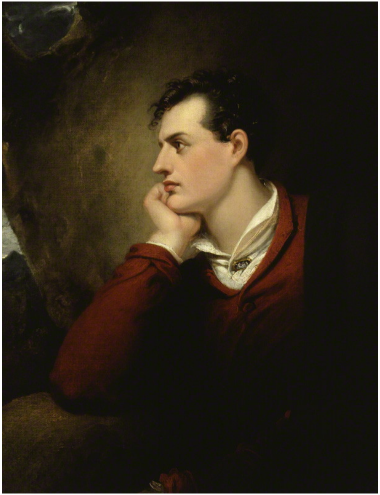
Ada Lovelace and the Logic of the Analytical Engine

Philip Welch, School of Mathematics.

University of BRISTOL

Annabella Milbanke 1792-1860



George Byron 1788-1824

Byron leaving England

Mrs Mardyn, Ld Byron

G Cruikshank fecit

April. 1816

FARE THEE WELL.

FARE thee well! and if for ever—
  Still for ever, fare *thee well*—
Even though unforgiving, never
  'Gainst thee shall my heart rebel.
Would that breast were bared before thee
  Where thy head so oft hath lain,
While that placid sleep came o'er thee
  Which thou ne'er canst know again:
Would that breast by thee glanc'd over,
  Every inmost thought could show!
Then, thou would'st at last discover
  'Twas not well to spurn it so—
Though the world for this commend thee—
  Though it smile upon the blow,
Even its praises must offend thee,
  Founded on another's woe—
Though my many faults defac'd me;
  Could no other arm be found
Then the one which once embraced me
  To inflict a cureless wound?
Yet—oh, yet—thyself deceive not—
  Love may sink by low decay,
But by sudden wrench, believe not,
  Hearts can thus be torn away;
Still thine own its life retaineth—
  Still must mine—though bleeding—beat,
And the undying thought which paineth
  Is—that we no more may meet.—
These are words of deeper sorrow
  Than the wail above the dead,
Both shall live—but every morrow
  Wake us from a widowed bed.—
And when thou would'st solace gather—
  When our child's first accents flow—
Wilt thou teach her to say "Father!"
  Though his care she must forego?
When her little hands shall press thee—
  When her lip to thine is press'd—
Think of him whose prayer shall bless thee—
  Think of him thy love had bless'd,
Should her lineaments resemble
  Those thou never more may'st see—
Then thy heart will softly tremble
  With a pulse yet true to me.
All my faults—perchance thou knowest—
  All my madness—none can know;
All my hopes—where'er thou goest—
  Wither—yet with *thee* they go—
Every feeling hath been shaken,
  Pride—which not a world could bow—
Bows to thee—by thee forsaken
  Ev'n my soul forsakes me now.
But 'tis done—all words are idle—
  Words from me are vainer still;
But the thoughts we cannot bridle
  Force their way without the will.
Fare thee well!—thus disunited—
  Torn from every nearer tie—
Seared in heart—and lone—and blighted—
  More than this I scarce can die.

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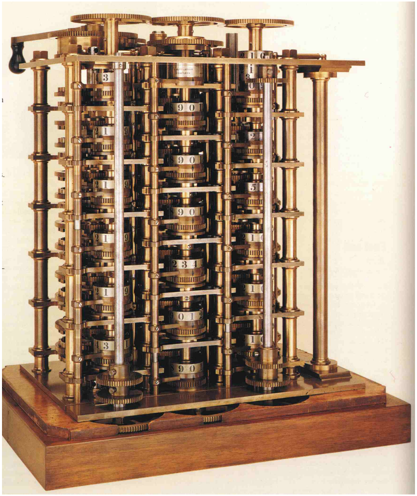
George Cruickshank 1816

