_\delta =_{\text{fin}} f_i(\delta + j)$$

for every limit $\delta \neq 0$, where $h(n+1) = (i, j)$.
Given f_k $(k \leq n)$, f_{n+1} exists by applying $\text{Unif}(\vec{C})$ to the colouring

$$\delta \longrightarrow f_i(\delta + j)$$

But now, for each limit $\delta \neq 0$, $(f_n \restriction \delta \ : \ n < \omega)$ determines $(f_n \restriction \delta + \omega \ : \ n < \omega)$. Hence,

$$(f_n \restriction \omega \ : \ n < \omega)$$

determines

$$(f_n \ : \ n < \omega),$$

and in particular $f_0 = g$. Hence $2^{\aleph_0} = 2^{\aleph_1}$. $\square$

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Forcing with symmetric systems of models as side conditions

Finite–support forcing iterations involving symmetric systems of models as side conditions are naturally useful in situations in which, for example, we want to force

- consequences of classical forcing axioms at the level of $H(\omega_2)$, together with
- $2^{\aleph_0}$ large.

The pure side condition forcing

$$\mathcal{P}_0 = (\{\mathcal{N} : \mathcal{N} \text{ a } T\text{–symmetric system}\}, \supseteq)$$

(for any fixed $T \subseteq H(\kappa)$) preserves CH:

This exploits the fact that given N, $N' \in \mathcal{N}$, $\mathcal{N}$ a symmetric system, if $N \cap \omega_1 = N' \cap \omega_1$, then there is an isomorphism

$$\Psi : (N; \in, \mathcal{N} \cap N) \longrightarrow (N'; \in, \mathcal{N} \cap N')$$

(Of course there is at most one isomorphism between $(N; \in)$ and $(N'; \in)$.)

Proof sketch: Suppose $(\dot{r}_\xi)_{\xi<\omega_2}$ are names for subsets of ω and $\mathcal{N} \Vdash_{\mathcal{P}_0} \dot{r}_\xi \neq \dot{r}_{\xi'}$ for all $\xi \neq \xi'$. For each ξ, let N_ξ be a sufficiently correct model such that $\mathcal{N}$, $\dot{r}_\xi \in N_\xi$.

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By CH we may find $\xi \neq \xi'$ such that there is an isomorphism

$$\Psi : (N_\xi; \in, T^*, \mathcal{N}, \dot{r}_\xi) \longrightarrow (N_{\xi'}; \in, T^*, \mathcal{N}, \dot{r}_{\xi'})$$

(where T^* is the satisfaction predicate for $(H(\kappa); \in, T)$). Then $\mathcal{N}^* = \mathcal{N} \cup \{N_\xi, N_{\xi'}\} \in \mathcal{P}_0$. But $\mathcal{N}^*$ is $(N_\xi, \mathcal{P}_0)$–generic and $(N_{\xi'}, \mathcal{P}_0)$–generic.

Now, let $n < \omega$ and let $\mathcal{N}'$ be an extension of $\mathcal{N}^*$. Suppose $\mathcal{N}' \Vdash_{\mathcal{P}_0} n \in \dot{r}_\xi$. Then there is $\mathcal{N}'' \in \mathcal{P}_0$ extending both $\mathcal{N}'$ and some $\mathcal{M} \in N_\xi \cap \mathcal{P}_0$ such that $\mathcal{M} \Vdash_{\mathcal{P}_0} n \in \dot{r}_\xi$. By symmetry, $\mathcal{N}''$ extends also $\Psi(\mathcal{M})$. But $\Psi(\mathcal{M}) \Vdash_{\mathcal{P}_0} n \in \Psi(\dot{r}_\xi) = \dot{r}_{\xi'}$.

