Conceptual realism: sets and classes

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Abstract

We outline an extension of Martin's view of a conceptual realism, to a Cantorian realm of absolute infinities. We then formulate a strong reflection principle within this framework to obtain extra-constructible large cardinals.

1 Introduction

This paper aims to address the topic of the foundations of mathematics[1] by considering a question in the foundations of set theory. I shall take it as a starting point that ZFC set theory is a foundation for mathematics, even if there are alternative foundational systems with probable benefits in terms of application, such as category theory for algebraic concepts *etc.* or the emergent homotopy type theory for a constructivist interpretation of mathematics. In general my viewpoint is that any such system can receive its interpretation in terms of set theory or, with some exceptions, variants thereof.

This paper, like Martin's [**?**] on which it is based, focusses on conceptual issues, rather than ontology. To continue stating my personal viewpoint: the Benacerrafian argument ([**?**]) concerning our putative causal relations with mathematical objects is quite decisive. However we have no need to interact, or perceive in some fashion those objects, in a way that is often famously attributed to Gödel due to several of his remarks.

What we do need, and have, are mathematical *concepts*. We have no need to locate the object $\aleph_1$, whatever that might be, in order to formulate the Continuum Problem. Gödel's emphasising the role of perception of mathematical

[1] A paper given at the *"Foundations of Mathematics: what are thay and what they for?"* conference, Cambridge July 2012

objects (*v.* [**?**] p.128), and the similarity to a kind of quasi-empirical research about 'objects' does not seem to advance us further.

My view today I think is closest to what Martin has identified in [**?**] when discussing Gödel's Conceptual Realism. The main tenor of [**?**] is that much, but not all, of Gödel's realism can be construed as being about concepts rather than objects. (For Gödel both concepts and objects have some real existence in some form.) Much of Martin's paper is taken up with discussing various aspects of Gödel's writings on this at various periods, in particular the two versions of the *"What is Cantor's Continuum Problem?"* ([**?**] and [**?**]), and the unpublished 1951 Gibb's Lecture, that we shall not repeat in full here. Briefly put his conclusions are that although Gödel made some oft quoted comments that support a view of robust realism, in any particular instance where, for example, Gödel is talking about using large cardinals to solve certain questions, in fact the realism of the objects in fact turns out to not play any substantial role.

> Probably there are other (hitherto unknown) axioms of set theory which a more profound understanding of the concepts underlying logic and mathematics would enable us to recognize as implied by these concepts (Gödel, [**?**], p182).

According to Martin this is all to do with understanding the concept of set, (perhaps with other concepts) rather than depending on there being an instance of the concept of set. He suggests that the only "perception ... of the objects of set theory" that plays a role in Gödel's account of actual and hoped-for set-theoretical knowledge is "perception" of an indirect concept of set that he adopts, and we outline below. One is tempted to dub this viewpoint "conceptual structuralism" except that that term has already been appropriated by Feferman ([**?**]), for a rather different purpose.

The structure of the paper is as follows: in the next section we look at Martin's indirect notion of "concept of set of x's" from [**?**]. This is interwoven with his extension of these ideas in [**?**], in particular we describe his route from "concept of natural number" to the analogous analysis of "concept of set." In Section 3 we outline how we wish to embrace an extension of Martin for a concept of universe together with its classes, naively thought of as Cantorian absolute infinities, $\mathscr{C}$. In Section 4 we discuss the nature of the collection $\mathscr{C}$. Our longer term aim is to provide some underpinning for reflection principles that are *extra-constructible*, that is that provide large cardinals sufficient to deliver Woodin's absoluteness results, and his program for Ω-logic. Such a Global Reflection Principle we have defined elsewhere ([**?**]), but extend some of this discussion in Section 5.

2 The concept of 'set of'

Martin distinguishes the two senses of 'concept of set' ([**?**] p.212), the latter he calls an "indirect sense" but to avoid a clash with Frege, says simply "my sense":

> My sense differs from the straightforward sense in that instances of a concept of set in the straightforward sense - the objects that fall under the concept - are sets (or, at least, what the concepts are count as sets). The instances of a concept of set in my sense are not sets. There are two versions of my sense. In one version the instances are concepts: straightforward-sense concepts of set. In the other version the instances might be described as set structures or universes of sets.

It is this final 'other version' that I shall want to mostly take here. We aim ultimately to extend Martin's notion of 'concept of set' in this second sense to a similarly indirect sense of 'concept of absolute infinity' (which I may abbreviate to 'concept of (proper) class' or perhaps to be more neutral to 'concept of a part (of V)').

However first (p.213, *ib.*) he would like to disabuse us of the notion that axioms compel an identification of sets:

> A concept of set expressed by axioms such as comprehension axioms cannot put any constraint on which objects count as sets and which do not. Such axioms put constraints on the isomorphism type of set theoretic structure ... a concept of set could count as concept of set in my [indirect] sense even if it determined completely what objects count as sets and what counts as the membership relation. A concept of this sort would have at most one instance: it would allow at most one structure to count as a set-theoretic universe ...

