Bounded Martin's Maximum, weak Erdős cardinals, and ψ_{AC}.

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Abstract

We prove that a form of the Erdős property (consistent with $V = L[H_{\omega_2}]$ and strictly weaker than the Weak Chang's Conjecture at ω_1), together with Bounded Martin's Maximum implies that Woodin's principle ψ_{AC} holds, and therefore $2^{\aleph_0} = \aleph_2$. We also prove that ψ_{AC} implies that every function $f : \omega_1 \to \omega_1$ is bounded by some canonical function on a club and use this to produce a model of the Bounded Semiproper Forcing Axiom in which Bounded Martin's Maximum fails.

1 Introduction

Recall the following bounded form of Martin's Maximum ([Fo-M-S]), the maximal forcing axiom for collections of $\aleph_1$–many antichains:

Definition 1.1 *Bounded Martin's Maximum (BMM) is the following statement:*

Suppose $\mathbb{P}$ is a stationary–set–preserving poset (i.e., every stationary subset of ω_1 remains stationary after forcing with $\mathbb{P}$) and $\langle A_i : i < \omega_1 \rangle$ is a sequence of maximal antichains of $\mathbb{P}$ of size at most $\aleph_1$. Then there is a filter $G \subseteq \mathbb{P}$ such that $G \cap A_i \neq \emptyset$ for all $i < \omega_1$.

Bounded forcing axioms, and BMM in particular, can be characterized as principles of generic absoluteness for Σ_1 formulas with parameters in H_{ω_2}. More precisely, the following holds ([B]):

Characterization 1.1 *BMM holds if and only if for every $a \in H_{\omega_2}$ and every Σ_1 formula $\varphi(x)$, $H_{\omega_2} \models \varphi(a)$ iff there is some stationary–set–preserving poset $\mathbb{P}$ such that $\Vdash_{\mathbb{P}} \varphi(\check{a})$.*

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We shall also consider the bounded forcing axiom obtained from replacing "stationary–set–preserving" by "semiproper" in the Definition 1.1. This is called *Bounded Semiproper Forcing Axiom* ($BSPFA$). $BSPFA$ can be characterized as a principle of generic absoluteness in a similar way as BMM. Specifically, in Characterization 1.1 one can replace "BMM" and "stationary–set–preserving" by "$BSPFA$" and "semiproper", respectively ([B]).

It turns out that $BSPFA$ is equiconsistent with the existence of a so–called Σ_2*–reflecting cardinal*, i.e. an inaccessible cardinal κ such that $V_\kappa \preccurlyeq_{\Sigma_2} V$, which is a large cardinal hypothesis of very low consistency strength (if λ is Mahlo, then there are stationarily many $\kappa < \lambda$ such that κ is Σ_2–reflecting in V_λ): starting from a ground model with such a cardinal one can force $BSPFA$ and, on the other hand, $BSPFA$– and even less– implies that ω_2 is Σ_2–reflecting in L ([G-S]). Unlike with $BSPFA$, very little is known about the consistency strength of BMM. No better lower bound than that for $BSPFA$ is known for BMM in general. Nevertheless, in Section 3 we prove that BMM, together with a weak large cardinal assumption, implies the consistency of quite a strong theory (this suggests that BMM could have large consistency strength by itself). This result is used there to build a model of $BSPFA$ and the negation of BMM starting from the assumption that a certain relatively weak theory is consistent. However, a proof of the inequivalence of $BSPFA$ and BMM starting from the optimal hypothesis– i.e., that of the consistency of ZFC + "There is a Σ_2–reflecting cardinal"– has not been found. In the other direction, the best results known so far are those of Hugh Woodin ([Wo], Corollary 10.99 and Remark 10.100), which give something like the existence of unboundedly many Woodin cardinals as an upper bound for the consistency strength of BMM.

Woodin introduced in [Wo], Definition 5.12, the following combinatorial principle:

Definition 1.2 ψ_{AC}: *If S and T are stationary and co-stationary subsets of ω_1, then there is an ordinal δ, a bijection $\pi : \omega_1 \to \delta$ and a club $C \subseteq \omega_1$ such that*

$$S \cap C = \{\nu < \omega_1 \,:\, ot(\pi``\nu) \in T\} \cap C$$

The following simple application of ψ_{AC} is due to Woodin ([Wo], Lemma 5.13).

Fact 1.2 ψ_{AC} *implies* $2^{\aleph_1} = \aleph_2$.

Proof: Fix a pairwise disjoint sequence $\langle S_\alpha \,:\, \alpha < \omega_1\rangle$ of stationary subsets of ω_1 and a stationary and co-stationary subset T of ω_1. For every nonempty $X \subseteq \omega_1$, $X \neq \omega_1$, let $S_X = \bigcup_{\alpha\in X} S_\alpha$. Then, S_X is a stationary and co-stationary subset of ω_1 and so, applying ψ_{AC} to S_X and T, we get an ordinal $\delta_X < \omega_2$, a bijection $\pi_X : \omega_1 \to \delta_X$ and a club $C_X \subseteq \omega_1$ such that $\{\nu < \omega_1 \,:\, ot(\pi_X``\nu) \in T\} \cap C_X = S_X \cap C_X$. Suppose $2^{\aleph_1} > \aleph_2$. Then there would be distinct $X$, $Y \subseteq \omega_1$ such that $\delta_X = \delta_Y$. Let $\alpha \in X \Delta Y$, and without loss of generality say that $\alpha$ is in $X$. Let $D = \{\nu < \omega_1 \,:\, \pi_X``\nu = \pi_Y``\nu\}$. $D$ is a club of $\omega_1$. Pick $\nu \in D \cap C_X \cap C_Y \cap S_\alpha$. Then, $ot(\pi_Y``\nu) = ot(\pi_X``\nu) \in T$, which is a contradiction, since $\nu \notin S_Y$. □

¿From the above proof it readily follows that ψ_{AC} implies that $L(\mathcal{P}(\omega_1)) \models AC$. ψ_{AC} also implies $2^{\aleph_0} = 2^{\aleph_1}$, which we shall have anyway– by MA_{ω_1}– in our context.

