P. D. Welch

Part I Construction of some semantical models:

(i) Kripkean fixed point models: Strong Kleene

(ii) Revision Theory: Herzberger sequences, Gupta–Belnap Revision Theory

(iii) Field's model(s).

Part II Analysis - Ramifications and connections

- Some mathematical analysis

- The 'interior' of a semantical model.

- connections: generalized recursion theory, proof theory, constructible sets, infinite computational models, (quasi) inductive definitions.

The Kripkean Strong Kleene model

Given an appropriate first order model $\mathcal{M}$ with a suitable language $\mathcal{L}$, we expand the latter to $\mathcal{L}_T$ by adding a unary predicate letter $T(v_0)$. The interpretation of T will be *partial*, and will be interpreted in $\mathcal{M}^+ = (\mathcal{M}, (E, A))$. We assume that the (denotations of the names of) sentences of $\mathcal{L}_T$ are elements of $|\mathcal{M}|$. and $E \cap A = \varnothing$.

- *Truth* or *Falsity* of $\mathcal{L}_T$-sentences in $\mathcal{M}^+$ is determined by the following positive inductive rules:

If σ is an atomic in $\mathcal{L}$, then σ is true (false) in $\mathcal{M}^+$ iff it is so in $\mathcal{M}$.
If σ is $T(\tau)$ then it is true (false) in $\mathcal{M}^+$ iff $\tau^{\mathcal{M}} \in E$ $(\tau^{\mathcal{M}} \in A)$;

$\neg\varphi$ is true in $\mathcal{M}^+$ iff φ is false ; $\neg\varphi$ is false in $\mathcal{M}^+$ iff φ is true;

$\varphi \vee \psi$ is true in $\mathcal{M}^+$ if one of φ, ψ is so true;

$\varphi \vee \psi$ is false in $\mathcal{M}^+$ iff both φ, ψ are so false.

$\exists v_0(v_0)$ is true in $\mathcal{M}^+$ iff, for some $a \in \mathcal{M}, \varphi(\overline{a})$ is true.

$\exists v_0(v_0)$ is false in $\mathcal{M}^+$ iff, for all $a \in \mathcal{M}, \varphi(\overline{a})$ is false.

The minimal fixed point

- We thus have an *inductive operator* $\Gamma = \Gamma_{\mathrm{sk}}: \ \Gamma((E,A)) = (E', A')$ which is *monotone* (meaning that if $(E,A) \subseteq (F,B)$ then $\Gamma((E,A)) \subseteq \Gamma((F,B))$).

- We then may iterate: $E_0 = A_0 = \varnothing$ and
$\Gamma((E_\alpha, A_\alpha)) = (E_{\alpha+1}, A_{\alpha+1}) \quad ; \quad (E_\lambda, A_\lambda) = \bigcup_{\alpha<\lambda}(E_\alpha, A_\alpha)$ (for λ a limit ordinal).

Then for some α_0 we reach $\Gamma((E_{\alpha_0}, A_{\alpha_0})) = (E_{\alpha_0}, A_{\alpha_0})$. We thus may define the *minimal partial fixed point*. We thus have for any τ denoting a sentence of $\mathscr{L}_T$, that

$$\left(\mathscr{M}^+, \ (E_{\alpha_0}, A_{\alpha_0})\right) \models \tau^{\mathscr{M}} \quad \text{if and only if} \quad \left(\mathscr{M}^+, \ (E_{\alpha_0}, A_{\alpha_0})\right) \models T\left(\tau^{\mathscr{M}}\right).$$

- If the structure $\mathscr{M}$ *admits a coding scheme* or in some way allows for diagonalisation, the usual Tarskian argument shows that the model $\left(\mathscr{M}^+, \ (E_{\alpha_0}, A_{\alpha_0})\right)$ cannot be *classical*, that is $A_{\alpha_0} = \mathrm{Sent}^{\mathscr{M}} \setminus E_{\alpha_0}$.

(This is Kripke's insight: we may nevertheless have partial fixed points.)