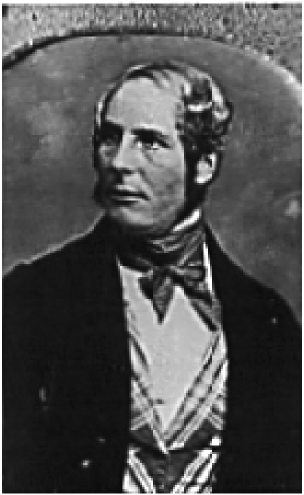
Augusta Ada Byron 1815-1852

Ada Byron, Age 17

Charles Babbage 1791-1871



Difference Engine No. 1 (portion) 1832



William King, Earl of Ockham,
later Lord Lovelace 1805-1893


Byron in Abyssinian dress

Ashley Combe, Somerset

“ ‘Mathematics’ should be written on my jaw ”

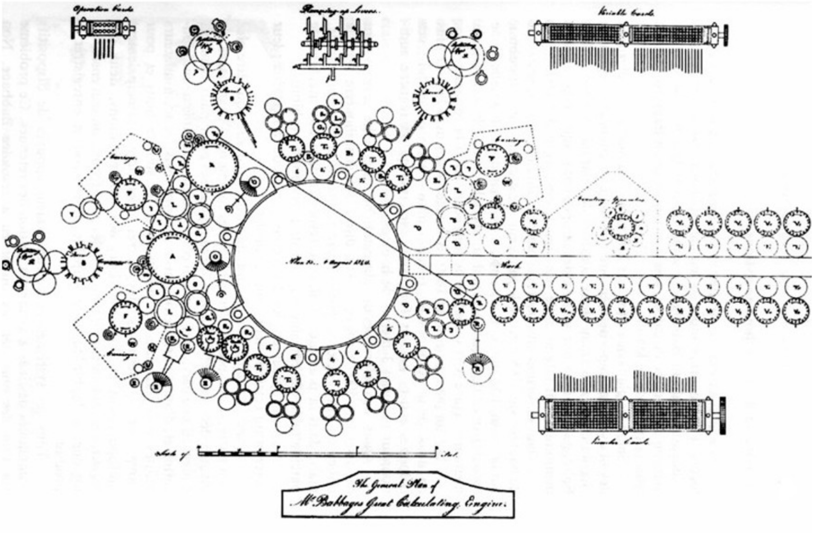
as Lady King, (Age 20)
by Margaret Carpenter, 1835

after a sketch by A.E. Chalon 1838. (Age 23)