What is ultimately at play here is the point Martin wishes to make that *instantiation* of a concept for mathematics (or set theory) is not needed: what we require is *uniqueness* (up to isomorphism) in order to make sense and understand concepts. He reads Gödel as primarily not needing instantiation in many crucial places: for example, he notes that neither it nor perception of objects plays any significant role in Gödel's justifications of strong axioms of infinity,

His primary point is perhaps plainly put ([**?**] p.215). Consider the Axiom of Extensionality: this axiom does not say what a set is, it only prescribes what it means for any two sets to be equal. The *concept* of set does not determine what it is for *an object* to be a set (as he states in [**?**]). The concept is sufficiently objective (or perhaps better, 'intersubjectively objective'): we understand it, talk about it, as no doubt they do on some other planet with discretely individuated intelligences. (It is not purely a sentimental impulse that has us engrave on steel plates pictures of Pythagoras' theorem and place them on the moon, or send them out on Voyager 23.)

In short we understand the concept 'set of' without having to *perceive* it in some Gödelian manner. Hence:

• Instantiation is not needed either in mathematics or in set theory; thus

• This is closer to a structuralist viewpoint. But to what kind of structuralist viewpoint? Martin says nothing further on this, and we shall return to this at a later stage.

2.1 The concept of *'natural number'*

In his paper for the Exploring the Frontiers of the Infinite Series [**?**] he considers two basic concepts, that of 'natural numbers' (or rather 'ω-sequence'), and 'set of'. He analyses successively these two cases, applying the same analysis in turn to each. He identifies three properties a basic concept may have:

(i) First order completeness: the concept is sufficiently clear and precise to determine truth values for all sentences expressible in the appropriate first order language associated with the concept. (Such truth values may be determined, but without our necessarily being able to know those truth values.)

(ii) Full determinateness: the concept fully determines what any instantiation would be like.

(iii) Categoricity.

A fourth item that might be added here, is whether the concept is a genuine mathematical one. A concept may be a genuine mathematical one, (such as the concept of an ω-sequence), but it may be that it is not fully determinate or first order complete (as he says may hold of the concept of set.)

As a first approach to a discussion of a concept of set, he addresses that of the concept of natural number. This concept yields IPA: Informal Peano Axioms, (not in the usual first or second order sense) which in turn yields categoricity of $\mathbb{N}$. However categoricity alone does not imply first order completeness: there may be no structure instantiating IPA. However he believes in full determinateness for $\mathbb{N}$ and ([**?**], p10):

> I believe that full determinateness of the concept is the only legitimate justification for the assertion that the concept is instantiate or that natural numbers exist.

Whilst neither endorsing or denying the last quotation, I'll go along with it for the present purposes. I shall cut short in any case discussion of the natural numbers for this paper.

2.2 The concept of *set of x' s*

He then applies a similar sequence of considerations for the concept of sets. For him the modern, iterative concept has four important components:

(1) the concept of natural number

(2) the concept of 'set of x's'

(3) the concept of transfinite iteration

(4) the concept of absolute infinity.

He remarks that (1) can be subsumed under (2) and (3). My remark is that (4) is perhaps not on everyone's list of components. He is thinking of the concept of sets as a concept akin to that of a 'structuralist's structure' and thus does not have to add anything as to what kind of things sets are. We adopt this view here. (Martin remains silent as to which flavour of structuralism's structure might be at play here, and we comment on this at the end of the section.) A set structure is then what is obtained by iterating the concept 'set of x's' absolutely infinitely many times letting 'x's' vary that part of the set structure formed by that stage of the induction.

We have only glimmerings of what goes on when considering subsets of $V_{\omega+1}$: is the Continuum Hypothesis true? Is every definable subset of the plane definably uniformisable? So we are hopelessly far from first order completeness. However, when considering subsets of V_ω we are, somewhat recently, in a better position. We now know that adding the assumption of Projective Determinacy to analysis, or to the theory of hereditarily countable sets give us as complete a picture of HC as PA does for V_ω = HF. Martin asks:

Question: Which informal axioms are implied by the concept of set?

He lists two (p.14).

(I) If a and b have the same members, then $a = b$.

(II) For any property P, there is a set whose members are those x's that have P.

The first is Extensionality, and the second is an Informal Comprehension Scheme: informal since “property” is not specified in generality. However any worries about too much informality here can be dispelled, since it will be clear that the few instances we shall use of the Scheme will use clear examples of properties. It is plausible, Martin says, that these axioms fully axiomatise the concept of set of x's.

Martin seeks to further soothe any worries that we need to specify what objects sets are in order to ‘fully understand’ the concept. He will ignore whatever structural constraints one may put on what sets actually are, other than the structural constraints of (I) and (II), and continues as follows:

Theorem 1 *(Essentially Zermelo) Axioms (I) and (II) are categorical: if* $(\mathfrak{V}_1, \in_1)$ *and* $(\mathfrak{V}_2, \in_2)$ *are two structures satisfying (I) and (II) with the same x's, then with each set* $b \in_1 \mathfrak{V}_1$ *of x's, we associate a set of x's,* $\pi(b) \in_2 \mathfrak{V}_2$.