Pros and Cons

Given diagonalisation:

- *Pros:* a) The naturalness of the scheme.

  b) A *liar sentence* $\lambda \equiv \neg T(\ulcorner \lambda \urcorner)$ is not in $E_{\alpha_0} \cup A_{\alpha_0}$ (or any other fixed point). This illustrates the true paradoxicality of the Liar.
  (A *truth-teller* $\tau \equiv T(\ulcorner \tau \urcorner)$ is also not in the minimal fixed point (but possibly in other f.p.'s).)
  Why? There are never *grounds* in the inductive scheme to add them to any $E_\beta \cup A_\beta$.
- *Cons:* a) The fixed point is not closed under FO logical consequence: there are many σ with $\sigma \vee \neg\sigma$ failing to gain a truth value. b) In particular:

  $\neg T(\ulcorner \lambda \urcorner) \vee T(\ulcorner \lambda \urcorner)$ or again $\neg(T(\ulcorner \lambda \urcorner) \vee T(\ulcorner \neg\lambda \urcorner))$

  are also *semantically defective*. In the object language $\mathscr{L}_T$ the latter fails to gain a truth value, but metalinguistically where we are using 2-valued classical logic we recognise its truth. Similarly defective is:

  $\forall v_0 (T(v_0) \longrightarrow T(v_0))$

• Note c) that we cannot express this semantic defectiveness within the object language.

An axiomatisation of the minimal fixed point

However:

- We may effect an axiomatisation of $E_\infty = E_{\alpha_0}$ by saying that it is the set of consequences of the true atomic and negation of atomic sentences of $\mathscr{L}$ in $\mathscr{M}$, and closed under the following rules of inference: $\vee$-Introd, $\neg\vee$-Introd, $\wedge$-Introd, $\neg\wedge$-Introd, $\neg\neg$-Introd, T-Introd, and $\neg T$-Introd, and

$$\frac{\varphi(v_0/c_m)}{\exists v_0\varphi} \qquad \frac{\neg\varphi(v_0/c_m)}{\neg\forall v_0\varphi} \qquad \frac{\{A(x/c_u) \mid u \in \mathscr{M}\}}{\forall xA} \qquad \frac{\{\neg A(x/c_u) \mid u \in \mathscr{M}\}}{\neg\exists xA}$$

- Note that if $\sigma \in E_\infty$ then so is $\sigma \leftrightarrow T(\ulcorner\sigma\urcorner)$. That is the T-biconditionals hold for any σ in the fixed point, but *not* for all σ. Also for $\sigma \in E_\infty$ we have the *Intersubstitutivity Principle* that σ can be substituted for any sub formula $T(\ulcorner\sigma\urcorner)$ or vice versa.

- Thought of as a rule of inference we see that E_∞ (indeed any f.p.) is closed under this scheme.

‘Defects’ of the Kripkean construction

To summarise:

- The f.p.’s of the Kripkean scheme validate the Intersubstitutivity Principle, but not the T-biconditional scheme.

- Indeed there is no useful conditional: since we do not have the law of excluded middle for the f.p.’s we do not have $\sigma \rightarrow \sigma$ say.

- There is no way to express within $\mathscr{L}_T$ that the liar sentence (or other similar sentences) are semantically defective, *i.e.* , they do not acquire a truth value in f.p.’s.

The Revision Theoretic construction

- Herzberger, Gupta, Belnap. For simplicity we now take $\mathcal{M} = (\mathbb{N}, +, \times, 0, \ldots)$, then $\mathcal{M}^+ = (\mathbb{N}, +, \times, 0, \ldots, H)$ with $T^{\mathcal{M}^+} = H$.

- We take a *fully interpreted T-predicate* with some (usually empty) assignment $H_0 \subseteq \mathrm{Sent}_{\mathcal{L}_T}$ which we then *revise* according to the standard Tarskian truth clauses:

$$H_{n+1} = \Gamma(H_n) = \{\ulcorner \sigma \urcorner \mid (\mathcal{M}^+, H_n) \models \sigma\}.$$

- $\left(\mathcal{M}^+, H_n\right) \models \sigma \Rightarrow \left(\mathcal{M}^+, H_{n+1}\right) \models T(\sigma)$ (to restate the last).
- Limit Rule (Herzberger):

$$H_\mu = \mathrm{Liminf}_{\alpha \to \mu} H_\alpha = \bigcup_{\beta<\mu} \bigcap_{\beta<\alpha<\mu} H_\alpha$$

Or:

$$\sigma \in H_\mu \Leftrightarrow \exists \beta < \mu \forall \alpha (\beta < \alpha < \mu \to \sigma \in H_\alpha)$$

- All *arithmetic truths* (so expressed in $\mathcal{L}$) are in H_1.
- Nothing monotonic here: $\lambda \in H_n \longrightarrow \lambda (\equiv \neg T(\lambda)) \notin H_{n+1} \longrightarrow \lambda \in H_{n+2}$.