We shall also consider a statement involving canonical functions for ordinals in ω_2. It is a generally well–known fact– see for example [J], p. 445 or [A1]– that given any ordinal $\beta < \omega_2$ and any surjective function $\pi : \omega_1 \to \beta$, the function $g : \omega_1 \to \omega_1$ given by $g(\nu) = ot(\pi``\nu)$ represents β in the standard generic ultrapower obtained from forcing with $\mathcal{P}(\omega_1)/NS_{\omega_1}$ (NS_{ω_1} is the nonstationary ideal over ω_1). Such a g is called a *canonical function for* β. These functions are canonical in the sense that, if π' is any other function from ω_1 onto β and g' is defined as g with π' instead of π, then $\pi``\nu = \pi'``\nu$ (and so $g(\nu) = g'(\nu)$) for all ν in some club of ω_1. By a canonical function we shall mean a canonical function for some ordinal less than ω_2. Several forms of Chang's Conjecture can be characterized in forms involving canonical functions. Thus, the Weak Chang's Conjecture at ω_1 ($wCC(\omega_1)$) can be characterized as the assertion that every function from ω_1 into ω_1 is dominated by some canonical function on a stationary subset of ω_1 (see [Do-K], Theorem 5.1). This is of course equivalent to the assertion that for every such f, $\mathcal{P}(\omega_1)/NS_{\omega_1}$ does not force that f represents in the standard generic ultrapower an ordinal above all $\alpha < \omega_2^V$. A stronger form of $wCC(\omega_1)$ is obtained from requiring that for every f as above, $\mathcal{P}(\omega_1)/NS_{\omega_1}$ forces that f represents an ordinal below ω_2^V in the generic ultrapower or, equivalently, that $\Vdash_{\mathbb{P}(\omega_1)/NS_{\omega_1}} j(\omega_1^V) = \omega_2^V$, where j denotes the standard generic elementary embedding. This form turns out to be equiconsistent with $wCC(\omega_1)$ ([Do-L]), and their consistency strength is exactly that of a so–called *almost* $<\omega_1$*–Erdős cardinal* ([Do-K], Theorems C and D), a weak form of ω_1–Erdős cardinal. The statement we shall consider is the yet stronger form of the above principles saying that every function from ω_1 into ω_1 is dominated on a club by a canonical function. This turns out to be much stronger than $wCC(\omega_1)$, since it implies that ω_2^V is an inaccessible limit of measurable cardinals in the Jensen core model K for measures of order zero ([D-Do]).

By virtue of Characterization 1.1, BMM is arguably a natural reflection principle for H_{ω_2}. Therefore it would be very nice to know whether it is strong enough to decide the size of the continuum. Let us briefly report on the advances made so far on this problem.

Hugh Woodin proved (see [Wo], Lemma 10.95) that, if BMM holds and either there is a measurable cardinal or NS_{ω_1} is precipitous, then ψ_{AC} holds and every function from ω_1 into ω_1 is bounded by some canonical function on a club.

Asperó and Stevo Todorčević proved (see [A1], Theorem 9.16) that this bounding condition also follows from BMM plus the hypothesis that the standard generic ultrapower obtained after forcing with $\mathcal{P}(\omega_1)/NS_{\omega_1}$ has a well–founded initial segment of length $\omega_2^V + 1$ (by the considerations about canonical functions in the above paragraph it certainly always has such a segment of length ω_2^V). Using this, Asperó proved that ψ_{AC} also follows from BMM together with the above well–foundedness hypothesis on the generic ultrapower and the assumption that $r^\sharp$ exists for every real r and the second uniform indiscernible is ω_2. He also proved that if $X^\sharp$ fails to exist for some $X \subseteq \omega_1$ and BMM holds, then $2^{\aleph_0} = \aleph_2$. For these results see [A2]. Ralf-Dieter Schindler then observed that in the last result above one can assume that $X^\sharp$ does not exist for some set X of ordinals; in fact, if this is the case and BMM holds, then ω_1 is accessible to reals. He observed that from Jensen's techniques for coding the universe with a real it follows that if $X^\sharp$ does not exist, then there is a set–sized stationary–set–preserving partial order which codes X into a real. By adding some information to X if necessary, we may assume that $\omega_1 = \omega_1^{L[X]}$. It follows that this partial order forces the existence of a real r such that $\omega_1 = \omega_1^{L[r]}$, which is a Σ_1 statement with ω_1 as parameter. Therefore, such a real r exists in case BMM is true.

Asperó improved part of the above mentioned result of Woodin by proving that BMM, together with the existence of an ω_1–Erdős cardinal, implies ψ_{AC}. Welch was able to weaken the large cardinal assumption in this result to that of the existence of a so-called *ω–closed cardinal* ([W]), and was finally able to weaken the assumption yet further to that of the existence of a cardinal κ satisfying the Erdős property $\kappa \longrightarrow (<\omega_1)^{<\omega}_{2^{\omega_1}}$. It is this result which we present in Section 2. On the other hand, Welch also observed that if BMM holds and for some $X \in H_{\omega_2}$ there is no inner model containing X and satisfying ZFC + "There exists κ such that $\kappa \longrightarrow (<\omega_1)^{<\omega}_{2^{\omega_1}}$", then $2^{\aleph_0} = \aleph_2$. This follows by noting that the arguments of [A2] apply also to K_X, the core model relative to X, where the assumptions allow for Strong Covering to hold over this model. In fact, such arguments allow one to show that the same conclusion holds if BMM holds and $X^\dagger$ does not exist for some $X \in H_{\omega_2}$.

We shall use a notion very slightly weaker than the almost $<\omega_1$–Erdős of [Do-K].

Definition 1.3 *For a cardinal κ, $\kappa \longrightarrow (<\omega_1)^{<\omega}_{2^{\omega_1}}$ iff the following hold: For any first order structure $\mathcal{A} = \langle L_\kappa[A], \in, A\rangle$ there is a sequence $\langle I_\alpha \mid \alpha < \omega_1\rangle$ such that $\forall \alpha, \beta < \omega_1, \forall \varphi(v_0, \ldots, v_{n+m-1}) \forall \vec{\xi} \in \omega_1^n \, \forall \vec{\gamma} \in I_\alpha^m \, \forall \vec{\gamma}' \in I_\beta^m$, $\vec{\gamma}$ and $\vec{\gamma}'$ strictly increasing,*

$$\mathcal{A}^+ \models \varphi(\dot{\vec{\xi}}, \vec{\gamma}) \leftrightarrow \varphi(\dot{\vec{\xi}}, \vec{\gamma}'),$$

where $\mathcal{A}^+ = \langle \mathcal{A}, \langle \xi \rangle_{\xi < \omega_1}\rangle$ and, for each $\xi < \omega_1$, $\dot{\xi}$ is a constant in an expanded language to be interpretated as ξ.

