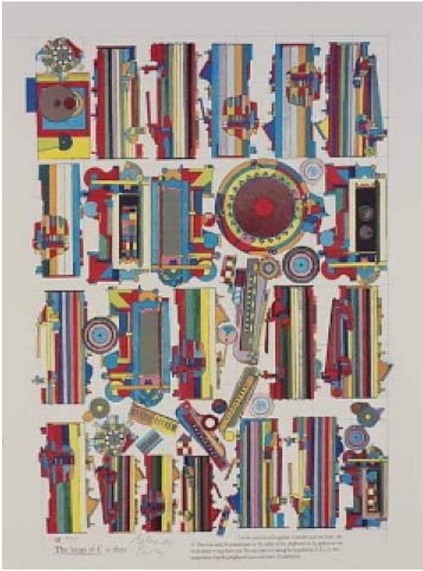
# Generalised transfinite Turing machines and strategies for games.

*P.D.Welch, University of Bristol.* *Chicheley Hall 2012*



• Theme: Connections between inductive operators, discrete transfinite machine models of computation, and determinacy.

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• Part I: ITTM description.
• Part II: Fixed points of operators yielding some strategies.
• Part III: Generalising Operators and Machines

# Part I: ITTM description[1]

• Allow a standard Turing machine to run transfinitely using one of the usual programs $\langle P_e \mid e \in \mathbb{N} \rangle$.
• Alphabet: $\{0, 1\}$;
• Enumerate the cells of the tape $\langle C_k \mid k \in \mathbb{N} \rangle$.
Let the current instruction about to be performed at time $\tau$ be $I_{i(\tau)}$;
Let the current cell being inspected be $C_{p(\tau)}$.

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Let the current cell being inspected be $C_{p(\tau)}$.
• Behaviour at successor stages $\alpha \to \alpha + 1$: as normal.
At limit times $\lambda$: (a) we specify cell values by:

$$C_k(\lambda) = \mathit{Liminf}_{\beta \to \lambda} C_k(\alpha)$$

(where the value in $C_k$ at time $\tau$ is $C_k(\tau)$).

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(where the value in $C_k$ at time $\tau$ is $C_k(\tau)$).
(b) we also (i) put the R/W to cell $C_{p(\lambda)}$ where

$$p(\lambda) = Liminf^{*}_{\alpha \mapsto \lambda} \{p(\beta) \mid \alpha < \beta < \lambda\};$$

(ii) set

$$i(\lambda) = Liminf_{\alpha \mapsto \lambda} \{i(\beta) \mid \alpha < \beta < \lambda\}.$$

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• Hamkins & Lewis proved there is a *universal machine*, an $S^m_n$*-Theorem*, and a *Recursion Theorem* for ITTM's, and a wealth of results on the resulting ITTM-*degree theory.*
• We may define halting sets:

$$H = \{(e,x) \mid e \in \mathbb{N}, x \in 2^{\mathbb{N}} \wedge P_e(x)\downarrow\}$$

$$H_0 = \{(e,0) \mid e \in \mathbb{N} \wedge P_e(0)\downarrow\}$$

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Q. What is $H$ or $H_0$?
Q. How long do we have to wait to discover if $e \in H_0$ or not?
Q. What are the ITTM (semi)-decidable sets of integers? Or reals?

# A Kleene Normal Form Theorem?

We'd like some type of a *"ITTM Normal Form Theorem"*:

## Theorem

*There is a* universal predicate $\mathfrak{T}$ *which satisfies* $\forall e \forall x$*:*

$$P_e(x) \downarrow z \quad \leftrightarrow \quad \exists y \in 2^{\mathbb{N}} [\mathfrak{T}(e, x, y) \wedge Last(y) = z].$$

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• However for this to occur we need to know whether the ordinal *length* of any computation is capable of being output or *written* by a(nother) computation.

# The $\lambda, \zeta, \Sigma$-Theorem[2]

### Theorem

*Let $\zeta$ be the least ordinal so that there exists $\Sigma > \zeta$ with the property that*

$$L_\zeta \prec_{\Sigma_2} L_\Sigma; \qquad (\zeta \textit{ is } \text{“}\Sigma_2\text{-extendible”.})$$

*(i) Then the* universal ITTM *on integer input first enters a loop at time $\zeta$.*

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*Let $\lambda$ be the least ordinal satisfying:*

$$L_\lambda \prec_{\Sigma_1} L_\zeta .$$

*(ii) Then* $\lambda = sup\{\alpha \mid \exists e\ P_e(0)\downarrow in\ \alpha\ steps\}$
$= sup\{\alpha \mid \exists e\ P_e(0)\downarrow y \in WO \wedge ||y|| = \alpha\}.$