Proof: Let P be the property of being an x such that $x \in_1 b$. By the Informal Comprehension Scheme there is a $c \in_2 \mathfrak{V}_2$ such that

$$\forall x[x \in_2 c \leftrightarrow P(x)].$$

Q.E.D.

Thus axioms (I) and (II) deliver categoricity, and the above is the basis of Zermelo's proof that any two models of ZFC (without *urelemente)* of the same ordinal height are isomorphic.

The notion (3) of transfinite iteration is just that of ordinals or even wellorderings. Martin points out that this makes one have confidence in the full determinateness of small transfinite ordinals or the levels of the L_α-hiearchy associated with them, and he further remarks that an Informal Wellfoundedness Axiom would play the role of Informal Comprehension Axiom here.

Indeed the same argument shows that if the $\alpha \to V_\alpha$ operation is iterated along the absolute infinity of *all* the ordinals, the universes obtained are categorical, and so unique up to isomorphism. As Martin has remarked elsewhere [**?**] the isomorphism argument following Zermelo works here too.

In short, what is unfolded from the iterative concept of set for Martin is the above categoricity fact. We did not need instantiation for the above argument, or indeed to know what *objects* $\{\emptyset\}$, or $\aleph_{23}$ *are.*

Gödel's concept of set (as an object) seems (following the quotations of Gödel that Martin considers) to be built out of a combination of Martin's sense of concept of set plus instantiation. Moreover he seems to believe in instantiation of the iterative concept of set as coming about through soundness of the primitive terms of set theory:

> For if the meanings of the primitive terms of set theory as explained on page 262 and in footnote 14 are accepted as sound, it follows that the set-theoretical concepts and theorems describe some welldetermined reality, in which Cantor's conjecture must be either true or false.(Gödel,[**?**] p260.)

Martin ([**?**] p220) doubts both the fact of instantiation of mathematical objects in general, and its importance for mathematics. For him a concept can be consistent and coherent without being instantiated. He gives the example of nominalists being right and there being only finitely many concrete objects, would have us being finitists about the natural numbers. But does that mean the concept of an ω-sequence was inconsistent or incoherent?

Nor do we, I take it to mean, actually need to assert that any *structure* such as $(\mathfrak{V}, \in)$ actually exists. Again this latter instantiation need not follow from the concept alone. Burgess in [**?**] analyses potential kinds of structuralism into three sorts, of which the first two, the 'eliminating objects' or *in re*, and 'natureless objects' or *ante rem*, (he calls them "hard-headed" and "mystical") are the most prevalent. He also identifies a third possible meaning, the 'arbitrary structure' (picked out by a use of the Hilbertian ϵ-symbol). His discussion centers around the idea introduced by Pettigrew [**?**] and also Shapiro [**?**] of using "an introduced parameter" as means of referring to mathematical concepts not only such as i or $\sqrt{2}$ but also the "the (algebraic) structure of the natural numbers" or the "real closed field" etc. The difficulties of extending structuralism to set theory to deal with all of V he says are well-known. Of the two (or three) kinds the 'mystical' option seems closest to what one might want (I hesitate to claim anything for Martin here) in that we are *talking about a special model whose distinctive metaproperty is to have no distinctive properties* in Burgess's words. Well, I said 'closest', but perhaps for many set theorists, this does not ring very close. Set theorists are probably either more 'formalist', and think of 'constructing' formally very distinctive models (probably by forcing), or else more 'realist' in attempting to ascertain the one true 'V' 's distinctive features. The latters' use of 'V' (as being the 'set of'/structure concept obtained by iterating power set along the ordinals) might be thought to set up 'V' as one of Pettigrew's 'distinctive free variables'. However this would not be within a strictly mathematical discussion, since we wish to restrict the domain of mathematics to sets, and not to include proper class entities such as V. A set theorist of the latter kind may well say "let 'V' be the universe of sets" and mean the one obtained by iterating power set along the ordinals, just as in the phrase "Let $\mathbb{N}$ be the natural number system": here $\mathbb{N}$ is then an example of one of Pettigrew's dedicated free variable.

However one overarching difficulty with using a structuralist approach to the universe of sets, is that usually in the structuralist analysis one has a multitude of potential "natural number structures" or ω-sequences, for example, to indicate one's intentions. But where are these when it comes to set theory? Where are all the V-structures? What is common-or-garden in mathematical activity, namely "taking a copy of the Klein 4-group" and manipulating it, becomes a different matter when "taking a copy of V"; this is not a standard mathematical activity. The difference being that V is the domain of mathematical discourse, but is perhaps not subject to all of the usual mathematical manipulations. Hence the Pettigrew/Shapiro analysis which I read as taking place within a realm where an "introduced parameter" is taken to denote a structural type within a *mathematical domain* of discourse, cannot really be applied to the universe V itself, if that universe is not an element of any such domain.