We thus require a sequence of sets of indiscernibles of increasing order type, so that the (model–theoretic) type of the different sets –using in addition countable ordinal parameters –is nevertheless the same. We thus have in essence also a further remarkability condition on each sequence I_α localised on countable ordinal parameters, although we shall see below at (0) of Theorem 2.2 that we can assume full remarkability in any case. An argument as in the proof of (0) of Theorem 2.2 actually shows that, given a structure $\mathcal{A} = \langle L_\kappa[A], \in, A\rangle$, a judicious choice of sets of indiscernibles for $\mathcal{A}$ ensures we can assume that, given any $\alpha < \omega_1$ and any $\gamma < sup(I_\alpha)$, $I_\alpha \backslash \gamma$ is a set of indiscernibles for $\langle \mathcal{A}, \langle \xi \rangle_{\xi<\gamma}\rangle$. These cardinals were defined in [Do-K]. It is easy to see that, given a cardinal κ, $\kappa \longrightarrow (<\omega_1)^{<\omega}_{2^{\omega_1}}$ holds in case for every colouring $c : [\kappa]^{<\omega} \to 2^{\omega_1}$ there are I_α $(\alpha < \omega_1)$ such that each I_α has order type α and $c``\bigcup_{\alpha<\omega_1}[I_\alpha]^n$ is constant for each n. Note that in general a κ as in Definition 1.3 will not be regular. The stronger version where κ is assumed regular and where in the structure $\mathcal{A}^+$ further constants are allowed from any ordinal $\lambda < \kappa$, (thus taken not only from ω_1) is the "almost $< \omega_1$–Erdős" of [Do-K]. The existence of such a cardinal is there shown equiconsistent with $wCC(\omega_1)$. Both the cardinals of Definition 1.3, and the almost $< \omega_1$–Erdős cardinals are thus weaker than an ω_1–Erdős cardinal, and thus than a measurable cardinal. It is not hard to see that the existence of a cardinal κ such that $\kappa \longrightarrow (<\omega_1)^{<\omega}_{2^{\omega_1}}$ implies the existence of the sharp of every subset of ω_1. On the other hand, if a set of ordinals B codes H_{ω_2} and $B^\sharp$ exists, then an absoluteness argument using the countable order types of the required I_α-sequences, shows that any Silver indiscernible κ for $L[B]$ will satisfy $\kappa \longrightarrow (<\omega_1)^{<\omega}_{2^{\omega_1}}$ in $L[B]$.

We shall prove that ψ_{AC} holds in case BMM holds and there is a κ satisfying the weak Erdős property of Definition 1.3, but first it will be convenient to give some preliminary general facts.

If $\omega_1 \subseteq \mathcal{X}$, $A \subseteq [\mathcal{X}]^{\aleph_0}$ is a *projective stationary subset (of* $[\mathcal{X}]^{\aleph_0}$*)* ([F-J]) if, for every stationary $S \subseteq \omega_1$, $\{X \in A : X \cap \omega_1 \in S\}$ is a stationary subset of $[\mathcal{X}]^{\aleph_0}$.

Given a set $\mathcal{X}$ and $A \subseteq [\mathcal{X}]^{\aleph_0}$, we define the following poset $\mathbb{P}_A$: $p \in \mathbb{P}_A$ if and only if p is a strictly $\subseteq$–increasing and $\subseteq$–continuous (i.e., if $\nu \in dom(p)$ is a limit ordinal, then $p(\nu) = \bigcup_{\nu'<\nu} p(\nu')$) $\alpha + 1$–sequence of elements of A for some countable ordinal α. $q \leq p$ if and only if $p \subseteq q$.

We shall use the following fact ([F-J]):

Lemma 1.3 *Let $\mathcal{X}$ be a set and let A be a stationary subset of $[\mathcal{X}]^{\aleph_0}$. Then $\mathbb{P}_A$ forces the existence of a strictly $\subseteq$–increasing and $\subseteq$–continuous sequence $\langle X_\nu : \nu < \omega_1\rangle$ of elements of A such that $\mathcal{X} = \bigcup_{\nu<\omega_1} X_\nu$. Suppose that $\omega_1 \subseteq \mathcal{X}$. Then $\mathbb{P}_A$ is stationary–set–preserving if and only if A is a projective stationary subset of $[\mathcal{X}]^{\aleph_0}$.*

From Lemma 1.3 it follows that the following bounded form of the Projective Stationary Reflection principle ([F-J])– which is equivalent to Todorčević's Strong Reflection Principle (see [F-J] for a proof)– is a consequence of BMM:

Definition 1.4 *$BPSR$: Suppose α is an ordinal, $a \in H_{\omega_2}$ and $A \subseteq [\alpha]^{\aleph_0}$ is a projective stationary subset of $[\alpha]^{\aleph_0}$ which is Σ_1 definable with a and α as parameters (i.e., there is some Σ_1 formula $\varphi(x, y, z)$ such that, for every set X, $X \in A$ iff $\models_1 \varphi(X, a, \alpha)$, where $\models_1$ is the definable satisfaction relation for Σ_1 formulas). Then there is some $\delta < \omega_2$ and some strictly $\subseteq$–increasing and*

$\subseteq$–continuous sequence $\langle X_\nu : \nu < \omega_1\rangle$ such that $\delta = \bigcup_{\nu<\omega_1} X_\nu$ and, for every $\nu < \omega_1$, $H_{\omega_2} \models \varphi(X_\nu, a, \delta)$.

Suppose BMM (or, more generally, $BPSR$) holds. In order to verify ψ_{AC} it suffices, by Lemma 1.3, to find some ordinal $\gamma \geq \omega_2$ so that, whenever S and T are stationary and co-stationary subsets of ω_1, $A = \{X \in [\gamma]^{\aleph_0} : X \cap \omega_1 \in S \text{ iff } ot(X) \in T\}$ is a projective stationary subset of $[\gamma]^{\aleph_0}$. Then, by $BPSR$ there is $\delta < \omega_2$ and a strictly $\subseteq$–increasing and $\subseteq$–continuous sequence $\langle X_\nu : \nu < \omega_1\rangle$ of countable subsets of δ such that $X_\nu \cap \omega_1 \in S$ iff $ot(X_\nu) \in T$ for each ν and $\delta = \bigcup_\nu X_\nu$. Let $\pi : \omega_1 \to \delta$ be any bijection and let $C \subseteq \omega_1$ be the club of all ordinals ν such that $\nu = \pi``\nu \cap \omega_1$ and $\pi``\nu = X_\nu$. Then, for every $\nu \in C$, $\nu \in S$ iff $ot(\pi``\nu) \in T$.