We have shown $\mathcal{N}^* \Vdash_{\mathcal{P}_0} \dot{r}_\xi \subseteq \dot{r}_{\xi'}$, and similarly we can show $\mathcal{N}^* \Vdash_{\mathcal{P}_0} \dot{r}_{\xi'} \subseteq \dot{r}_\xi$. Contradiction since $\mathcal{N}^*$ extends $\mathcal{N}$ and $\xi \neq \xi'$. $\square$

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In typical forcing iterations with symmetric systems as side conditions, $2^{\aleph_0}$ is large in the final extension. Even if $\mathcal{P}_0$ can be seen as the first stage of these iterations, the forcing is in fact designed to add reals at (all) subsequent successor stages.

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In this work we implemented this idea. Our test problem: Measuring

Definition

Measuring holds if and only if for every sequence $\vec{C} = (C_\delta : \delta \in \omega_1)$, if each C_δ is a closed subset of δ in the order topology, then there is a club $C \subseteq \omega_1$ such that for every $\delta \in C$ there is some $\alpha < \delta$ such that either

- $(C \cap \delta) \setminus \alpha \subseteq C_\delta$, or
- $(C \setminus \alpha) \cap C_\delta = \emptyset$.

We say that C *measures* $\vec{C}$.

Natural forcing for adding a club measuring a given $\vec{C}$ by initial segments is proper and adds no new reals. On the other hand it is not known if these forcings can (consistently) be iterated without adding new reals. Strongest failures of Club–Guessing known to be within reach of current techniques for iterating proper forcing without adding reals are in the region of $\neg$ WCG (Shelah, NNR revisited).

Measuring implies $\neg$ WCG: Suppose $\vec{C} = (C_\delta : \delta \in \mathrm{Lim}(\omega_1))$ ladder system and $C \subseteq \omega_1$ is a club measuring $\vec{C}$. Then, for every $\delta \in C$, if δ is a limit point of limit points of C, then a tail of $C \cap \delta$ is disjoint from C_δ since $\mathrm{ot}(C_\delta) = \omega$.

Question

(Moore) Is Measuring consistent with CH?

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Our main theorem:

Theorem

(A.–Mota) (CH) Let $\kappa > \omega_2$ be a regular cardinal such that $2^{<\kappa} = \kappa$. There is then a partial order $\mathcal{P}$ with the following properties.

(1) *$\mathcal{P}$ is proper and $\aleph_2$–Knaster.*
(2) *$\mathcal{P}$ forces the following statements.*
    (a) *Measuring*
    (b) *CH*
    (c) *$2^\mu = \kappa$ for every uncountable cardinal $\mu < \kappa$.*

Question

(Moore) Does Measuring imply that there are non-constructible reals?

Trees on $\aleph_2$ and GCH

This is joint work with Mohammad Golshani, completed about a month ago.

Let κ be a regular uncountable cardinal.

- A κ–tree is a tree T of height κ all of whose levels are smaller than κ. A κ–*Aronszajn tree* is a κ–tree which has no κ–branches.
- A κ–*Souslin tree* is a κ–tree which has no κ–branches and no antichains of size κ.
- If $\kappa = \lambda^+$, a κ–Aronszajn tree T is said to be *special* if there exists a function $f : T \to \lambda$ such that $f(x) \neq f(y)$ whenever $x, y \in T$ are such that $x <_T y$. We say that f *specializes* T.
- The special Aronszajn tree property at $\kappa = \lambda^+$, $\mathsf{SATP}(\kappa)$, is the statement "there exist κ–Aronszajn trees and all such trees are special".

Aronszajn trees were introduced by Aronszajn (1930's?), who proved the existence, in ZFC, of a special $\aleph_1$–Aronszajn tree. Later, Specker (1949) showed that $2^{<\lambda} = \lambda$ implies the existence of special λ^+–Aronszajn trees for λ regular, and Jensen (1972) produced special λ^+–Aronszajn trees for singular λ in L.

Baumgartner, Malitz and Reinhardt (1970) showed that Martin's Axiom + $2^{\aleph_0} > \aleph_1$ implies SATP($\aleph_1$), and hence Souslin's Hypothesis at $\aleph_1$ as well. Later, and as already mentioned, Jensen (1974) produced a model of GCH in which SATP($\aleph_1$) holds.