H-Stability sets

The *stable truth set* (based on H_0) is: $H_\infty = \text{Liminf}_{\alpha\to\text{On}} H_\alpha$.

- Commonly one takes $H_0 = \varnothing$.
- (Herzberger) There is some least pair $\rho, \pi < \omega_1$ so that $H_\rho = H_\infty$, and $H_{\rho+\pi.\iota} = H_\rho$ all $\iota < \text{On}$.

Arithmetic quasi-inductive definitions

(Gupta-Belnap, Burgess) Given any predicate $G(v_0)$ defined by some arithmetic formula $\Phi_G(v_0, G)$ we may define

$$G_0 = \varnothing; \quad \Gamma(G_\alpha) = G_{\alpha+1} =_{\text{df}} \{n \mid \langle \mathbb{N}, G_\alpha \rangle \models \Phi[n, G_\alpha / T]\}$$

$$Lim(\mu): \ G_\mu = \text{Liminf}_{\alpha\to\mu} G_\alpha; \qquad G_\infty = \text{Liminf}_{\alpha\to\text{On}} G_\alpha$$

(AQI - Burgess)

Call a set $Y \subseteq \mathbb{N}$ *AQI*, if Y is (1-1) reducible to some such G_∞ given by an arithmetic Φ_G.

Generalized Revision Theories of Truth

Gupta, Belnap, defined much wider variants of revision sequences: they

- quantified over all possible starting arrangements $G_0, B_0 \subseteq \mathscr{M}$;
- Belnap advocated complete freedom of choice of *limit rule*.

The 'true' sentences are those that survive, and become stably true, in *all* such varied revision sequences. This implicit quantification over the whole real continuum, entails that the stable truths for them form Π^1_2-complete sets of integers.

- Gupta further defined a strengthened notion of truth 'categorical truth', and this notion turned out to be Π^1_3 complete over $\mathbb{N}$.

What's in or out of a Herzberger Revision sequence.

- As for Strong Kleene/Kripke, most self-referential sentences do not have stable semantic values. For *stable* σ, $T(\sigma)\leftrightarrow\sigma$ is in H_∞, but the full T-biconditional scheme is not.

- The Intersubstitutivity Principle holds in full (note $\sigma, T(\sigma)$ always have the same semantic value at stage $\infty = On$).

Field's construction: objectives

Construct a theory of truth with a 2-place conditional operator $\curvearrowright$ in which

- We have the *Principle of Intersubstitutivity* (it is harmless as regards truth value to substitute $T(\ulcorner\sigma\urcorner)$ for σ anywhere and v.v.).

- We have the full T-biconditionals $T(\ulcorner\sigma\urcorner) \curvearrowleft\curvearrowright \sigma$ for all sentences σ.

- We may express the defectiveness of the liar, or strengthened liars, or other sentences, thus hoping to create a *revenge immune* system.

Field's construction

- Takes place over a model (any countable acceptable model, or $\mathbb{N}$, or could be V_κ), but take $\mathcal{M} = (\mathbb{N}, +, \times, \ldots, T)$ in a language $\mathcal{L}_{T,\curvearrowright}$.
- It seeks to assign semantic values from $\{0, \frac{1}{2}, 1\}$ to sentences σ of $\mathcal{L}_{T,\curvearrowright}$ in a recursive fashion defining models $\langle \mathcal{M}_\alpha \mid \alpha \in \mathrm{On} \rangle$. The domain of these models is constant as $\mathbb{N}$.
- We describe $\mathcal{M}_\alpha$, assuming, for $\delta < \alpha$ that we have an assignment $|\cdot|_\delta$ of semantic values to any sentence, and in particular to any conditional $\sigma \curvearrowright \tau$ (we'll write $|\sigma|_\delta$ or $|\sigma \curvearrowright \tau|_\delta = j$ for some $j \in \{0, \frac{1}{2}, 1\}$). *Firstly*:

$$
\text{(i)} \quad |A \curvearrowright B|_\alpha \begin{array}{lll} = & 1 & \text{if } \exists\beta < \alpha \forall\gamma \in [\beta, \alpha)(|A|_\gamma \leq |B|_\gamma) \\ = & 0 & \text{if } \exists\beta < \alpha \forall\gamma \in [\beta, \alpha)(|A|_\gamma > |B|_\gamma) \\ = & \frac{1}{2} & \text{otherwise.} \end{array}
$$

(ii) Reset all truth values to 1/2: $T(\ulcorner \sigma \urcorner) = 1/2$;
(iii) Construct the least Kripkean Strong Kleene fixed point, with resulting semantic values $|\sigma|_\alpha$.