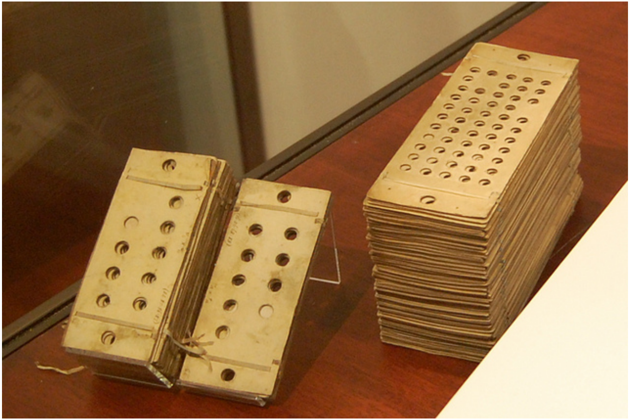
J.R. Jacquard 1752-1834

A Jacquard Loom

The Analytical Engine

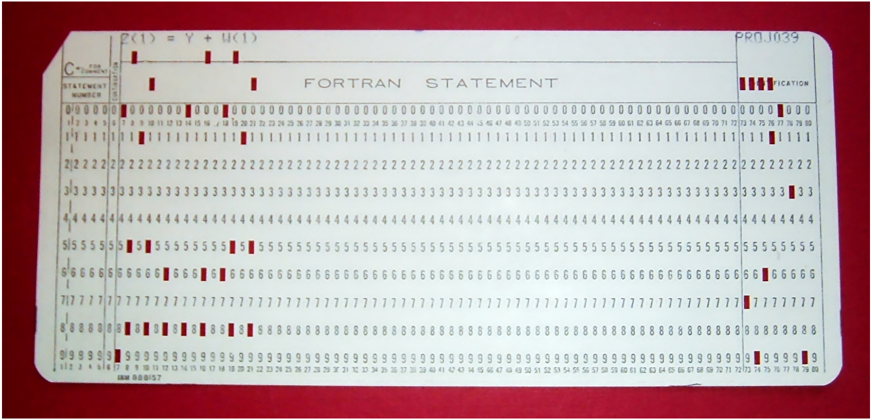


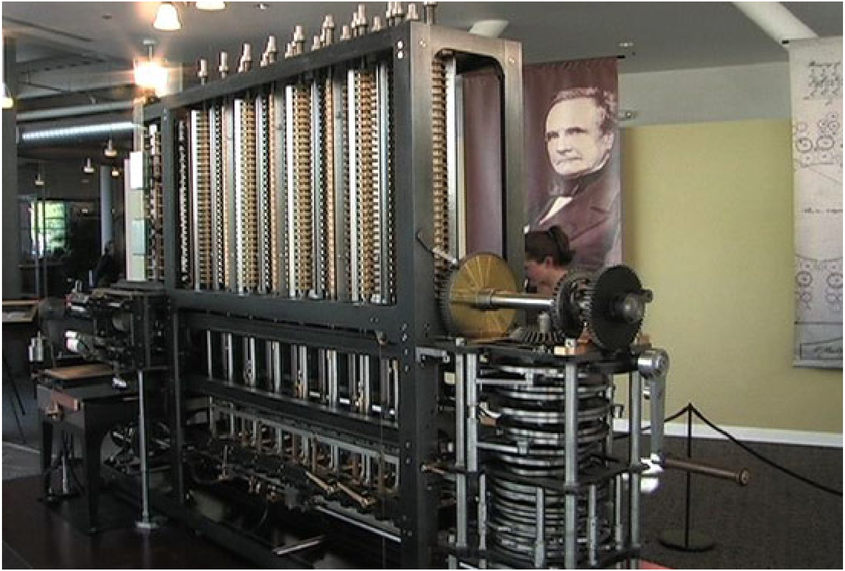
Punched Cards: Operator & Variable



Lovelace: *"The bounds of* arithmetic *were outstepped the moment of applying the cards had occured"*

And 150 years later …





Luigi Federico Menabrea, 1805-1893

& later Prime Minister of Italy

Menabrea's article - 1842

- He describes the Analytical Engine in great detail - not the would be physical construction, but the methodology of the computations.

- He gives three examples of calculations. We'll look at that for a simple (2×2) simultaneous equation with unknowns x and y:

$$\begin{aligned} mx + ny &= d \\ m'x + n'y &= d' \end{aligned}$$

This will have solution:

$$x = \frac{dn' - d'n}{n'm - nm'} \quad ; \quad y = \frac{d'm - dm'}{mn' - m'n}$$