For every ordinal γ and for every pair $\langle S, T\rangle$ of stationary subsets of ω_1 let

$$\mathcal{S}^\gamma_{S,T} = \{X \in [\gamma]^{\aleph_0} : X \cap \omega_1 \in S \text{ and } ot(X) \in T\}$$

It easily follows from the above remark that in order to prove– in the presence of $BPSR$– ψ_{AC} it is enough to prove that there is some ordinal $\gamma \geq \omega_2$ so that $\mathcal{S}^\gamma_{S,T}$ is a stationary subset of $[\gamma]^{\aleph_0}$ for all S and T which are stationary subsets of ω_1.

The rest of the paper is organized as follows: The main result of Section 2, Theorem 2.2, is due, in the present form, to Welch. From this theorem we derive, by the observations in the above paragraphs, that BMM, together with the existence of a cardinal κ such that $\kappa \longrightarrow (<\omega_1)^{<\omega}_{2^{\omega_1}}$, implies ψ_{AC}. In Section 3 we prove that ψ_{AC} implies the bounding of every function from ω_1 into ω_1 by some canonical function, from which it follows that ψ_{AC} has large consistency strength. We also prove that the above bounding condition implies a strong form of the principle $P_2(\omega_1)$ of [Do-L]. This result is essentially due to Paul Larson. Then we use these results and the ones in Section 2 to produce, assuming the consistency of a theory slightly weaker than ZFC + "There exists a Mahlo almost $<\omega_1$–Erdős cardinal", a model in which $BSPFA$ holds but BMM fails. Except for the result of P. Larson mentioned above, and Fact 3.3 (due to Welch), the results in Section 3 are due, in their present form, to Asperó.

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2 Bounded Martin's Maximum in the presence of a weak Erdős cardinal

We make two preliminary observations about the weak Erdős property under discussion. Firstly, a proof that the least κ satisfying $\kappa \longrightarrow (<\omega_1)^{<\omega}_{2^{\omega_1}}$ is singular is essentially contained in [Ba-G]. We add to this the remark:

Lemma 2.1 *Let κ be least with the weak Erdős property $\kappa \longrightarrow (<\omega_1)^{<\omega}_{2^{\omega_1}}$. Then κ is a strong limit cardinal.*

Proof: Suppose for a contradiction that $\delta < \kappa$ is least with $2^\delta \geq \kappa$; pick an injective function $G : \kappa \longrightarrow \mathcal{P}(\delta)$. We claim that any structure $\mathcal{A}$ that has G as a predicate may be assumed to have its indiscernibility sequences $\langle I_\alpha \rangle_{\alpha<\omega_1}$ chosen so that $I_\alpha \subseteq \delta$ for all $\alpha < \omega_1$. As any structure of the kind $\mathcal{A}' = \langle L_\delta[A'], \in, A' \rangle$ can be construed as an initial segment of a suitable $\mathcal{A}$, we have in effect indiscernible sequences for such structures on δ, so that δ also has the weak Erdős property.

To show this claim, let $\langle I_\alpha \rangle_{\alpha<\omega_1}$ be a sequence for $\mathcal{A}$ as given by Definition 1.3 with $\min(I_1)$ minimal. This ensures that each I_α is remarkable (see (0) of the proof of Theorem 2.2). We assume for a contradiction that $I_\alpha \not\subseteq \delta$ for some α. Since the order type of I_α is a limit ordinal, we can take $\xi < \zeta < \zeta'$ from I_α above δ. Then we argue, by our assumed remarkability, that if $\tau < \delta < \min(I_\alpha)$ is least in $G(\xi)\Delta G(\zeta)$, then τ is also least in $G(\xi)\Delta G(\zeta')$ and in $G(\zeta)\Delta G(\zeta')$. This is a contradiction. $\Box$

Theorem 2.2 *Suppose there is a cardinal κ such that $\kappa \longrightarrow (<\omega_1)^{<\omega}_{2^{\omega_1}}$. Then there is an inaccessible cardinal $\gamma < \kappa$ such that, whenever S and T are stationary subsets of ω_1, $\mathcal{S}^\gamma_{S,T}$ is a stationary subset of $[\gamma]^{\aleph_0}$.*

Proof: Assume, towards a contradiction, that the theorem fails. By the last lemma, if κ is chosen least with the stated weak Erdős property, then it is a strong limit. We may therefore let $A \subseteq \kappa$ be such that $H_\kappa = L_\kappa[A]$ and set $\mathcal{A} = \langle L_\kappa[A], \in, A \rangle$. Choose for $\gamma < \kappa$ $\mathcal{A}$-least pairs (S_γ, T_γ) and functions $F_\gamma : [\gamma]^{<\omega} \to \gamma$ so that

$$\forall X \in [\gamma]^\omega \, F_\gamma``[X]^{<\omega} \subseteq X \longrightarrow (X \cap \omega_1 \in S_\gamma \to ot(X) \notin T_\gamma)$$

Let $\langle I_\alpha \,|\, \alpha < \omega_1 \rangle$ be a sequence satisfying the Definition 1.3 and with ωth element of I_2 minimal amongst all such sequences. By a standard argument, this choice ensures that the following remarkability condition applies:

For all $\alpha < \omega_1$, $p, m, n < \omega$, $\vec{\xi} \in \omega_1^p$, $\vec{\gamma} \in I^m_\alpha$, all strictly increasing sequences $\vec{\gamma}_0 \in I^n_\alpha$ and $\vec{\gamma}_1 \in I^n_\alpha$ such that $max(\vec{\gamma}) < min(\vec{\gamma}_0)$, $min(\vec{\gamma}_1)$ and all $\mathcal{A}$–terms $t(x_0, \ldots x_{p+m+n-1})$,

(0) if $t^{\mathcal{A}}(\vec{\xi}, \vec{\gamma}, \vec{\gamma}_0)$ is defined and is an ordinal below $min(\vec{\gamma}_0)$, then $t^{\mathcal{A}}(\vec{\xi}, \vec{\gamma}, \vec{\gamma}_0) = t^{\mathcal{A}}(\vec{\xi}, \vec{\gamma}, \vec{\gamma}_1)$.