Field's construction contd.

- $\sigma \wedge (\top \curvearrowright \sigma)$ expresses in Field's model "σ is true at this stage and was so before".
- Compare with Herzberger sequence "$\sigma \wedge T(\sigma)$".

For our purposes here, we may define:

$$F_\beta =_{\mathrm{df}} \left\{ \langle \ulcorner A \curvearrowright B \urcorner, 1 \rangle : |A \curvearrowright B|_\beta = 1 \right\} \cup \left\{ \langle \ulcorner A \curvearrowright B \urcorner, 0 \rangle : |A \curvearrowright B|_\beta = 0 \right\}.$$

At limit stages Field uses the Liminf ruling to give semantic values to conditionals. Hence $F_\mu = \mathit{Liminf}_{\alpha \to \mu} F_\alpha$ too.

Determinateness

Field uses this to express ‘determinateness’ : $D(\sigma) \equiv \sigma \wedge (\top \frown \sigma)$.

- Applied to the Liar this becomes $D(\lambda) \equiv \lambda \wedge (\top \frown \lambda)$. But this *always* has semantic value 0. So the Liar is *determinately false* even if it itself is given sem. value 1/2. Field would say then that this expresses within $\mathscr{L}_{T,\frown}$ the indeterminateness of the Liar.

- But this is insufficient to give revenge immunity: $\lambda_1 \equiv \neg D(T(\lambda_1))$:

$$||D(\lambda_1)|| = 1/2 \text{ but } ||\neg DD(T(\lambda_1))|| = 1.$$

- We thus have a hierarchy of *determinateness operators* and parallel *liars*:

$$D^{\alpha+1}(\sigma) := D(D^{\alpha}(\sigma)) \qquad \text{and} \qquad \lambda_\beta \equiv \neg D^{\beta}(T(\lambda_\beta) \text{ for } \alpha, \beta < ???$$

Questions

- Q1 Can we describe or characterise either the Herzbergerian *stable truth* set H_∞ or the Fieldian model's *ultimate truth* set F_∞?

- Q2 Is there some axiomatisation of either H_∞ or F_∞ - thus something corresponding to *KF* for the Kripkean Strong Kleene minimal fixed point (or Cantini's *VF* for the Kripkean super valuation version)?

- Q3 Can we give some definite meaning to Field's "path independent hierarchies"? Can we calculate the length of possible determinate hierarchies? Can we find strengthened liar sentences of the model that diagonalise past them?

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Intricately tied up with the Gödel hierarchy of constructible sets:

$$L_0 = \varnothing; L_{\alpha+1} = \mathrm{FODef}(\langle L_\alpha, \in\rangle); \quad L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha; \qquad L = \bigcup_{\alpha\in\mathrm{On}} L_\alpha.$$

(Kripke-Platek set theory KP)

A subset of ZF which restricts Separation to Δ_1-expressions, and Replacement to Σ_1. A model of KP is called *admissible*.
α is called *admissible* if L_α is an admissible set.

Fact:
The least $\alpha > \omega$ with L_α admissible, is ω_1^{ck} - the least *non-recursive ordinal*.

Admissible ordinals τ_α

Let τ_α enumerate the admissible ordinals in increasing order.

HYP - the hyperarithmetic sets of integers.

- It is possible to give a characterisation of the sets, H, of integers in $L_{\omega_1^{ck}}$: such a set H may be construed as constructed from computable sets by a computable process of taking unions and complements; this can be ordered schematically by a *finite path tree* $T = T(H)$ of rank some recursive ordinal $\beta < \omega_1^{ck}$.

- We may thus construe a typical HYP set as one for which we have a computable protocol for its construction, and for testing whether $?n \in H$?

- This analysis is due to Kleene, and was formerly carried out using a system $\mathscr{O} \subseteq \mathbb{N}$ of numbers standing as *notations* for the recursive ordinals: this yields $\langle H_a \mid a \in \mathscr{O} \rangle$ and any HYP set H is (1-1) reducible to some H_a.