However Burgess has other reasons for doubting that this form of structuralism can be deployed in the case of set theory. He continues (his emphasis):

> But if that is how set theory is conceived, then there seems to be no room for the activity, important to many set theorists, of going back to an intuitive notion of set motivating the axioms in order to motivate more axioms to settle questions not settled by the existing axioms. Structuralism here ties set theory to a particular axiom system in a way that seems to *block the road of inquiry.*

The difficulty about there being 'no room' seems to be alleviated if one allows for the fact that we are currently at a stage of enquiry where we have no definite knowledge about this 'special model' (or the equivalence class up to isomorphism of this special model). Structuralists' arguments as applied to the natural number structure or the real continuum structure are being applied to concepts that are well-trodden and enjoy virtual unanimity of conception amongst mathematicians. These are structures that appear to mathematicians to be plenitudinously instantiate with all their multiple isomorphic copies. Set Theory, and V, does not have the same status. We have instead an intuitive notion of ordinal, and of informal recursion along On. If we allow ourselves to apply the latter to the power set operation then this gives us our 'set-of' concept, our 'structure'. With that pre-formal perspective, we then formalise the subject and then afterwards our view of V evolves, as we continue to discover more about its properties and potential embedding spectra (a.k.a. potentially new axioms of infinity.)

In conclusion the Martinian concept of set-structure does not seem to fit squarely with the structuralists' notion of structure, and thus Burgess's comments, directed against structuralists, that the latters' activity works against the set theorist seems to miss the target, if that set theorist is of the Martinian persuasion.

3 Stepping up to other absolute infinities.

To set the record straight Martin states that he is dubious about the notion of absolute infinities (p19, [**?**]). This is precisely the point where we want to step up and beyond. Yet it would seem that he might accept the following argument concerning mappings between the ordinal classes without difficulty.

Just as the argument that for any two $\mathfrak{V}_1 = (V_1, \in_1)$, $\mathfrak{V}_2 = (V_2, \in_2)$ obtained by iterating the V_α function throughout all the absolute infinity of ordinals, we have an isomorphism $\pi : (V_1, \in_1) \to (V_2, \in_2)$ (Thm **??**), then we see that $\pi \restriction \mathrm{On}^{\mathfrak{V}_1} : \mathrm{On}^{\mathfrak{V}_1} \cong \mathrm{On}^{\mathfrak{V}_2}$ where $\mathrm{On}^{\mathfrak{V}_i}$ is the absolute infinity of von Neumann ordinals in the model $\mathfrak{V}_i$.

We want to take a Cantorian view, perhaps even a naive view, about absolute infinities. We recognise the *logical* necessity of such: the Russell, Burali-Forti, Cantor arguments force these upon us. If we wish to see what follows as a *logical necessity* from the concept of set (1)-(4) then a consequence of this is acknowledging these arguments. Purloining some terminology from mereology, we may view absolute infinities as the *parts of* V, or rather what is left after we have identified the 'set-sized' parts of V with the corresponding set of V. We continue to use the word 'part' or '(class-sized) part' or 'absolute infinity' but these would seem little different from 'proper class', if the latter are distinguished from properties.

We should like to take a view-point that sees the universe V of sets identified as the realm of all mathematical discourse. Like Cantor we could restrict mathematics to the world of sets, and so elements of V. We don't regard the absolute infinities, such as V itself for example, as strictly mathematical objects or even structures within mathematics. (Very little of mathematics seems to be restricted with this view *pace* a few 'large categories'.)

However, of the parts of V the ordinals occupy a special place. [2] Cantor one assumes would have thought so, and we too see the ordinals as the quintessentially transfinite objects that give set theory (beyond the hereditarily finite sets) its character. Without ω and at least the countable ordinals there is little set theory. We should like to list the concept of ordinal number amongst the 'fundamental concepts' that Martin mentions as named by Feferman [**?**], and that he himself calls 'basic.' This might seem controversial, since Martin only wants

[2] The centrality of the ordinals to Cantor, and to modern set theory is emphasised in [**?**].

to allow concepts that are to some extent atomic, that is not built out of other concepts, and for this he mentions only natural number and the set concept, but would not, presumably, include the concept of von Neumann ordinal which requires the notion of 'transitive set.' However I note that when Martin comes to consider the concept of ω*-sequence* (as opposed to just simply natural numbers), he remarks that although one can define such from sets, he will take the concept of ω-sequence as basic and that consists of some objects coming equipped with a successor function *etc.* or alternatively a successor *relation.* For us we should have to take an ordinal as some objects, together with a predecessor relation, with the additional well ordering requirement.

Whether much turns on our selecting the ordinal concept as basic, I am not sure, but from the ordinals much can be derived[3] when we consider the addition of power set operations and replacement: the V_α hierarchy itself is obtained by iterating the power set operation along the ordinals. In our Cantorian, pre-theoretic thinking, the ordinals, like the natural numbers, are determinate. Before Cantor the natural numbers would have constituted an 'absolute infinity' - he showed us otherwise. Later we come to formalise our set theory and eventually contemplate strong axioms of infinity within the language of that theory, but these do not affect ordinals - they are not 'longer' because we discover/posit/assert that there are inaccessible or measurable cardinals (which are in any case cardinal-theoretic properties, not ordinal-theoretic ones) any more than the natural numbers are 'longer than we thought' because of the Skewes number.