Columns on which are inscribed the primitive data	Number of the operations	Cards of the operations		Variable cards			Statement of results
		No. of the Operation-cards	Nature of each operation	Columns acted on by each operation	Columns that receive the result of each operation	Indication of change of value on any column	
${}^1V_0 = m$	1	1	$\times$	${}^1V_0 \times {}^1V_4 =$	${}^1V_6$	$\left\{ {}^1V_0 = {}^1V_0;\ {}^1V_4 = {}^1V_4 \right\}$	${}^1V_6 = mn'$
${}^1V_1 = n$	2	”	$\times$	${}^1V_3 \times {}^1V_1 =$	${}^1V_7$	$\left\{ {}^1V_3 = {}^1V_3;\ {}^1V_1 = {}^1V_1 \right\}$	${}^1V_7 = m'n$
${}^1V_2 = d$	3	”	$\times$	${}^1V_2 \times {}^1V_4 =$	${}^1V_8$	$\left\{ {}^1V_2 = {}^1V_2;\ {}^1V_4 = {}^0V_4 \right\}$	${}^1V_8 = dn'$
${}^1V_3 = m'$	4	”	$\times$	${}^1V_5 \times {}^1V_1 =$	${}^1V_9$	$\left\{ {}^1V_5 = {}^1V_5;\ {}^1V_1 = {}^0V_1 \right\}$	${}^1V_9 = d'n$
${}^1V_4 = n'$	5	”	$\times$	${}^1V_0 \times {}^1V_5 =$	${}^1V_{10}$	$\left\{ {}^1V_0 = {}^0V_0;\ {}^1V_5 = {}^0V_5 \right\}$	${}^1V_{10} = d'm$
${}^1V_5 = d'$	6	”	$\times$	${}^1V_2 \times {}^1V_3 =$	${}^1V_{11}$	$\left\{ {}^1V_2 = {}^0V_2;\ {}^1V_3 = {}^0V_3 \right\}$	${}^1V_{11} = dm'$
	7	2	$-$	${}^1V_6 - {}^1V_7 =$	${}^1V_{12}$	$\left\{ {}^1V_6 = {}^0V_6;\ {}^1V_7 = {}^0V_7 \right\}$	${}^1V_{12} = mn' - m'n$
	8	”	$-$	${}^1V_8 - {}^1V_9 =$	${}^1V_{13}$	$\left\{ {}^1V_8 = {}^0V_8;\ {}^1V_9 = {}^0V_9 \right\}$	${}^1V_{13} = dn' - d'n$
	9	”	$-$	${}^1V_{10} - {}^1V_{11} =$	${}^1V_{14}$	$\left\{ {}^1V_{10} = {}^0V_{10};\ {}^1V_{11} = {}^0V_{11} \right\}$	${}^1V_{14} = d'm - dm'$
	10	3	$\div$	${}^1V_{13} \div {}^1V_{12} =$	${}^1V_{15}$	$\left\{ {}^1V_{13} = {}^0V_{13};\ {}^1V_{12} = {}^1V_{12} \right\}$	${}^1V_{15} = \frac{dn' - d'n}{mn' - m'n} = x$
	11	”	$\div$	${}^1V_{14} \div {}^1V_{12} =$	${}^1V_{16}$	$\left\{ {}^1V_{14} = {}^0V_{14};\ {}^1V_{12} = {}^0V_{12} \right\}$	${}^1V_{16} = \frac{d'm - dm'}{mn' - m'n} = y$
1	2	3	4	5	6	7	8

Bernoulli Numbers

- These are a sequence of fractions $B_1, B_3, B_5, \ldots B_{2n+1}, \ldots$ They crop up again and again in analysis and number theory. There are many ways to define them.

- We can define them through a *power series*

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + B_1 \frac{x^2}{2!} + B_3 \frac{x^4}{4!} + B_5 \frac{x^6}{6!} + \cdots$$

Or *explicitly*:

$$B_{2m+1} = \sum_{k=0}^{m} \sum_{v=0}^{k} (-1)^v \binom{k}{v} \frac{v^m}{k+1}.$$

Lovelace chose:

$$B_{2m+1} = -\sum_{k=0}^{m-1} \binom{2m+1}{2k+1} \frac{B_{2k+1}}{2m-2k+1}.$$

Her choice looks like:

$$B_{2m+1} = a_1 B_1 + a_3 B_3 + \cdots + a_{2m-1} B_{2m-1}$$

Rearranging:

Rearranging:

$$\begin{aligned} B_3 &= a_1 B_1 \\ B_5 &= d_1 B_1 + d_3 B_3 \\ B_7 &= e_1 B_1 + e_3 B_3 + e_5 B_5 \end{aligned}$$

- So her programme to compute B_7 proceeds by computing first B_1, then B_3 from B_1; then B_5 from B_3 *and* B_1 *etc.*

- This is known as a *course-of-values recursion* because it requires the complete course of *all* the previous values to get the next one.