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The situation at $\aleph_2$ turned out to be more complicated. Jensen (1972) proved that the existence of an $\aleph_2$–Souslin follows from each of the hypotheses $\mathrm{CH} + \diamondsuit(\{\alpha < \omega_2 \mid \mathrm{cf}(\alpha) = \omega_1\})$ and $\square_{\omega_1} + \diamondsuit(\{\alpha < \omega_2 \mid \mathrm{cf}(\alpha) = \omega\})$. The second result was improved by Gregory (1976); he proved that GCH together the existence of a non–reflecting stationary subset of $\{\alpha < \omega_2 \mid \mathrm{cf}(\alpha) = \omega\}$ yields the existence of an $\aleph_2$–Souslin tree.

Laver and Shelah (1981) produced, relative to the existence of a weakly compact cardinal, a model of $\mathrm{ZFC} + \mathrm{CH}$ in which $\mathrm{SATP}(\aleph_2)$ holds. But in their model $2^{\aleph_1} > \aleph_2$, and the following remained a major open problem (s. e.g. Kanamori–Magidor 1977):

Question

Is ZFC+GCH consistent with the non–existence of $\aleph_2$–Souslin trees?

In December 2017, while visiting Golshani in Tehran, we started thinking about combining the ideas from Measuring + CH with the Laver–Shelah construction for SATP($\aleph_2$). We eventually succeeded:

Theorem** (A.–Golshani) Suppose GCH holds and κ is a weakly compact cardinal. Then there exists a set–generic extension of the universe in which

1. GCH holds,
2. $\kappa = \aleph_2$, and
3. SATP($\aleph_2$) holds (and hence there are no $\aleph_2$–Souslin trees).

Our argument can be easily extended to the successor of any regular cardinal.

Our large cardinal assumption is optimal:

- ⋆ Rinot (2017) proved that GCH+ Souslin's Hypothesis at $\aleph_2$ implies $\neg\square(\omega_2)$; on the other hand, Todorčević (1987) proved that $\neg\square(\omega_2)$ implies that ω_2 is weakly compact in L.

Sketch of definition of forcing

Let κ be weakly compact and assume GCH. Let $\mathbb{P} = \mathrm{Col}(\aleph_1, <\kappa)$ be the Lévy collapse turning κ into $\aleph_2$. Let $G_{\mathbb{P}}$ be $\mathbb{P}$–generic over V.

We define in $V[G_{\mathbb{P}}]$,
by induction on β, a sequence $\langle \mathbb{Q}_\beta \mid \beta \leq \kappa^{++} \rangle$ of forcing notions.

Given $\beta \leq \kappa$, a condition in $\mathbb{Q}_\beta$ is a $\mathbb{P}$–name $\dot{q}$ such that $\mathbb{P}$ forces that $\dot{q}$ is a pair $q = (f_q, \tau_q)$ such that:

(1) f_q is a countable function such that $\mathrm{dom}(f_q) \subseteq \beta$ and such that for all $\alpha \in \mathrm{dom}(f_q)$, $f_q(\alpha)$ is a countable partial function $f_q : \kappa \times \omega_1 \longrightarrow \omega_1$.

(2) τ_q is a countable collection of pairs (N, γ) such that
- N is an elementary submodel of $H(\kappa^{++})$ such that $|N| = \aleph_1$ and N is closed under ω–sequences, and
- $\gamma \leq \beta$ and γ is in the closure of $N \cap \beta$.

(3) For all $\alpha < \beta$, $q \restriction \alpha \in \mathbb{Q}_\alpha$.