Proof of (0): (This is somewhat of a standard argument, but at the referee's suggestion we sketch a proof.) Let $I_\alpha = \langle \gamma^\alpha_i \,|\, i < \omega\alpha \rangle$ for $0 < \alpha < \omega_1$ be enumerated in ascending order. Suppose (0) fails. By indiscernibility and the independence of the type over countable parameters, there is a term t and a finite sequence $\vec{\xi}$ of countable ordinals such that for every α $(0 < \alpha < \omega_1)$,

$t^{\mathcal{A}}(\vec{\xi}, \gamma_0^\alpha, \dots \gamma_{m-1}^\alpha, \gamma_{i_0}^\alpha, \dots \gamma_{i_{n-1}}^\alpha)$ is an ordinal in $\gamma_{i_0}^\alpha$ for $m-1 < i_0 < \dots < i_{n-1} < \omega\alpha$, and $t^{\mathcal{A}}(\vec{\xi}, \gamma_0^\alpha, \dots \gamma_{m-1}^\alpha, \gamma_{i_0}^\alpha, \dots \gamma_{i_{n-1}}^\alpha) \neq t^{\mathcal{A}}(\vec{\xi}, \gamma_0^\alpha, \dots \gamma_{m-1}^\alpha, \gamma_{i_n}^\alpha, \dots \gamma_{i_{2n-1}}^\alpha)$ for $m-1 < i_0 < \dots < i_{2n-1}$.

Now define for each $\alpha < \omega_1$ and $k < \omega\alpha$, $\eta_k^\alpha = t^{\mathcal{A}}(\vec{\xi}, \gamma_0^\alpha, \dots \gamma_{m-1}^\alpha, \vec{\delta}_k^\alpha)$, where $\vec{\delta}_k^\alpha = \langle \gamma_{m-1+k+j}^\alpha \,|\, j \leq l \rangle$. Then the usual arguments show that we must have $k < l \to \eta_k^\alpha < \eta_l^\alpha$ for all α. Also, each $\langle \eta_k^\alpha \,|\, k < \omega\alpha \rangle$ is a sequence of indiscernibles for $\mathcal{A}$, and moreover it is easy to check, using the independence of the type of the I_α sets, that these new sets also have the same model–theoretic type over countable parameters. But then, letting $J_\alpha = \langle \eta_k^\alpha \,|\, k < \omega\alpha \rangle$ for all α, we have that $\langle J_\alpha \,|\, \alpha < \omega_1 \rangle$ is a sequence of indiscernible sets of the prescribed kind. However, the ωth element of J_2 is below γ_ω^2, contrary to our original choice of $\langle I_\alpha \,|\, \alpha < \omega_1 \rangle$. **Q.E.D.(0)**

Also by standard arguments, if the sequence is chosen as above, then each $\gamma \in I_\beta$ is inaccessible. By the indiscernibility of the sets I_α with respect to countable parameters, we see that:

(1) $\forall \alpha, \beta < \omega_1 \, \forall \gamma \in I_\alpha, \delta \in I_\beta \quad (S_\gamma, T_\gamma) = (S_\delta, T_\delta)$

So let S, T be this common value. Let $Y_0 = Sk^{\mathcal{A}}(I_1)$ denote the Skolem hull of I_1 in $\mathcal{A}$. Then $\exists \alpha_0 < \omega_1 \, Y_0 \cap \omega_1 = \alpha_0$. We can find $\alpha_0 \leq \alpha < \omega_1$ so that

(a) $\alpha \in S$,

(b) $Sk^{\mathcal{A}}(\alpha \cup I_1) \cap \omega_1 = \alpha = Sk^{\mathcal{A}}(\alpha \cup I_\beta) \cap \omega_1$ for any $\beta < \omega_1$.

The existence of α such that (a) and the first equality of (b) hold follows from the usual closure argument; we inductively increase the intersection of the hull with ω_1: let $\alpha_{\zeta+1} = \omega_1 \cap Sk^{\mathcal{A}}(\alpha_\zeta + 1 \cup I_1))$, and set $\alpha_\lambda = \sup_{\zeta < \lambda} \alpha_\zeta$ for limit $\lambda < \omega_1$. As the sequence α_ζ is continuous, there will be a least ζ_0 with $\alpha_{\zeta_0} \in S$. The second equality in (b) follows from the fact that all I_β have the same type over $\mathcal{A}$ expanded with constants for all countable ordinals.

Let $H : \omega_1 \longrightarrow \omega_1$ be the function given by $H(\delta) = ot(sup(I_\delta) \cap Sk^{\mathcal{A}}(\alpha \cup I_\delta))$. Then $H(\delta) \geq \delta$ for all $\delta < \omega_1$.

(2) If $\bar{\beta} < \beta < \omega_1, \quad ot(sup(I_\beta \restriction \omega\bar{\beta}) \cap Sk^{\mathcal{A}}(\alpha \cup I_\beta \restriction \omega\bar{\beta})) = H(\bar{\beta})$

Proof of (2): This follows from (ii) in the Definition 1.3. Since $\omega\beta$, $\omega\bar{\beta}$ are limit ordinals, if t is an $\mathcal{A}$-term, $\vec{\xi} < \alpha, \vec{\gamma} \in I_\beta \restriction \omega\bar{\beta}, \vec{\delta} \in I_{\bar{\beta}}$, then $t^{\mathcal{A}}(\vec{\xi}, \vec{\delta})$ defines an ordinal below $sup(I_{\bar{\beta}})$ if and only if $t^{\mathcal{A}}(\vec{\xi}, \vec{\gamma})$ defines an ordinal below $sup(I_\beta \restriction \omega\bar{\beta})$. If t_1, t_2 are two further terms and $\vec{\gamma}_i \in I_\beta \restriction \omega\bar{\beta}$, and $\vec{\delta}_i \in I_{\bar{\beta}}$ for $i = 1, 2$, then we have similarly that if R denotes one of the three relations $<$, $>$, or $=$, then

$$t_1^{\mathcal{A}}(\vec{\xi}, \vec{\delta}_1) \, R \, t_2^{\mathcal{A}}(\vec{\xi}, \vec{\delta}_2) \Leftrightarrow t_1^{\mathcal{A}}(\vec{\xi}, \vec{\gamma}_1) \, R \, t_2^{\mathcal{A}}(\vec{\xi}, \vec{\gamma}_2)$$

where $\vec{\gamma}_i$, $\vec{\delta}_i$ $(i = 1, 2)$ are ordered the same way. (That is, if $\vec{\delta}_0 \cup \vec{\delta}_1$ is enumerated as $\delta_0 < \cdots < \delta_k$ (and similarly for $\vec{\gamma}_0 \cup \vec{\gamma}_1$) and if $\vec{\delta_0} = \langle \delta_{n_i} | i < l \rangle$ then $\vec{\gamma_0} = \langle \gamma_{n_i} | i < l \rangle$, and similarly for $\vec{\gamma_1}$and $\vec{\delta_1}$). Clearly from this the equality of the order types of the ordinals in the respective hulls follows. **Q.E.D.(2)**