One additional caveat in the above discussion is that our phrasing "the ordinals are determinate", cannot be meant in the strong sense of Martin: *"a concept is fully determinate if it is determined, in full detail, what a structure instantiating it would be like.*([**?**],p5) since he only seems to accept the determinateness of small countable ordinals. Martin does not mind if someone takes full determinateness of a concept to imply instantiations of it exist. His objection is that the concept of set, and presumably the concept of ordinal in generality is so fully determinate.

We take in this paper the view that we do have *sufficient determinateness* of von Neumann ordinals: these are the transitive sets wellordered by $\in$. The fact that we use the concept of set to state this definition, should not mean that we do not fully understand this. There may be uncountable ordinals, inaccessible initial ordinals, *etc.* and these varying 'details' beyond the purely ordinal-theoretic, may be what Martin views as insufficiently determining the concept. However the base concept of the von Neumann ordinal as just defined allows one *given*

[3] again see Jensen [**?**]

any putative instantiation of it, to tell, figuratively speaking, whether it is, or is not, an ordinal. [4] There is a world of difference between asserting this sufficient determinateness of the von Neumann ordinal concept, and, say, that of the concept of power set of $V_{\omega+1}$. We are perhaps cutting the division between instantiations and determinateness in a different way to Martin: whereas he does not mind if determinateness is taken to imply instantiation, we are not saying this for the sufficient determinateness, or whatever amount of determinateness one wants to call it, that determines our description of von Neumann ordinal (again modulo understanding the 'set of' concept). Thus again: sufficient determinateness should not in general imply instantiation.

We therefore let $\mathscr{C}$ denote the collection of the parts of the domain of the universe $\mathfrak{V}$. When talking about a structure with its parts as a predicate such as $\mathfrak{V} = (V, \mathscr{C}, \in)$ we are thinking of a two sorted language with variables $x, y, z, \ldots$ for sets in V, and $X, Y, Z, \ldots$ for the parts in $\mathscr{C}$.

Theorem 2 *If we have two structures of sets $\mathfrak{V}_i = (V_i, \in_i)$ $(i = 1,2)$ satisfying Martin's (1) and (2) above, with collections of parts $\mathscr{C}_i$, we may define an isomorphism $\pi : (V_1, \in_1) \to (V_2, \in_2)$ as before. π then extends to an isomorphism:*

$$\pi : (V_1, \mathscr{C}_1, \in_1) \cong (V_2, \mathscr{C}_2, \in_2).$$

Proof: Let $(V_1)_\alpha$ denote the set of $\mathfrak{V}_1$-sets of rank α in the sense of $\mathfrak{V}_1$ (and similarly $(V_2)_\beta$ etc). It suffices to show for every part $X \subseteq V_1$ (thus X is in $\mathscr{C}_1$) there is a $Y \subseteq V_2$ (and so in $\mathscr{C}_2$) with $\pi(X \cap (V_1)_\alpha) = Y \cap (V_2)_\beta$ where $\alpha \in_1 \mathrm{On}^{\mathfrak{V}_1}$ and $\beta \in_2 \mathrm{On}^{\mathfrak{V}_2}$ with $\pi(\alpha_1) = \beta$; and conversely - since then we may define $\pi(X) = \bigcup_{\alpha \in_1 \mathrm{On}^{\mathfrak{V}_1}} \pi(X \cap (V_1)_\alpha)$. *etc.*, thereby yielding $\pi(X)$ is in $\mathscr{C}_2$. Q.E.D.

Here we are taking the 'informal union' of the sets of the form $\pi(X \cap (V_1)_\alpha)$. However we are not declaring this union to be a 'set' or any such, so no formal axiom is needed. This is unproblematic as it is simply taking a union (or fusion if you will) of the parts $\pi(X \cap (V_1)_\alpha)$ and thus is a part of V_2. A point to be mentioned is that we obtain the map π from Martin's argument at Theorem **??** above which turned on a use of his Informal Comprehension Scheme: nothing further is needed to extend the map to the parts of each universe (the 'informal union' being only an instance of Informal Comprehension). We thus have an extended categoricity theorem for the concept of 'set-structure with its parts.'

[4] We might even argue this as in a set theory class: the notion of "ordinal" is simple; it is a Δ_0 concept in ZF and hence is absolute. Whereas a major part of the indeterminacy of the power set operation is its highly non-absolute nature, which figures in its, necessarily, Π_1 definition.

Much of the above could be given a simple, and natural, explanation in a formal second order logical framework in which the relations $\in_1$ and $\in_2$ are expressed predicates, but we are intentionally restricting our appeal to second order formal methods, and giving an account of informal reasoning that leads to the formalisations that we currently now have.

4 What is the character of $\mathscr{C}$?

I shall write from now on $(\mathfrak{V}, \mathscr{C}, \in)$ since we have argued that this is a conceptual structure unique up to isomorphism. We think of elements of $\mathscr{C}$ as the absolutely infinite parts of V. *Prima facie* there may seem not much that can be said. But there is more to the unfolding of the concept set of/part of.

As we adopted a non-instantiative approach to V we need not feel queasy that we are positing new, instantiated (and large) entities: just as we have adopted a view of $(V, \in)$ as a structure unique up to isomorphism, and seen how we can extend that to a view of V together with its parts, we do not have to say anything further about ontological committment beyond what we have discussed here: we have just taken Martin's concept of set as a set-structure, and considered as parts of V the absolute infinities that perforce must be associated with it.