(4) For all $\alpha \in \mathrm{dom}(f_q)$,

   (a) $\mathrm{cf}(\alpha) = \kappa$,
   (b) $\mathbb{Q}_\alpha \lessdot \mathbb{Q}_{\beta'}$ for all $\beta' \in [\alpha, \beta)$, and
   (c) for all x, $y \in \mathrm{dom}(f_q(\alpha))$, if $(f_q(\alpha))(x) = (f_q(\alpha))(y)$, then $q \restriction \alpha$ does not force that x and y are comparable in $\dot{T}_\alpha$ (where $\dot{T}_\alpha$ is, in $V^{\mathbb{Q}_\alpha}$, a κ–Aronszajn tree given by a suitable book-keeping; we assume all trees are on $\kappa \times \omega_1$ with ρ-th level $\{\rho\} \times \omega_1$).

(5) Suppose (N_0, γ_0), $(N_1, \gamma_1) \in \tau_q$, $\alpha \in N_0 \cap \min\{\gamma_0, \gamma_1\}$, $\alpha' \in N_1 \cap \min\{\gamma_0, \gamma_1\}$, and there is an isomorphism $\Psi_{N_0,N_1} : (N_0, \in) \longrightarrow (N_1, \in)$ which is *sufficiently* correct and such that $\Psi_{N_0.N_1}(\alpha) = \alpha'$. Then the natural restriction of $q \restriction \alpha$ is isomorphic to the natural restriction of $q \restriction \alpha'$ to N_1.

The extension relation:

Given $q_1, q_0 \in \mathbb{Q}_\beta$, $q_1 \leq_\beta q_0$ if and only if:

(A) $\mathrm{dom}(f_{(\dot{q}_0)_{G_\mathbb{P}}}) \subseteq \mathrm{dom}(f_{(\dot{q}_1)_{G_\mathbb{P}}})$

(B) $f_{(\dot{q}_0)_{G_\mathbb{P}}}(\alpha) \subseteq f_{(\dot{q}_1)_{G_\mathbb{P}}}(\alpha)$ for all $\alpha \in \mathrm{dom}(f_{(\dot{q}_0)_{G_\mathbb{P}}})$.

(C) $\tau_{(\dot{q}_0)_{G_\mathbb{P}}} \subseteq \tau_{(\dot{q}_1)_{G_\mathbb{P}}}$

Main facts

(1) Due to the strong symmetry in clause (5) of the definition, it is probably not the case that $\mathbb{Q}_\beta \lessdot \mathbb{Q}_{\beta'}$ for all $\beta < \beta'$. On the other hand:

- $\mathbb{Q}_\beta \lessdot \mathbb{Q}_{\beta+1}$ for all $\beta < \kappa^{++}$.
- There is a κ–club C such that $\mathbb{Q}_\beta \lessdot \mathbb{Q}_{\beta'}$ for all $\beta < \beta'$ in $C \cup \{\kappa^{++}\}$.

(2) For all $\beta \leq \kappa^{++}$ such that $\mathrm{cf}(\beta) \geq \kappa$, $\mathbb{Q}_\beta$ is ω_1–strategically closed; in particular, $\mathbb{Q}_\beta$ does not add reals and hence preserves CH.

(3) $\mathbb{Q}_{\kappa^{++}}$ adds $\aleph_2$–many new subsets of ω_1, but not more than that; in particular, $\mathbb{Q}_{\kappa^{++}}$ preserves $2^{\aleph_1} = \aleph_2$ [essentially the same argument we saw on slide 7].

(4) If $\mathbb{Q}_{\kappa^{++}}$ has the κ–c.c. then it forces $\mathrm{SATP}(\aleph_2)$.

The κ–chain condition

Lemma

For every $\beta \leq \kappa^{++}$, $\mathbb{P} * \mathbb{Q}_\beta$ has the κ–c.c (equivalently, it is κ–Knaster (since $\kappa \longrightarrow (\kappa)^2_2$)).

This is the most involved part of the proof, and the only place where we use the weak compactness of κ. The proof is by induction on β.