By an argument as in the above proof, we can also establish the following:

(3) $\forall \eta < \gamma \leq \beta$, $sup(I_\beta \restriction \omega\eta) \cap Sk^{\mathcal{A}}(\alpha \cup I_\beta \restriction \omega\eta)$ is an initial segment of $sup(I_\beta \restriction \omega\gamma) \cap Sk^{\mathcal{A}}(\alpha \cup I_\beta \restriction \omega\gamma)$.

Let $C = \{\eta < \omega_1 \mid H``\eta \subseteq \eta\}$. C is a club of ω_1. Pick $\eta \in C \cap T$. Let $I = I_{\eta+1} = \langle \iota_\xi \mid \xi < \omega(\eta+1) \rangle$. Let $I' = I \restriction \omega\eta = \langle \iota_\xi \mid \xi < \omega\eta \rangle$; let $\iota = \iota_{\omega\eta}$. Now let $X = \iota \cap Sk^{\mathcal{A}}(\alpha \cup I')$. This X will give the desired contradiction.

(4) $X \cap \omega_1 = \alpha \in S$, $ot(X) = \eta \in T$ and $F_\iota``[X]^{<\omega} \subseteq X$.

Proof of (4): That $X \cap \omega_1 = \alpha$ follows from (b). By (2), (3) and the definition of X we have:

$$\begin{aligned} ot(X) &= ot(\iota \cap Sk^{\mathcal{A}}(\alpha \cup I')) \\ &= ot(sup(I') \cap Sk^{\mathcal{A}}(\alpha \cup I')) \\ &= \sup_{\bar{\eta}<\eta} ot(sup(I' \restriction \omega\bar{\eta}) \cap Sk^{\mathcal{A}}(\alpha \cup I' \restriction \omega\bar{\eta})) \\ &= \sup_{\bar{\eta}<\eta} ot(sup(I_{\bar{\eta}}) \cap Sk^{\mathcal{A}}(\alpha \cup I_{\bar{\eta}})) \\ &= \sup_{\bar{\eta}<\eta} H(\bar{\eta}) = \eta. \end{aligned}$$

Let $Y = Sk^{\mathcal{A}}(\alpha \cup I' \cup \{\iota\})$. But, by the remarkability property (2) and the fact that ι is a limit ordinal, $Y \cap \iota = X$. As $F_\iota \in Y$ we have $F\iota``[X]^{<\omega} \subseteq X$. **Q.E.D.(4) & Theorem 2.2**

Corollary 2.3 *Suppose there is a cardinal κ such that $\kappa \longrightarrow (< \omega_1)^{<\omega}_{2^{\omega_1}}$. If BPSR holds, then so does ψ_{AC}.*

3 Separating bounded forcing axioms.

In [F-M-S] a model of Martin's Maximum was produced by forcing over a universe with a supercompact cardinal and obtaining a forcing extension in which the seemingly weaker Semiproper Forcing Axiom holds. In this model, a reflection principle holds which implies that every stationary–set–preserving poset is actually semiproper, and so it follows that in fact Martin's Maximum holds there. Saharon Shelah showed in [S] that this argument was unnecessary by proving that the Semiproper Forcing Axiom already implies that every stationary–set–preserving poset is semiproper, so that it is equivalent to Martin's Maximum. A natural question is whether this equivalence still holds between bounded forms of these two axioms. It turns out that this is not the case at the level of $\aleph_2$. Specifically, call $BSPFA_{<\aleph_3}$ and $BMM_{<\aleph_3}$ the principles that allow families of maximal antichains of size at most $\aleph_2$ in the formulation of $SPFA$ and of MM, respectively. Then it is not hard to see that $BMM_{<\aleph_3}$ implies that NS_{ω_1} is saturated (see [A-B], Fact 4.3), and so it is much stronger than $BSPFA_{<\aleph_3}$, which can be forced over L; in fact, $BSPFA_{<\aleph_3}$ is equiconsistent with a large cardinal hypothesis that relativizes down to L ([Mi], Theorem 4.2; see also [A1], Theorem 4.28 and Corollary 4.29). Until now, the situation for the usual bounded forms, i.e. those allowing collections of maximal antichains of size at most $\aleph_1$, was totally unclear, since little is known about the consistency strength of BMM (see the discussion before Definition 1.2). In this Section we give a partial solution to this problem.

Welch showed that a slight modification of the proof of Theorem 2.2 proves, in any model of $BPSR$ in which there is a cardinal with the weak Erdős property of that theorem, that every function from ω_1 into ω_1 is bounded on a club by a canonical function. This established, by the result of [D-Do] mentioned in Section 1, that $Con(ZFC + BMM + \kappa \longrightarrow (<\omega_1)^{<\omega}_{2^{\omega_1}})$ implies $Con(ZFC$ + "There is an inaccessible limit of measurable cardinals"). Then, Asperó improved an earlier result of Peter Koepke saying that $wCC(\omega_1)$ follows from ψ_{AC}: he observed that the above bounding principle follows outright from ψ_{AC}. This shows, again by the result of [D-Do], that ψ_{AC} also has this very large consistency strength.

Fact 3.1 shows that the bounding principle follows from the weakening of ψ_{AC} obtained from replacing the equality in the conclusion of Definition 1.2 by $\subseteq$.