- $\mathscr{O}$ essentially is a *tree* of numbers under a suitable partial ordering $<_{\mathscr{O}}$. There are many maximal paths through $\mathscr{O}$ that result in Π^1_1-sets, and indeed $\mathscr{O}$ is itself a complete Π^1_1-set.

More on the least Kripkean Strong Kleene fixed point

Facts:

• Any Π^1_1 set of integers is $\Sigma_1(L_{\omega_1^{ck}}, \in)$.

• It is possible to express the Gödel codes of the extension of the least Kripkean fixed point as a Π^1_1 set of integers. Indeed both it, and:

$$T = T^1_{\omega_1^{ck}}, \text{ the } \Sigma_1\text{-theory of } (L_{\omega_1^{ck}}, \in)$$

are *complete* Π^1_1 sets of this form and are *recursively isomorphic*, that is there is a paper and pencil algorithm for converting members of one set into members of the other in a bijective fashion:

$$E_\infty \equiv_1 T^1_{\omega_1^{ck}}$$

Determinateness hierarchies

Q1 How long can a determinateness hierarchy be?
Q2 Can we use such considerations to avoid "revenge problems", or is it the case that we may define 'super-liars' whose self-referentiality cannot be captured by some determinateness operator such as some D^{α}?
Field distinguishes *internally* and *externally definable paths* through his model. What would constitute an internal path? A (copy of) ω is such a path, and then we can define $\langle D^{`n'}(v_0) \mid `n \in \omega' \rangle$ as an iteration of D *along* this path. Such a path is far from maximal, and we could define easily a liar λ_{ω} that diagonalises past all such D^n, and comes back to haunt us as a sentence whose indeterminateness is not expressible by any sentence involving (finitely many) of these $D^{`n'}$, or any other $D^{`n''}$.
So what paths can be defined, and which are 'internal' to the model $\mathscr{M}_{\infty}$ for which we may define such iterations and such liars?

First step: the lengths of the hierarchies & stability set results

- The Herzberger and Fieldian hierarchies are of the same length

Let (ζ, Σ) be the lex. least pair of ordinals with $L_\zeta \prec_{\Sigma_2} L_\Sigma$.

Theorem

Let T^2_ζ be the Σ_2-$Th(L_\zeta, \in)$.
Let H_∞ be the stable truth set of Herzberger.
Let F_∞ be Field's 'ultimate truth' set. Then:

$$T^2_\zeta \equiv_1 H_\infty \equiv_1 F_\infty$$

(ii) Indeed ∞ can be replaced with ζ: $F_\infty = F_\zeta = F_\Sigma$ and likewise for H_∞.
(Burgess: $H_\infty \leq_1 T^2_\zeta$)

Second step: analysis of the $\langle H_\alpha \rangle$ and $\langle F_\alpha \rangle$ hierarchies

- Both the L_α and H_α hierarchies are iterated Tarskian definability. They should be related.

Uniform Definability Theorem -H

(i) There is a single uniform method of arithmetically defining the whole sequence $\langle H_\gamma \mid \gamma < \beta \rangle$from H_β for any $\beta < \Sigma$. This method is uniform in the sense that it is independent of β.
(ii) The same as (i) with the Fieldian sets F_γ replacing H_γ.

Theorem

For all $\alpha < \Sigma$: $\quad T_\alpha$ is uniformly r.e. in $H_{\alpha+1}$.

Doing this for $\langle F_\alpha | \alpha < \Sigma \rangle$

- F_α's jump through not *successive* levels of the L_α but through *successive admissible* levels L_{τ_α}.

Nevertheless:

Uniform Definability Theorem - F

Just the same as the theorem for the H's: just replace 'H' by 'F' everywhere.

Theorem

For all $\alpha < \Sigma$: $\quad T_{\tau_\alpha}$ is uniformly r.e. in $F_{\alpha+1}$.