The case when β is a limit ordinal with $\mathrm{cf}(\beta) < \kappa$ uses

- the fact that if two conditions q and q' are compatible in $\mathbb{Q}_\alpha$, then they have a greatest lower bound $q \oplus_\alpha q'$ (obtained essentially from closing under relevant isomorphisms Ψ_{N_0,N_1}) together with
- $\kappa \longrightarrow (\kappa)^2_{\mathrm{cf}(\beta)}$.

The hardest case is the case $\mathrm{cf}(\beta) = \kappa$. For this case we use an adaptation of the following key separation argument from Laver–Shelah.

Lemma

(Laver–Shelah) Suppose κ *is weakly compact,* $\mathbb{P} = \mathrm{Coll}(\omega_1, <\kappa)$*, and in* $V^{\mathbb{P}}$*,* $(\mathcal{Q}_\beta)_{\beta \leq \tau}$ *is a countable support iteration such that for all* $\beta < \tau$*,* $\mathcal{Q}_{\beta+1} = \mathcal{Q}_\beta * \dot{\mathcal{R}}_\beta$*, where* $\dot{\mathcal{R}}_\beta$ *is the natural forcing for specializing some given* κ*–Aronszajn tree* $\dot{T}_\beta$*. Then* $\mathbb{P} * \mathcal{Q}_\beta$ *is* κ*–c.c. for all* $\beta \leq \tau$*.*

Proof sketch: Given conditions $\langle p^L, q^L \rangle$, $\langle p^R, q^R \rangle$, $\alpha \in \mathrm{dom}(f_{q^L}) \cap \mathrm{dom}(f_{q^R})$, $x \in \mathrm{dom}(f_{q^L}(\alpha))$ and $y \in \mathrm{dom}(f_{q^R}(\alpha))$ (x and y may or may not be equal), we say that

$\langle p^L, q^L \restriction \alpha \rangle$, $\langle p^R, q^R \restriction \alpha \rangle$ *separate* x *and* y *by means of* $\bar{x}$, $\bar{y}$

if there is $\bar{\rho} < \lambda$, together with $\zeta \neq \zeta'$ in ω_1, such that letting $\bar{x} = (\bar{\rho}, \zeta)$ and $\bar{y} = (\bar{\rho}, \zeta')$, $q^L_\lambda \restriction \alpha \Vdash_\alpha \bar{x} <_{\dot{T}_\alpha} x$ and $q^R_\lambda \restriction \alpha \Vdash_\alpha \bar{y} <_{\dot{T}_\alpha} y$.

Let $\sigma = (\langle p^*_\lambda, q^*_\lambda \rangle \mid \lambda < \kappa)$ be a sequence of conditions. Let $\mathcal{F}$ be the weak compactness filter on κ (i.e., $\mathcal{F}$ is the filter generated by the sets $\{\alpha < \kappa \mid (V_\alpha, \in, A \cap V_\alpha) \models \phi\}$, for $A \subseteq V_\kappa$ and for a Π^1_1 sentence ϕ over $(V_\kappa, \in, A)$). $\mathcal{F}$ is a proper normal filter on κ.

Given $X \in \mathcal{F}^+$, say that $(\langle p^L_\lambda, q^L_\lambda \rangle \mid \lambda < \kappa)$, $(\langle p^R_\lambda, q^R_\lambda \rangle \mid \lambda < \kappa)$ *is a separating pair for* $(\langle p^*_\lambda, q^*_\lambda \rangle \mid \lambda < \kappa)$ if for all $\lambda \in X$:

Let $\sigma = (\langle p^*_\lambda, q^*_\lambda \rangle \mid \lambda < \kappa)$ be a sequence of conditions. Let $\mathcal{F}$ be the weak compactness filter on κ (i.e., $\mathcal{F}$ is the filter generated by the sets $\{\alpha < \kappa \mid (V_\alpha, \in, A \cap V_\alpha) \models \phi\}$, for $A \subseteq V_\kappa$ and for a Π^1_1 sentence ϕ over $(V_\kappa, \in, A)$). $\mathcal{F}$ is a proper normal filter on κ.