Fact 3.1 *Suppose that for all stationary and co-stationary S, $T \subseteq \omega_1$ there is $\delta < \omega_2$, a bijection $\pi : \omega_1 \to \delta$ and a club $C \subseteq \omega_1$ such that $S \cap C \subseteq \{\nu < \omega_1 : ot(\pi``\nu) \in T\} \cap C$. Then, every function $f : \omega_1 \to \omega_1$ is bounded on a club by a canonical function.*

Proof: Let S be any stationary and co-stationary subset of ω_1. Let $C = \{\nu < \omega_1 : f``\nu \subseteq \nu\}$. Applying the weak form of ψ_{AC} to the pairs $(S, C\backslash S)$ and $(\omega_1\backslash S, C \cap S)$, we get ordinals α_0, $\alpha_1 < \omega_2$, canonical functions g_0 and g_1 for α_0 and α_1, respectively, and clubs C_0, $C_1 \subseteq \omega_1$ such that for all $\nu \in S \cap C_0$ and all $\nu' \in C_1\backslash S$, $g_0(\nu) \in C\backslash S$ and $g_1(\nu') \in C \cap S$. Let $\alpha = max\{\alpha_0, \alpha_1\}$, let g be a canonical function for α and let D be a club of ω_1 such that for all $\nu \in D$, $g_0(\nu)$, $g_1(\nu) \leq g(\nu)$. Notice that, as $S \cap (C\backslash S)$ and $(\omega_1\backslash S) \cap (C \cap S)$ are both empty, $\omega_1 < \alpha_0$, α_1, and we may assume that $\nu < g_0(\nu)$, $g_1(\nu)$ for all $\nu \in D$. Now let $\nu \in D \cap C_0 \cap C_1$, and without loss of generality say that $\nu \in S$. Then $\nu < g_0(\nu) \in C$, so that $f(\nu) < g_0(\nu) \leq g(\nu)$. This shows that g dominates f on $D \cap C_0 \cap C_1$. □

Remark 3.2 *Using an argument similar to the proof of Fact 3.1, one can prove that ψ_{AC} is equivalent to the apparently stronger version of it asserting that for all S and T as in the Definition 1.2 there are unboundedly many ordinals δ below ω_2 for which the conclusion of the Definition 1.2 holds.*

Next we present a result of P. Larson showing that the bounding condition on functions $f : \omega_1 \to \omega_1$ implies a certain weak form of Chang's Conjecture from [Do-L] stronger than $wCC(\omega_1)$.

Definition 3.1 *(a) $P_2(\omega_1)$: For every club $E \subseteq [\omega_2]^{\aleph_0}$, $\{ot(X) : X \in E\}$ is a stationary subset of ω_1.*

(b) $P_2(\omega_1)^+$: For every club $E \subseteq [\omega_2]^{\aleph_0}$, there are club–many ordinals ν in ω_1 such that $\{ot(X) : X \in E, X \cap \omega_1 = \nu\}$ is a stationary subset of ω_1.

(c) $P_1(\omega_1)$: For every club $E \subseteq [\omega_2]^{\aleph_0}$, $\{ot(X) : X \in E\}$ contains a club subset of ω_1.

(d) $P_1(\omega_1)^+$: For every club $E \subseteq [\omega_2]^{\aleph_0}$, there are club–many ordinals ν in ω_1 such that $\{ot(X) : X \in E, X \cap \omega_1 = \nu\}$ contains a club subset of ω_1.

(a) and (c) are defined in [Do-L], Definition 2.17. There, in Section 8, it is shown, among other things, that $P_2(\omega_1)$ and $P_1(\omega_1)$ are both equiconsistent with the existence of a so-called *nearly* $< \omega_1$*–Erdős* cardinal. $wCC(\omega_1)$ follows from – and is strictly weaker than – $P_2(\omega_1)$. Actually all four of the above principles are equiconsistent. This follows from an observation of Welch that a modification of the proof of Theorem 8.14 of [Do-L] shows:

Fact 3.3 $Con(ZFC +$ *there exists a nearly* $< \omega_1$*-Erdős cardinal*$) \longrightarrow Con(ZFC + P_1(\omega_1)^+)$.

which

In an earlier version of this paper we proved that ψ_{AC} implies $P_2(\omega_1)^+$. There we asked whether $P_2(\omega_1)$ already follows from the bounding principle. Paul Larson proved that it does. Now we present a variant of his proof which establishes the ostensibly stronger $P_2(\omega_1)^+$.

Fact 3.4 *If every function from ω_1 into ω_1 is bounded on a club by a canonical function, then $P_2(\omega_1)^+$ holds.*

Proof: Fix $F : [\omega_2]^{<\omega} \to \omega_2$ and let $E = \{\alpha < \omega_2 : F``[\alpha]^{<\omega} \subseteq \alpha\}$. E is a club of ω_2. Let S be any stationary subset of ω_1 and let C be any club of ω_1. We inductively build five sequences $(f_n)_n$, $(\alpha_n)_n$, $(\pi_n)_n$, $(g_n)_n$ and $(\eta_n)_n$ as follows:

Let $f_0 : \omega_1 \to \omega_1$ be given by $f_0(\nu) = min(C \backslash (\nu + 1))$. By the bounding principle there is an ordinal α_0, $\omega_1 \leq \alpha_0 < \omega_2$, such that every canonical function for α_0 dominates f_0 on a club. Let $\pi_0 : \omega_1 \to \alpha_0$ be any surjective function and let g_0 be the canonical function for α_0 given by π_0, i.e., $g_0(\nu) = ot(\pi_0``\nu)$ for all ν. Now fix n and suppose α_n has been found. As in the $n = 0$ case let π_n be any function from ω_1 onto α_n and let g_n be the canonical function for α_n given by π_n. Let $f_{n+1} : \omega_1 \to \omega_1$ be given by $f_{n+1}(\nu) = min(C\backslash(g_n(\nu)+1))$. Finally let α_{n+1} be any ordinal below ω_2 such that every canonical function for α_{n+1} dominates f_{n+1} on a club and such that there is some $\eta_n \in E$, $\alpha_n < \eta_n < \alpha_{n+1}$. Again, such an α_{n+1} exists by the bounding principle. $\alpha := sup_n \alpha_n$ belongs to E. From this, from the definition of α_n and from the considerations on canonical functions in Section 1, it follows that, letting $\pi : \omega_1 \to \alpha$ be any surjective function, there is a club $D \subseteq \omega_1$ such that for every $\nu \in D$, $(\pi``\nu) \cap \omega_1 = \nu$ and $\pi``\nu$ is closed under F and, for each n, $f_n(\nu) < g_n(\nu) < f_{n+1}(\nu)$ and $\pi``\nu \cap \alpha_n = \pi_n``\nu = \pi_{n+1}``\nu \cap \alpha_n$. But then, if $\nu \in D$, $ot(\pi``\nu)$ is equal to $sup_n ot(\pi_n``\nu)$, and also equal to $sup_n f_n(\nu)$, which is an ordinal in $C$. Now pick $\nu \in D \cap S$ and let $X = \pi``\nu$. X is closed under F, $X \cap \omega_1 = \nu$ and $ot(X) \in C$. Since C was an arbitrary club of ω_1, this proves that $A = \{ot(X) : X \cap \omega_1 \in S \text{ and } F``[X]^{<\omega} \subseteq X\}$ is a stationary subset of ω_1. Since the mapping sending $\xi \in A$ to $X \cap \omega_1$ (where X is any set of order type ξ witnessing that ξ belongs to A) is regressive, there is a stationary $B \subseteq A$ and a fixed $\nu \in S$ such that for every $\xi \in B$ there is some X closed under F such that $X \cap \omega_1 = \nu$ and $ot(X) = \xi$. Since S was an arbitrary stationary subset of ω_1, this proves that there are club–many $\nu < \omega_1$ such that $\{ot(X) : F``[X]^{<\omega} \subseteq X, \text{ and } X \cap \omega_1 = \nu\}$ is a stationary subset of ω_1. □