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- As a corollary one derives the Normal Form Theorem and:

## Corollary

$$H_0 \equiv \Sigma_1\text{-}Th(L_\lambda).$$

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# Part II: Over $\mathbb{N}$ $\quad \Pi^1_1$-$IND = \Game\Sigma^0_1$

- The *game* quantifier $\Game$:

## Definition

A set $A \subseteq \mathbb{N}$ is $\Game\Gamma$ if there is $B \in \mathbb{N} \times \mathbb{R}$ so that:

$$n \in A \iff \text{Player } I \text{ has a winning strategy in } G_{B_n}$$

where $B_n = \{x \in \mathbb{R} \mid B(n, x)\}$.

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## Theorem (Folklore)

*A set $A \subseteq \mathbb{N}$ is (mon.-$\Pi^1_1$)-IND iff it is $\Game\Sigma^0_1$ iff it is $\Sigma_1(L_{\omega_1^{ck}})$.*

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$Z_2 \nvdash \Sigma^0_4$-*Det.*

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*$Z_2 \nvdash \Sigma^0_4$-Det.*

Q Are strategies for $\Sigma^0_3$ sets ITTM-semi-decidable? Thus: are they $\Sigma_1(L_\Sigma)$?

## Definition

Let "ITTM" abbreviate: "$\forall X(H_0^X$ exists )"
("the complete ITTM-semi-decidable-in-$X$ set exists").

[3] Welch *" Weak Systems of Determinacy and Arithmetical Quasi-Inductive Definitions", JSL, June 2011*

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*The theories:*

$$\Pi^1_3\text{-}\mathsf{CA}_0, \Delta^1_3\text{-}\mathsf{CA}_0 + \Sigma^0_3\text{-}\mathsf{Det}, \Delta^1_3\text{-}\mathsf{CA}_0 + \mathsf{ITTM}, \Delta^1_3\text{-}\mathsf{CA}_0$$

*are in strictly descending order of strength*[3].

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# Open Questions

Q1 *Give another description of the least* $\beta$ *over which strategies for* $\Sigma^0_3$ *sets are definable.*

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*Conjecture 1:* Lubarsky's 'Feedback-ITTM's' are related to this.

# Lubarsky's Feedback ITTM's[4]

Lubarsky has suggested a variant of the HL-machine which is allowed to make calls to sub-routines to obtain the answer to the question:

"*Does the ITTM-computation $P_e$ with the current real $x$ of the scratch tape halt or loop?*".

• In other words one considers ITTM-computations recursive in $H$.

• Clearly such an FITTM may make an infinite chain of such calls in which case Lubarsky calls the computation "freezing".

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• Clearly such an FITTM may make an infinite chain of such calls in which case Lubarsky calls the computation "freezing".

• The natural *Conjecture 1* emerges that any winning strategy for a $\Sigma^0_3$ game which wins for player $I$ can be written by an FITTM.

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*(i) If $\gamma < \beta$ is least with $L_\gamma \prec_{\Sigma_1} L_\beta$ then $\gamma$ is the closure ordinal of monotone-$\Game\Sigma^0_3$-Inductive Operators.*
*(ii) $\Sigma_1$-Th$(L_\gamma)$ is a* complete $\Game\Sigma^0_3$ set.

# Part III: Hypermachines[5]

• Can we find 'machines' that will lift the $\Sigma_2$ "*Liminf*" of [HL] to a $\Sigma_n$-rule at limit stages?

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*For any $n \geq 2$ there is such a $\Sigma_n$ limit rule, so that for a machine running under such a rule, the universal $\Sigma_n$-machine on integer input first enters a loop at the least $\zeta(n)$ s.t. $\exists \Sigma(n) > \zeta(n)$ with*

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• Such machines then compute, taken as a whole, all the reals of the least $\beta$-model of analysis $2^\omega \cap L_{\beta_0}$.

• Then, *e.g.* using Montalban-Shore, strategies for $n$-$\Sigma^0_3$ games are computable by the $\Sigma_{n+2}$-machines.

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Q4 *Quantify* $Det(\boldsymbol{\Gamma_n})$.

A sample theorem of what is known:

### Theorem

*ZFC* $+$ *Det*$(\boldsymbol{\Gamma_2}) \Rightarrow$ *There is an inner model with a proper class of strong cardinals*[6].

[6] W: *Determinacy in Strong Cardinal Models*, JSL, June 2011