Given $X \in \mathcal{F}^+$, say that $(\langle p^L_\lambda, q^L_\lambda \rangle \mid \lambda < \kappa)$, $(\langle p^R_\lambda, q^R_\lambda \rangle \mid \lambda < \kappa)$ *is a separating pair for* $(\langle p^*_\lambda, q^*_\lambda \rangle \mid \lambda < \kappa)$ if for all $\lambda \in X$:

(1) $\mathrm{dom}(p^L_\lambda) = \mathrm{dom}(p^R_\lambda)$ and $\mathrm{dom}(f_{q^L_\lambda}) = \mathrm{dom}(f_{q^R_\lambda})$

(2) Both of $\langle p^L_\lambda, q^L_\lambda \rangle$ and $\langle p^R_\lambda, q^R_\lambda \rangle$ extend $\langle p^*_\lambda, q^*_\lambda \rangle$.

(3) The following holds for all $\lambda < \lambda'$ in X and all $\alpha \in \mathrm{dom}(f_{q^L_\lambda}) \cap \mathrm{dom}(f_{q^L_{\lambda'}})$.

  (a) $p^L_\lambda \restriction \lambda \times \omega_1 = p^R_\lambda \restriction \lambda \times \omega_1 = p^L_{\lambda'} \restriction \lambda' \times \omega_1 = p^R_{\lambda'} \restriction \lambda' \times \omega_1$ and $f_{q^L_\lambda}(\alpha) \restriction \lambda \times \omega_1 = f_{q^R_\lambda}(\alpha) \restriction \lambda \times \omega_1 = f_{q^L_{\lambda'}}(\alpha) \restriction \lambda' \times \omega_1 = f_{q^R_{\lambda'}}(\alpha) \restriction \lambda' \times \omega_1$.

  (b) $\rho < \lambda'$ for every $(\rho, \zeta) \in (\mathrm{dom}(p^L_\lambda) \cup \mathrm{dom}(p^R_\lambda)) \setminus \lambda$ and $\rho < \lambda'$ for every $(\rho, \zeta) \in (\mathrm{dom}(f_{q^L_\lambda}(\alpha)) \cup \mathrm{dom}(f_{q^R_\lambda}(\alpha))) \setminus \lambda$.

(4) For all $\alpha \in \mathrm{dom}(f_{q^L_\lambda})$, $x \in \mathrm{dom}(f_{q^L_\lambda}(\alpha)) \setminus (\lambda \times \omega_1)$ and $y \in \mathrm{dom}(f_{q^R_\lambda}(\alpha')) \setminus (\lambda \times \omega_1)$, $\langle p^L_\lambda, q^L_\lambda \restriction \alpha \rangle$ and $\langle p^R_\lambda, q^R_\lambda \restriction \alpha \rangle$ separate x and y by means of some pair $\bar{x}$, $\bar{y}$.

(5) For all $\lambda < \lambda'$ in X, $\alpha \in \mathrm{dom}(f_{q^L_\lambda}) \cap \mathrm{dom}(f_{q^L_{\lambda'}})$, $x \in \mathrm{dom}(f_{q^L_\lambda}(\alpha)) \setminus (\lambda \times \omega_1)$ and $y \in \mathrm{dom}(f_{q^R_\lambda}(\alpha)) \setminus (\lambda \times \omega_1)$ there are $x' \in \mathrm{dom}(f_{q^L_{\lambda'}}(\alpha)) \setminus (\lambda' \times \omega_1)$ and $y' \in \mathrm{dom}(f_{q^R_{\lambda'}}(\alpha)) \setminus (\lambda' \times \omega_1)$ such that $\bar{x}' = \bar{x}$ and $\bar{y}' = \bar{y}$.

The following claim is immediate.

Claim

Let $X \in \mathcal{S}$ and suppose $\sigma^L = (\langle p^L_\lambda, q^L_\lambda\rangle \mid \lambda \in X)$, $\sigma^R = (\langle p^R_\lambda, q^R_\lambda\rangle \mid \lambda \in X)$ is a separating pair for σ. Then for all $\lambda < \lambda'$ in X, $\langle p^L_\lambda, q^L_\lambda\rangle$ and $\langle p^R_{\lambda'}, q^R_{\lambda'}\rangle$ are compatible conditions.