Question: Which informal axioms follow from the concepts of 'set of/part of' or 'set of/absolute infinity of'?

We ask this question deliberately to mirror the same question of Martin's above concerning the concept of 'set of' alone. Should we be adopting some kind of Informal Comprehension Scheme involving a properties scheme with both sets and parts of V? Well we could, but we have stated that we should like to hold back from too much overtly informal second order reasoning. Thus we might make the following observations about sets and absolute infinities directly: clearly

$$\{(x, x) \mid x \in V\} \text{ and } \{(y, x) \mid y \in x \in V\}$$

are both absolute infinities (here "(y, x)" denotes the usual ordered pair of y and x and later (z, y, x) for ordered triple). Continuing with this idea, and allowing sets to reappear also as parts of V we might be tempted to argue that if X and Y are absolute infinities, then there is some part of V that is their intersection: some Z so that $Z = X \cap Y$. This is informal reasoning, rather than a formalised axiom. Similarly one could claim that a finite number of instances of informal arguments establishes the following informal, but more, or less, intuitive, principles:

(i) For any two parts $X, Y \in \mathscr{C}$ there is a part of the universe $Z \in \mathscr{C}$ which is the collection of all those t which are both in X and in Y.

We have expressed this in English to emphasise the informal nature of the reasoning leading to this conclusion. Similarly:

(ii) For any $X, Y \in \mathscr{C}$ the collection of those t in X but not Y forms a part of V.

Still intending informality, but less ponderously expressed:

(iii) $\forall X \exists Y (Y = V \setminus X)$
(iv)$\forall X, Y \exists Z (Z = X \times Y)$
(v) $\forall X \exists Y (Y = \mathrm{dom}(X))$
(vi) $\forall X \exists Y \forall xyz((x, y, z) \in X \leftrightarrow (z, x, y) \in Y)$
(vii) $\forall X \exists Y \forall xyz((x, y, z) \in X \leftrightarrow (x, z, y) \in Y)$.

Just as Martin would invoke a small number of instances of the informal notion of 'Property' in his Informal Comprehension Scheme (and those properties that he does invoke are defined from the structures involved, which he claims legitimates their use, *ib.* p.16), so we are using a small number of instances of rudimentary reasoning about parts. What we have done is to show that whatever the collection of parts $\mathscr{C}$ is, a small number of instances of informal reasoning leads from simply given parts to other parts, and in particular from absolute infinities to parts (that in some cases are also absolute infinities - but may not be). Of course whatever $\mathscr{C}$ is, if we accept the above we have shown:

Proposition 1
$\mathfrak{V} = (V, \mathscr{C}, \in)$ *satisfies the formal von Neumann-Bernays-Gödel axioms.*

since (i)-(vi) capture Bernays' finite axiomatisation of NBG.[5]

If the reader does not wish to accept this last move, then this will not harm what follows.

[5]Bernays was giving an alternative treatment to von Neumann's presentation [**?**]. von Neumann embraced outright a view of sets and classes - expressed in functional terms, where a domain of functions (*'I Dinge'*)and a domain of arguments (*'II Dinge'*) are postulated, with both intersecting in a domain of argument-functions (*'I.II Dinge'*). An argument-function, that is essentially a characteristic function, is called a set. By distinguishing between functions and argument-functions we avoid the usual Russell and Burali-Forti paradoxes.

5 Global reflection principles

On its own the iterative concept of set says nothing about Reflection, but it is perhaps remarkable that first order reflection is a theorem of ZF due independently to Montague and Levy.

Gödel again:

> "All the principles for setting up the axioms of set theory should be reducible to Ackermann's principle: The Absolute is unknowable. The strength of this principle increases as we get stronger and stronger systems of set theory. The other principles are only heuristic principles. Hence, the central principle is the reflection principle, which presumably will be understood better as our experience increases. Meanwhile, it helps to separate out more specific principles which either give some additional information or are not yet seen clearly to be derivable from the reflection principle as we understand it now." (Wang [**?**].)

Peter Koellner in [**?**] suggests that *intrinsic reflection theorems* are those that derive from the iterative concept of set and moreover these are bound in strength by that of an ω-Erdős cardinal. Such cardinals are consistent with $V = L$ and hence are *intra-constructible* (our term, not his). Koellner in this paper seeks to analyse some suggestions for reflection principles of Tait [**?**] who proposed some as giving large cardinal strength that of measurable cardinals. However Koellner shows that Tait's principles are either inconsistent or intra-constructible. Koellner gives a heuristic argument as to why all intrinsic reflection theorems are intra-constructible.