- An *(ordinary) recursion* would just have defined B_3 in terms of B_1, and B_5 in terms of B_3, B_7 in terms of B_5, that is using *only* the most immediate previous value *etc.*

$$\begin{aligned} B_3 &= a_1 B_1 \\ B_5 &= d_3' B_3 \\ B_7 &= e_5' B_5 \end{aligned}$$

Lovelace's programme to compute Bernoulli Numbers

Diagram for the computation by the Engine of the Numbers of Bernoulli. See Note G. (page 722 *et seq.*)

Number of Operation	Nature of Operation	Variables acted upon	Variables receiving results	Indication of change in the value on any Variable	Statement of Results	Data. $^1V_1$	Data. $^1V_2$	Data. $^1V_3$	Working Variables. $^0V_4$	$^0V_5$	$^0V_6$	$^0V_7$	$^0V_8$	$^0V_9$	$^0V_{10}$	$^0V_{11}$	$^0V_{12}$	$^0V_{13}$	Result Variables. $^1V_{21}$	$^1V_{22}$	$^1V_{23}$	$^1V_{24}$
						○ 0 0 1	○ 0 0 2	○ 0 0 4	○ 0 0 0	○ 0 0 0	○ 0 0 0	○ 0 0 0	○ 0 0 0	○ 0 0 0	○ 0 0 0	○ 0 0 0	○ 0 0 0	○ 0 0 0	○ B_1 in a decimal fraction.	○ B_3 in a decimal fraction.	○ B_5 in a decimal fraction.	○ 0 0 0
						1	2	n											B_1	B_3	B_5	B_7
1	×	$^1V_2 \times {}^1V_3$	$^1V_4, {}^1V_5, {}^1V_6$	$\left\{\begin{matrix} ^1V_2 = {}^1V_2 \\ ^1V_3 = {}^1V_3 \end{matrix}\right\}$	$=2n$	...	2	n	$2n$	$2n$	$2n$											
2	−	$^1V_4 - {}^1V_1$	$^1V_4$	$\left\{\begin{matrix} ^1V_4 = {}^2V_4 \\ ^1V_1 = {}^1V_1 \end{matrix}\right\}$	$=2n-1$	1	...	...	$2n-1$													
3	+	$^1V_5 + {}^1V_1$	$^2V_5$	$\left\{\begin{matrix} ^1V_5 = {}^2V_5 \\ ^1V_1 = {}^1V_1 \end{matrix}\right\}$	$=2n+1$	1	...	...	...	$2n+1$												
4	÷	$^2V_4 \div {}^2V_5$	$^1V_{11}$	$\left\{\begin{matrix} ^3V_5 = {}^0V_5 \\ ^3V_4 = {}^0V_4 \end{matrix}\right\}$	$=\frac{2n-1}{2n+1}$	...	...	...	0	0	...	...	...	...	...	$=\frac{2n-1}{2n+1}$						
5	÷	$^1V_{11} \div {}^1V_2$	$^2V_{11}$	$\left\{\begin{matrix} ^1V_{11} = {}^2V_{11} \\ ^1V_2 = {}^1V_2 \end{matrix}\right\}$	$=\frac{1}{2} \cdot \frac{2n-1}{2n+1}$	...	2	...	...	...	...	...	...	...	...	$=\frac{1}{2} \cdot \frac{2n-1}{2n+1}$						