Hence, it suffices to prove that there is $\sigma^L = (\langle p^L_\lambda, q^L_\lambda\rangle \mid \lambda \in X)$, $\sigma^R = (\langle p^R_\lambda, q^R_\lambda\rangle \mid \lambda \in X)$, a separating pair for σ. But this follows essentially from applying the following claim countably many times, using the normality of $\mathcal{F}$.

Claim

Suppose $X \in \mathcal{F}^+$, $(\langle p^L_\lambda, q^L_\lambda\rangle \mid \lambda \in X)$ *and* $(\langle p^R_\lambda, q^R_\lambda\rangle \mid \lambda \in X)$ *are sequences of conditions satisfying (1)–(3),* $\alpha \in \mathrm{dom}(f_{q_\lambda})$ *for all* λ*, and* $(x_\lambda)_{\lambda \in X}$ *and* $(y_\lambda)_{\lambda \in X}$ *are such that* $x_\lambda \in \mathrm{dom}(f_{q^L_\lambda}(\alpha)) \setminus (\lambda \times \omega_1)$ *and* $y_\lambda \in \mathrm{dom}(f_{q^R_\lambda}(\alpha)) \setminus (\lambda \times \omega_1)$ *for all* λ*. Then there are* $X' \subseteq X$, $X' \in \mathcal{F}^+$*, and* $(\langle p^{LL}_\lambda, q^{LL}_\lambda\rangle \mid \lambda \in X')$, $(\langle p^{RR}_\lambda, q^{RR}_\lambda\rangle \mid \lambda \in X')$ *such that*

- *for each* $\lambda \in X'$, $\langle p^{LL}_\lambda, q^{LL}_\lambda\rangle \leq \langle p^L_\lambda, q^L_\lambda\rangle$ *and* $\langle p^{RR}_\lambda, q^{RR}_\lambda\rangle \leq \langle p^R_\lambda, q^R_\lambda\rangle$,
- $(\langle p^{LL}_\lambda, q^{LL}_\lambda\rangle \mid \lambda \in X')$, $(\langle p^{RR}_\lambda, q^{RR}_\lambda\rangle \mid \lambda \in X')$ *satisfies (1)–(3), and*
- *for each* $\lambda \in X'$, $\langle p^{LL}_\lambda, q^{LL}_\lambda \restriction \alpha\rangle$ *and* $\langle p^{RR}_\lambda, q^{RR}_\lambda \restriction \alpha\rangle$ *separate* x_λ *and* y_λ.

The claim is proved via a Π^1_1 reflection argument: There is a measure 1 set C in $\mathcal{F}$ of $\lambda < \kappa$ for which there is a model M such that $M \cap \kappa = \lambda$ and such that, for relevant α,

- $M \cap (\mathbb{P} * \mathbb{Q}_\alpha) \lessdot \mathbb{P} * \mathbb{Q}_\alpha$ and
- $M \cap (\mathbb{P} * \mathbb{Q}_\alpha)$ forces, over V, that $\dot{T}_\alpha \cap M$ has no λ–branches.

Using this idea one can find suitable conditions

$$\langle p^{LL}_\lambda, q^{LL}_\lambda \rangle \leq \langle p^{L}_\lambda, q^{L}_\lambda \rangle$$

and

$$\langle p^{RR}_\lambda, q^{RR}_\lambda \rangle \leq \langle p^{R}_\lambda, q^{R}_\lambda \rangle$$

forcing ‘conflicting’ information regarding the projections of x and y to some level below λ (if this were not possible, we would be able to find λ–branches through $\dot{T}_\alpha \cap M$ in the $M \cap (\mathbb{P} * \mathbb{Q}_\alpha)$–extension). $\square$