Path Hierarchies revisited

- Field is seeking to find *'path-dependent hierarchies'* that are based on bivalently definable paths by some binary predicate $A(x,y)$ so that one can define iterates of D along the path: $D^x(v_0), \ldots D^y(v_0)$ etc. These are *internally defined.*

- What do we mean by internal?
By example, of the Kripkean Strong Kleene minimal fixed point E_∞, and in $\mathscr{L}_T$, it can be shown:
For any wellorder $R \in$ HYP, there is a $P_R(v_0, v_1)$ defying R (meaning that $nRm \longleftrightarrow P_R(\bar{n}, \bar{m}) \in E_\infty$) and so that

$$m \in Field(R) \longrightarrow \forall q \in \mathbb{N} \left(P_R(\bar{q}, \bar{m}) \in E_\infty \vee P_R(\bar{q}, \bar{m}) \in A_\infty \right).$$

Such R we may dub as being "internal to the model". Consequently we could define using $D_H(\sigma) \equiv \sigma \wedge T(\sigma)$ iterations along P_R.

- But actually he, and we, shall want *'path-independent hierarchies'* - not ones that are constricted to 'ordinals' bivalently definable within the model.

For Field's model

- What is internal? These will be wellorderings R for which there is a $P_R(v_0, v_1)$ so that $nRm \longleftrightarrow ||P_R(\bar{n}, \bar{m})|| = 1$ and as before that

$$m \in Field(R) \longrightarrow \forall q \in \mathbb{N} \left(||P_R(\bar{q}, \bar{m})|| \in \{0, 1\} \right).$$

- Idea: we use *sentences as notations* for ordinals. If $||A|| = 1$, then let

$\rho(A)$ be the least μ such that $\forall \nu > \mu \; |A|_\nu$ is constantly 1 (or constantly 0).

A will be a notation for $\rho(A)$.

Lemma

There is a predicate $P_\prec$ so that:

$$\begin{aligned} \|P_\prec(\ulcorner A \urcorner, \ulcorner B \urcorner)\| &= 1 \text{ iff } \rho(A)\downarrow, \rho(B)\downarrow \wedge \rho(A) \leq \rho(B) \\ &= 0 \text{ iff } \rho(A)\downarrow, \rho(B)\downarrow \wedge \rho(A) > \rho(B) \\ &= \tfrac{1}{2} \text{ otherwise.} \end{aligned}$$

Lemma

$$o.t.(\prec) = \sup \left\{ o.t.(R) : R \text{ is internally definable} \right\} = \zeta.$$

Spector Classes and PreWellOrders.

Spector Classes

A class $\Gamma \subseteq P(\mathbb{N})$ is called a *Spector class* if it satisfies [*roughly*]:
(i) [Some basic closure conditions], closure under $\forall^{\mathbb{N}}, \exists^{\mathbb{N}} \ldots$
(ii) Has an *enumerating set*
(iii) Satisfies the *prewellordering property* PWO(Γ) : for any $A \in \Gamma$, there is $R \in \Gamma \cap P(\mathbb{N}) \times P(\mathbb{N})$ with R a PWO of A.

This is what we are using:
(i) that Π^1_1 sets form a Spector class: we may find a Π^1_1 PWO of length ω_1^{ck} ordering the places where sentences are put into E_∞.
(ii) that the class Γ_F of sets of integers definable using stabilization of predicates in Field's model also form such a Spector class: The Field($\preceq$) defined above is a complete and so universal set of sentence-codes, and $\preceq$ is itself a PWO of this field defined in the same way; that is PWO(Γ_F) holds.

Ineffable Liars - the *dénouement*

We may define for *any* sentence C:

$$D^C(A) \equiv \forall B\Big(P_{\prec}(B,C) \to (\forall y(y = \ulcorner D^B(A)\urcorner \to T(y)))\Big).$$

Theorem on Ineffable Liars

There are sentences $C \in \mathscr{L}^+$ so that for any determinateness predicate D^B with $B \in \text{Field}(\preceq)$ $\|D^B(\lambda_C)\| = \frac{1}{2}$.

Thus the defectiveness of $\lambda_C :\equiv \neg D^C(T(\lambda_C))$ is not measured by any such determinateness predicate definable within the $\mathscr{L}^+$ language. It is an "ineffable liar".

Other notations for $\alpha < \zeta$

- Kleene's $\mathcal{O}$ is more usually cited as a notation system for ordinals below ω_1^{ck}. This set and the associated ordering $<_{\mathcal{O}}$ on it, can be enumerated as follows:

Kleene's $\mathcal{O}$

(i) 0 receives notation 1;
Assume all ordinals $< \gamma$ have received a notation.
(ii) If $\gamma = \beta + 1$ and x is a notation for β then 2^x is a notation for γ ; put $x <_{\mathcal{O}} 2^x$
(iii) IF $\mathrm{Lim}(\gamma)$ and y is such that $\forall n \varphi_y(n)\downarrow$ AND $\forall i < j \big(\varphi_y(i) <_{\mathcal{O}} \varphi(j)\big)$ are already enumerated AND $\{\varphi_y(n)\}$ are notations for an increasing sequence of ordinals with supremum γ THEN y is a notation for γ.