6	−	$^0V_{13} - {}^2V_{11}$	$^1V_{13}$	$\left\{\begin{matrix} ^2V_{11} = {}^0V_{11} \\ ^0V_{13} = {}^1V_{13} \end{matrix}\right\}$	$=-\frac{1}{2} \cdot \frac{2n-1}{2n+1}$	...	...	...	...	...	...	...	...	...	...	0		$=-\frac{1}{2} \cdot \frac{2n-1}{2n+1} = A_0$				
7	−	$^1V_3 - {}^1V_1$	$^1V_{10}$	$\left\{\begin{matrix} ^1V_2 = {}^1V_3 \\ ^1V_3 = {}^2V_2 \end{matrix}\right\}$	$=n-1 (=3)$	1	...	n	...	...	...	...	...	...	n							
8	+	$^1V_2 + {}^0V_7$	$^1V_7$	$\left\{\begin{matrix} ^1V_2 = {}^1V_2 \\ ^0V_7 = {}^1V_7 \end{matrix}\right\}$	$=2+U=2$	...	2	...	...	...	...	2										
9	÷	$^1V_6 \div {}^1V_7$	$^3V_{11}$	$\left\{\begin{matrix} ^1V_6 = {}^1V_6 \\ ^0V_{11} = {}^3V_{11} \end{matrix}\right\}$	$=\frac{2n}{2} = A_1$	...	...	...	...	...	$2n$	2	...	...	...	$\frac{2n}{2} = A_1$						
10	×	$^1V_{12} \times {}^3V_{11}$	$^1V_{12}$	$\left\{\begin{matrix} ^1V_{21} = {}^1V_{21} \\ ^3V_{11} = {}^3V_{11} \end{matrix}\right\}$	$=B_1 \cdot \frac{2n}{2} = B_1 A_1$	...	...	...	...	...	...	...	...	...	...	$\frac{2n}{2} = A_1$	$B_1 \cdot \frac{2n}{2} = B_1 A_1$	......	B_1			
11	+	$^1V_{12} + {}^1V_{13}$	$^2V_{12}$	$\left\{\begin{matrix} ^1V_{12} = {}^0V_{12} \\ ^1V_{13} = {}^2V_{13} \end{matrix}\right\}$	$=-\frac{1}{2} \cdot \frac{2n-1}{2n+1} + B_1 \cdot \frac{2n}{2}$	...	...	...	...	...	...	...	...	...	...	......	0	$\left\{-\frac{1}{2} \cdot \frac{2n-1}{2n+1} + B_1 \cdot \frac{2n}{2}\right\}$				
12	−	$^1V_{10} - {}^1V_1$	$^2V_{10}$	$\left\{\begin{matrix} ^1V_{10} = {}^2V_{10} \\ ^1V_1 = {}^1V_1 \end{matrix}\right\}$	$=n-2=2$	1	...	...	...	...	...	...	...	...	$n-2$							
13	−	$^1V_6 - {}^1V_1$	$^2V_6$	$\left\{\begin{matrix} ^1V_6 = {}^2V_6 \\ ^1V_1 = {}^1V_1 \end{matrix}\right\}$	$=2n-1$	1	...	...	...	...	$2n-1$											
14	+	$^1V_1 - {}^1V_7$	$^2V_7$	$\left\{\begin{matrix} ^1V_1 = {}^1V_1 \\ ^1V_7 = {}^2V_7 \end{matrix}\right\}$	$=2+1=3$	1	...	...	...	...	...	3										
15	[illegible]	[illegible]	[illegible]	$\left\{\begin{matrix} ^2V_6 = {}^2V_6 \\ \text{[illegible]} \end{matrix}\right\}$	$2n-1$ [illegible]						[illegible]	[illegible]	$2n-1$									