I find it difficult to see how higher order reflection principles such as those of Bernays which deal with Π^1_n or even Π^m_n reflection schemes, follow from the iterative concept of set. If one takes a Zermelian approach [**?**] which involves a never-ending tower of normal domains indexed by inaccessible cardinals then this potentialist never-to-be-completed universe of sets and domains hardly leaves scope for higher type quantification over 'everything'. Hence it is better to adopt, as Koellner does, an 'actualist' stance where the universe of V is built by iterating the rank function along the absolute infinity of On and that is it: we have the concept of a set structure, that is a universe, and over this we may consider higher type quantifications leading to the satisfaction of some higher type sentence Ψ say. However it is hard to see how we can properly formulate the truth conditions for such a formula Ψ with the tools at hand. The second (or higher) quantifiers have to range over something. One can perhaps do something with the iterative concept plus plural quantification plus reflection thereof, but the higher order reflection needed to get Π^m_n reflection and thence Π^m_n-indescribable cardinals (still intra-constructible) needs further concepts.

Reflection from the iterative concept of sets with classes.

We consider $\mathfrak{V} = (V, \mathscr{C}, \in)$ together with its parts. We make no assumptions concerning it being or not being an NBG-model. We let $\mathscr{L}^+$ be the usual first order language of set theory, augmented with second order variables $X_1, X_2, \ldots$ but without second order quantification. The interpretation of the second order variables from a formula φ in $\mathscr{L}^+$ is that the X_i range over the parts in $\mathscr{C}$.

Formula-by-formula reflection now is unexceptional: fix an $i \le \omega$, then for any $\varphi \in \Sigma_1$:

$$\forall\alpha\exists\beta > \alpha : \forall \vec{x}_i \in V_\beta \forall \overrightarrow{X}_j \in \mathscr{C} : \varphi(\vec{x}_i, \vec{X}_j)^{(V,\mathscr{C},\in)} \leftrightarrow \varphi(\vec{x}_i, \overrightarrow{X_j \cap V_\beta}))^{(V_\beta, V_{\beta+1}, \in)}.$$

Here we have identified the parts of V_β with $\mathscr{P}(V_\beta) = V_{\beta+1}$. This is consonant with what we have done: $V_{\beta+1} = \{X \cap V_\beta \mid X \in \mathscr{C}\}$. Here the strength is rather weak, we have something less than Π^1_1-indescribability, and so are firmly intra-constructible.

However now let us express the ineffability of V together with its parts $\mathscr{C}$ by asking that we have a rich form of reflection that mirrors the whole of $(V, \mathscr{C}, \in)$ down to some $(V_\beta, \mathscr{D}, \in)$ in some very uniform way. We express this by asserting the explicit existence of a *connection*, or *reflecting map* j as follows:

$$\exists j : (V_\beta, \mathscr{D}, \in) \longrightarrow_{\Sigma_1} (V, \mathscr{C}, \in) \qquad (*)$$

where $j \restriction V_\beta = \mathrm{id} \restriction V_\beta$, and the elementarity is Σ_1 in the language $\mathscr{L}^+$.

1) Just as On is a class in $\mathscr{C}$ we have that β is a 'part' of V_β and so is naturally in $\mathscr{D}$. Notice that $j(\beta) = \mathrm{On}$. (This is because

$$\forall\tau(\tau \text{ is an ordinal } \leftrightarrow \tau \in \beta)^{(V_\beta, \mathscr{D}, \in)}$$

is a Π_1 formula about the class β and and so goes up to $(V, \mathscr{C}, \in)$ about $j(\beta)$ which must then equal On.)

2) More generally for $X \in \mathscr{D}$ $j(X) \cap V_\beta = X$.

3) The assumed elementarity (and ZFC holding in $(V, \in)$) ensures that β is a strongly inaccessible cardinal, however as yet nothing has been posited that goes beyond reflection principles obtained by other approaches: $\mathscr{D}$ may be a thin collection of parts of V_β: we may have nothing much more than strong elementarily between $(V_\beta, \in)$ and $(V, \in)$ than is commonplace and still intra-constructible. However now allow $\mathscr{D}$ to contain $\mathscr{P}(\beta) \cap L$, and then we shall be

able to deduce the existence of an embedding $\pi : L \longrightarrow L$. Such is already an extra-constructible principle. We may if we wish think of allowing β and so $\mathscr{D}$ to vary in $(*)$: whenever $\mathscr{D}$ contains $\mathscr{P}(\gamma) \cap M$ and $\exists k : (V_\gamma, \mathscr{D}, \in) \longrightarrow_{\Sigma_1} (V, \mathscr{C}, \in)$ with $k \restriction V_\gamma = \mathrm{id}$ *etc.* for M some canonical inner model like L then we shall typically have some principle as: $\pi : M \longrightarrow M$. The 'fatter' the model M the stronger the principle. For L and other models, such embeddings in fact engender very canonical embeddings derived from iterated ultrapowers. We make no attempt here to require that our embeddings $j, k, \ldots$ be canonical in any way; indeed this would go against our view of asserting downwards reflection or resemblance between $(V, \mathscr{C}, \in)$ and some initial segment. Our Global Reflection Principle GRP_0 is simply the limiting principle of the above spectrum, namely take $\mathscr{D}$ as large as possible:

$$\exists j : (V_\beta, V_{\beta+1}, \in) \longrightarrow_{\Sigma_1} (V, \mathscr{C}, \in) \qquad (\mathrm{GRP}_0)$$

GRP_0 then asserts that there is some strong resemblance between the universe and its parts with this V_β and its parts - at least as far as existential statements about sets are concerned. Whilst the assertion of j's existence is an assertion that there is a $\mathscr{P}(V_\beta)$-sized collection of ordered pairs $(X, j(X))$ of classes these can be thought of as a single $Z = \{(y, X) \mid y \in j(X)\}$. We may thus view j either (I) as a plurality of a small number of parts of V of a particular kind[6], or else (II) the result of a single Σ^1_1-assertion about the existence of such a Z.