We remark that we do not have any proof of $P_1(\omega_1)^+$ from the club bounding principle. Now we come to the main result in this Section. Theorem 3.5 presents the first known model in which $BSPFA$ holds but BMM fails.

Given cardinals λ and κ, $\kappa \longrightarrow (<\omega_1)^{<\omega}_\lambda$ means that for every colouring $c : [\kappa]^{<\omega} \to \lambda$ there is a sequence $\langle I_\alpha : \alpha < \omega_1\rangle$ of subsets of κ such that each I_α has order type α and $c``\bigcup_{\alpha<\omega_1}[I_\alpha]^n$ is constant for each $n < \omega$.

Theorem 3.5 *Con(ZFC + There are cardinals $\lambda < \kappa$ such that λ is Σ_2–reflecting in V_κ, $V_\kappa \models ZFC$ and $\kappa \longrightarrow (<\omega_1)^{<\omega}_\lambda$) implies Con(ZFC + BSPFA + $\neg$BMM).*

Proof: Let V be a ground model in which there are cardinals $\lambda < \kappa$ such that $V_\kappa \models ZFC$, λ is Σ_2–reflecting in V_κ and $\kappa \longrightarrow (<\omega_1)^{<\omega}_\lambda$ but in which ZFC + "There are cardinals $\lambda < \kappa$ such that λ is Σ_2–reflecting in V_κ, $V_\kappa \models ZFC$ and $\kappa \longrightarrow (<\omega_1)^{<\omega}_\lambda$" is inconsistent. Very much as in [G-S]– see also [A-B], Lemma 2.2 and Corollary 2.3–, perform an RCS–iteration of semiproper forcing notions and obtain a semiproper poset $\mathbb{P} \subseteq V_\lambda$ forcing $BSPFA$ over V_κ (in [G-S] a countable support iteration is used to force the Bounded Proper Forcing Axiom, i.e., the bounded forcing axiom for proper posets; here we do the same, except that we use an RCS–iteration instead to deal with semiproper forcing). Let G be generic for $\mathbb{P}$ over V. Of course, G is also $\mathbb{P}$–generic over V_κ and, in $V_\kappa[G]$, $\omega_1 = \omega_1^V$, $\omega_2 = \lambda$ and $BSPFA$ holds. Also, since $\mathbb{P}$ is of small size, standard methods (*cf* [Do-L], Theorem 4.2) show that $\kappa \longrightarrow (<\omega_1)^{<\omega}_{2^{\omega_1}}$ holds in $V[G]$, so that by Theorem 2.2 the following holds in $V_\kappa[G]$: there is an inaccessible cardinal γ such that for all S and T which are stationary subsets of ω_1, $\mathcal{S}^\gamma_{S,T}$ is a stationary subset of $[\gamma]^{\aleph_0}$.

Claim 3.6 $V_\kappa[G] \models \neg BPSR$

Proof: Otherwise, in $V_\kappa[G]$ ψ_{AC} holds (by Corollary 2.3), and so every function from ω_1 into ω_1 is bounded on a club by a canonical function (by Fact 3.1). By [D-Do], this implies that $\omega_2^{V_\kappa[G]} = \lambda$ is an inaccessible limit of measurable cardinals in K as computed in $V_\kappa[G]$. Of course, every measurable cardinal δ is Mahlo (and so there are regular cardinals $\delta' < \delta$ such that $V_{\delta'} \preccurlyeq_{\Sigma_2} V_\delta$) and satisfies the extra Erdős property. This shows that, contrary to our assumption on V, ZFC + "There is a measurable cardinal" is consistent there. □
□

Notice that, by reflection with λ, the hypothesis of Theorem 3.5 implies that V_κ is closed under sharps.

Theorem 3.5 was first proven by Asperó from the stronger assumption that ZFC + "There exists a nearly $<\omega_1$–Erdős cardinal" is consistent. That proof used the fact that ψ_{AC} implies the consistency at least of the existence of a nearly $<\omega_1$–Erdős cardinal– since it implies $P_2(\omega_1)^+$– in order to prove Claim 3.6. Another proof of Claim 3.6 was found by Welch after he observed that BMM and the existence of a cardinal with the weak Erdős property of Definition 1.3 implies the club bounding condition on functions $f : \omega_1 \longrightarrow \omega_1$. The current proof, using Corollary 2.3 and Fact 3.1, is due to Asperó. Note, in answer to a question of the referee, that we can formulate a proof another way: assuming say there exists an inner model with a measurable cardinal, there is a least inner model of ZFC + "$V = K$" containing a cardinal κ which is Mahlo and almost $<\omega_1$–Erdős (i.e., $\kappa \longrightarrow (<\omega_1)^{<\omega}_\xi$ for all $\xi < \kappa$). Then of course there exists $\lambda < \kappa$ which is Σ_2–reflecting in V_κ, so that we may take this model as V and consider the above RCS-forcing as one of length λ over this inner model.

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