15	÷	${}^2V_6 \div {}^2V_7$	${}^1V_8$	$\left\{\begin{array}{c}{}^2V_6={}^2V_6\\{}^2V_7={}^2V_7\end{array}\right\}$	$=\frac{2n-1}{3}$	...	...	...	...	...	$2n-1$	3	$\frac{2n-1}{3}$									
16	×	${}^1V_8 \times {}^3V_{11}$	${}^4V_{11}$	$\left\{\begin{array}{c}{}^1V_8={}^0V_8\\{}^3V_{11}={}^4V_{11}\end{array}\right\}$	$=\frac{2n}{2}\cdot\frac{2n-1}{3}$	...	...	...	...	...	...	3		0	...	...	$\frac{2n}{2}\cdot\frac{2n-1}{3}$					
17	−	${}^2V_6 - {}^1V_1$	${}^3V_6$	$\left\{\begin{array}{c}{}^2V_6={}^3V_6\\{}^1V_1={}^1V_1\end{array}\right\}$	$=2n-2$	1	...	...	...	...	$2n-2$											
18	+	${}^1V_1 + {}^2V_7$	${}^3V_7$	$\left\{\begin{array}{c}{}^2V_7={}^3V_7\\{}^1V_1={}^1V_1\end{array}\right\}$	$=3+1=4$	1	...	...	...	...	...	4										
19	÷	${}^3V_6 \div {}^3V_7$	${}^1V_9$	$\left\{\begin{array}{c}{}^3V_6={}^3V_6\\{}^3V_7={}^3V_7\end{array}\right\}$	$=\frac{2n-2}{4}$	...	...	...	...	...	$2n-2$	4	...	$\frac{2n-2}{4}$	...	$\left\{\begin{array}{c}\frac{2n}{2}\cdot\frac{2n-1}{3}\cdot\frac{2n-2}{3}\\=A_2\end{array}\right\}$						
20	×	${}^1V_9 \times {}^4V_{11}$	${}^5V_{11}$	$\left\{\begin{array}{c}{}^1V_9={}^0V_9\\{}^4V_{11}={}^5V_{11}\end{array}\right\}$	$=\frac{2n}{2}\cdot\frac{2n-1}{3}\cdot\frac{2n-2}{4}=A_3$	...	...	...	...	...	...	...	...	0	...							
21	×	${}^1V_{22} \times {}^5V_{11}$	${}^6V_{11}$	$\left\{\begin{array}{c}{}^1V_{22}={}^1V_{22}\\{}^0V_{12}={}^0V_{12}\end{array}\right\}$	$=B_3\cdot\frac{2n}{2}\cdot\frac{2n-1}{3}\cdot\frac{2n-2}{3}=B_3A_3$	...	...	...	...	...	...	...	...	...	...	0	B_3A_3	...	...	B_3		
22	+	${}^3V_{12} + {}^2V_{13}$	${}^3V_{13}$	$\left\{\begin{array}{c}{}^3V_{12}={}^0V_{12}\\{}^2V_{13}={}^3V_{13}\end{array}\right\}$	$=A_0+B_1A_2+B_3A_3$	...	...	...	...	...	...	...	...	...	...	......	0	$\{A_3+B_2A_2+B_3A_3\}$				
23	−	${}^2V_{10} + {}^1V_1$	${}^3V_{10}$	$\left\{\begin{array}{c}{}^3V_{10}={}^3V_{10}\\{}^1V_1={}^1V_1\end{array}\right\}$	$=n-3=1$	1	...	...	...	...	...	...	...	...	$n-3$							

Here follows a repetition of Operations thirteen to twenty-three.

24	+	${}^4V_{12} + {}^0V_{24}$	${}^1V_{24}$	$\left\{\begin{array}{c}{}^4V_{12}={}^0V_{12}\\{}^0V_{24}={}^1V_{24}\end{array}\right\}$	$=B_7$	...	...	...	...	...	...	...	...	...	...	......	......	......	...	...	...	B_7
25	+	${}^1V_1 + {}^1V_3$	${}^1V_3$	$\left\{\begin{array}{c}{}^1V_1={}^1V_1\\{}^1V_3={}^1V_3\end{array}\right\}$ ${}^4V_6={}^0V_6$ ${}^1V_7={}^0V_7$	$=n+1=4+1=5$ by a Variable-card. by a Variable-card.																	

Was the Analytical Engine just a larger Difference Engine?