Which of these two viewpoints should we take? In the former viewpoint, if we were to replace the plural viewpoint by some standard quantification over classes we *prima facie* should have the assertion of j's existence as a third order expression. As it stands it is either a superplural of pluralities of sets from V, which we may wish to reason away as being no more than a plural; or else if we adopt a mereological view of the classes of V as the parts of V, then we have the assertion of such a j as being of the type that could be expressed as the existence of a plurality of parts. One concern that might be nagging is that we might by reflecting on pluralities of parts of V, be motivating ourselves into inconsistency by having what is tantamount to allowing third order principles to reflect upon: as Reinhardt pointed out, (*v.* [**?**]) third order principles *with third order parameters allowed* quickly becomes inconsistent. However this is not the case, even with this talk of pluralities of parts, the GRP_0 is not reflecting on third order parameters.

However our view is that pluralities do not really earn a place in this discussion: the attempt to circumvent class talk by grafting on this linguistically derived convention goes completely against the grain of our previous arguments:

[6] Both these views are discussed further in[**?**]

the coded second version (II) seems more preferable. Here this becomes more simply a single second order assertion about a number of relations between all elements of $V_{\kappa+1}$ and some corresponding elements of the collection of absolute infinities $\mathscr{C}$. Moreover it does not require that membership in $\mathscr{C}$ be determinate in any way: it asserts only that between the parts of V_κ (namely the elements of $V_{\kappa+1}$) and a small set of absolute infinities that there is such a relational link that allows the first order Σ_1 elementarity to hold. We do not have to quantify over all of $\mathscr{C}$ or even know that we have a "completed" domain $\mathscr{C}$ to potentially quantify over (as we would if we wished to speak about higher order reflection above that even at the level Π^1_2). We may have such a reflection principle without even knowing all of $\mathscr{C}$, or that falling under the concept of 'part of V' is determinate.

Proposition 2 ([?]) $\mathrm{GRP}_0 \Longrightarrow$ *There is an absolute infinity of measurable Woodin cardinals.*

We thus have a strongly *extra-constructible principle*, but moreover one which delivers the proper class of large cardinals needed for Woodin's work. To state only some consequences of this, by the work of Martin & Steel [**?**], and of Woodin (see [**?**]), the above proves the following:

Corollary 1 GRP_0 *implies (i)* Projective Determinacy, *moreover (ii)* $\mathrm{AD}^{L(\mathbb{R})}$, *and (iii) no statement of analysis can be forced to change its truth value by Cohen style set forcing.*

We finally remark that GRP_0 does not imply further large cardinals beyond the following: the principles are consistent relative to that of ZFC and the assertion of the existence of 'weakly sub-compact cardinals' (from which they are derived) but they do not imply any form of sub- or supercompact cardinal. They thus seem to sit at a watershed between those weaker large cardinals and those that imply there are $\mathscr{L}$-elementary embeddings $j : V \longrightarrow M$ with critical point some κ so that $j(\kappa^+) > \sup j``\kappa^+$. (All weaker large cardinals have equality here.) This may look like an arcane technicality, but this 'jump' discontinuity is at the base of many arguments involving, for example, supercompact cardinals and in particular forcing arguments. It is in some sense a natural threshold, but it is somewhat hard to assess exactly its significance.

Thus GRP_0 is strong: it implies the existence of large cardinals needed for Woodin's program, (but some that cannot as yet be incorporated inside canonical inner models as built by the inner model program) but it is not extravagantly strong.

6 Conclusions

We have argued that the natural extension of the concept 'set of' (in the Martinian fashion) to include the logically necessary 'absolute infinities' following on from a Cantorian viewpoint, yields a conceptual framework which in turn entails, it can be argued, an informal axiom scheme of comprehension in the form of the Bernays finite axiomatisation of NBG. We have done this in order to avoid requiring the existence of either sets or of classes as instantiated mathematical objects.

A strong reflection principle, the Global Reflection Principle is then introduced. This can be viewed as the limiting principle of a spectrum of weaker principles starting from the intra-constructible and passing through 'small' large cardinals embedding properties of L [7] Such principles do require the assertion of the existence of a connection or map exemplifying the reflection of simple existential assertions between the universe V together with its absolutely infinite parts, and those of some one V_β together with its collection of parts which we have identified in the strongest case of GRP_0 with $V_{\beta+1}$. GRP_0 then yields proper classes of sufficiently large cardinals to use Martin & Steel's result that Projective determinacy holds, Woodin's results that $\mathrm{AD}^{L(\mathbb{R})}$, and that both these statements as well as any other statements of analysis cannot be changed by Cohen style set-forcing techniques.

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