- But can we extend this notation system to ζ? Yes!

Other notations for $\alpha < \zeta$

$\mathcal{O}^+$

(i) 0 receives notation 1;
Assume all ordinals $< \gamma$ have received a notation.
(ii) If $\gamma = \beta + 1$ and x is a notation for β then 2^x is a notation for γ ; put $x <_{\mathcal{O}^+} 2^x$
(iii) IF Lim(γ) and y is such that $\forall n \varphi_y(n)\downarrow$ AND $\forall i < j \big(\varphi_y(i) <_{\mathcal{O}^+} \varphi(j)\big)$ are already enumerated AND $\{\varphi_y(n)\}$ are notations for an increasing sequence of ordinals with supremum γ THEN y is a notation for γ.

- What has changed is that we consider running our y'th Turing Machine transfinitely as an Infinite Time Turing Machine (ITTM) of Hamkins & Kidder. This makes perfect sense and defines a tree $\mathcal{O}^+ \supset \mathcal{O}$ with maximal branches of height ζ.

A "halting" set

$\widetilde{K}$

$\widetilde{K} =_{df} \left\{ e : \text{ the } e\text{'th ITTM function } \varphi_e(e)\downarrow \text{ eventually to a settled output} \right\}.$

Theorem

$$F_\infty \equiv_1 H_\infty \equiv_1 T^2_\zeta \equiv_1 \widetilde{K}.$$

Is H_∞ an inductive set?

Proposition

There is a generalised quantifier Q *so that* H_∞ *is positive* Q*-elementary inductive.*

- Thus instead of thinking of the stable truths in this quasi-inductive fashion it is possible to view this as a monotone inductive set of sentences - albeit with an operation not defined in standard FO logic.

How much of second order number theory?

To say Φ is a 'Π^1_1 definable operator' is to say '$n \in \Phi(X)$' is a Π^1_1-relation of n and X. (The Fieldian successor step is such.)

Π^1_1QI

Let Π^1_1QI denote the assertion that for evey $X \subseteq \mathbb{N}$, for every Π^1_1 definable operator $\Phi : \mathscr{P}(\mathbb{N}) \longrightarrow \mathscr{P}(\mathbb{N})$, the stability set $\Phi^\infty(X)$ exists.

Theorem (W)

$$\Pi^1_3\text{-CA}_0 \quad >> \quad \Delta^1_3\text{-CA}_0 + \Sigma^0_3\text{-Det} \quad >> \quad \Delta^1_3\text{-CA}_0 + \Pi^1_1\text{QI} \quad >> \quad \Delta^1_3\text{-CA}_0$$

are in strictly descending order of strength, in that each proves the existence of (many) β-models of the next.

And for proof theory?

Our pair (ζ, Σ) is lexicographically least so that $L_\zeta \prec_{\Sigma_2} L_\Sigma$. L_Σ is thus the least β-model of (lightface) Π^1_3-CA_0.

• Thus it might be that to give an ordinal analysis for Π^1_3-CA_0, one may first need to analyse, *e.g.* , $\Pi^1_1\mathsf{QI}$-definitions as a stepping stone.

Conclusions for semantic truth constructions

- That revision theoretic processes are complex from a definability point of view. In terms of complexity it matters not whether the operator is recursive (ITTM), arithmetic (Herzberger) or Π^1_1 (Field). The complexity arises in exactly the same way: from the $\liminf$ limit rule.

- The examples we've seen of very different successor stage rules under the $\liminf$ rule yield *recursively isomorphic* stability sets (whether as 'halting' sets for ITTM's, ultimate truths for Field or Stable truths for Herzberger.) The $\liminf$ operation is far from neutral, and is acting as a kind of a generalised infinitary ω-rule.

- The complexity that arises is well into second order number theory and well beyond a) the subsystems needed for any purely mathematical theorems known (other than determinacy) and b) current proof-theoretic ordinal analysis. This should give pause for thought if we are simply trying to find a predicate of sentences for expressing first order truth about $(\mathbb{N}, +, \times, \ldots)$ say, or to define a new conditional.