- The Difference Engine, although it had the same arithmetical operations as the Analytical Engine, was *logically* on a lower plane.

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The DE could perform the following *operations*:

- The arithmetical operations $+, \times, -$ are operations.
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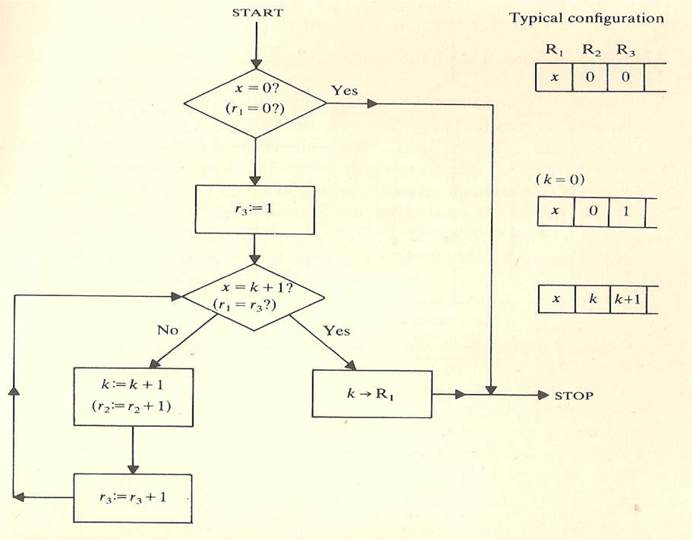
- *(Iteration)* Any finite iteration of an operation is an operation.

(Today we should say that the DE could compute any *primitively recursive* function.) But the Analytical Engine additionally had:

- *Conditional iteration* If P is an operation and T is a test on register contents, then the result of iterating P until T succeeds is an operation.

Flowchart for a Shepherdson-Sturgis-Minsky Register Machine



Was the Analytical Engine 'Universal'?

- A *universal* computer is one equipped with a *universal program* that is one that could theoretically emulate the action of every other program. Turing, defining the term, proved that his 'machines' were universal.

- What Babbage and Lovelace lacked, was a *coding method*, a way of coding up program instructions by numbers, which a universal machine could decode and then simulate.

- Kurt Gödel (1930) gave a way of coding alphabets, then words, then sentences for use in his famous Incompleteness Theorems. Turing then used this idea to code up programs (1936). Using this a Shep.-S-M Register machine only needs a half dozen Registers to be a fully universal machine.

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So the answer is "Yes": the Analytical Engine, using Turing and Gödel's ideas, can be seen to be universal.

Moreover as Turing showed, the programs could also be treated as data, and simple be stored inside the machine prior to operation.

The vistas beyond

Neither Babbage, nor Lovelace knew of universality, but both saw potentialities beyond the calculations they programmed.

Babbage's thesis:
"These two memoirs [Menabrea's and Lovelace's articles] furnish a complete demonstration - *that the whole of the development and operations of analysis are capable of being executed by machinery."* (Babbage's italics.)

Do machines think? “Lovelace’s Objection”

Lovelace:
“It is desirable to guard against the possibility of exaggerated ideas that might arise as to the powers of the Analytical Engine [which] has no pretensions whatever to *originate* anything. It can do whatever we *know how to order it* to perform . . . . but it has no power of *anticipating* any analytical relations or truths.”

• This is discussed in a famous paper by Turing in the journal *Mind* on the possibility of machines having intelligent thought.

But again. . . .

Lovelace:
"Again it might act upon things other than *number*, ... Supposing for instance , that the fundamental relations of pitched sounds in the science of harmony and of musical composition were susceptible of such expression and adaptations, the engine might compose elaborate and scientific pieces of music of any degree of complexity or extent."

The effect on computing machinery on pure mathematics

Lovelace again:
“It is however pretty evident, on general principles, that in devising for mathematical truths a new form in which to record and throw themselves out for actual use, views are likely to be induced, which should again react on the more theoretical phase of the subject”