WEAK SYSTEMS OF DETERMINACY AND ARITHMETICAL QUASI-INDUCTIVE DEFINITIONS.

P. D. WELCH   DRAFT 23.04.10

Abstract. We locate winning strategies for various Σ_3^0-games in the L-hierarchy in order to prove the following:

THEOREM 1. KP $+\Sigma_2$-Comprehension $\vdash \exists \alpha L_\alpha \models$ "Σ_2-KP $+\mathbf{\Sigma_3^0}$-Determinacy ."

Alternatively: Π_3^1-CA$_0$ $\vdash$ "there is a β-model of Δ_3^1-CA$_0 + \mathbf{\Sigma_3^0}$Determinacy." The implication is not reversible. (The antecedent here may be replaced with $\Pi_3^1(\Pi_3^1)$-CA$_0$: Π_3^1 instances of Comprehension with only Π_3^1-lightface definable parameters - or even weaker theories.)

THEOREM 2. KP $+\Delta_2$-Comprehension $+ \Sigma_2$-Replacement $+$AQI $\nvdash \Sigma_3^0$-Determinacy .

(Here AQI is the assertion that every arithmetical quasi-inductive definition converges.) Alternatively:

Δ_3^1CA$_0$ + AQI $\nvdash \Sigma_3^0$-Determinacy.

Hence the theories: Π_3^1-CA$_0$, Δ_3^1-CA$_0 + \Sigma_3^0$-Det, Δ_3^1-CA$_0$ + AQI, and Δ_3^1-CA$_0$ are in strictly descending order of strength.

§**1. Introduction.** The work in this paper was initially at least, motivated by trying to see how the theory of arithmetical quasi-inductive definitions (AQI as defined below) fits in with other subsystems of second order number theory. We had been working with one example of such a definition, essentially a recursive quasi-inductive definition, and had calculated certain ordinals where such definitions reached fixed points or exhibited a looping convergence [21] and [20]. Earlier J. Burgess [2] had in fact distilled from H. Herzberger's notion of a revision sequence [9] the notion of arithmetically quasi-inductive, and shown that the same ordinals appeared. (Herzberger's notion was connected with a "truth operator" and thus, strictly speaking is not arithmetical, but just beyond; however this only makes for a trivial difference.) Other examples of constructions involving such quasi-inductive definitions have appeared in the theory of truth [5], and in theoretical computer science: S.Kreutzer in [12] uses essentially arithmetical quasi-inductive definitions to formulate a notion of semantics for partial fixed point logics over structures with infinite domains in order to separate away this logic from inflationary fixed point logic.

Here however we rather mention some of the possibilities that connect these concepts with potential proof theoretical results on the way to looking at ordinal notation systems

for Π^1_3-CA$_0$: for Π^1_2-CA$_0$, by work of Rathjen [16], [15], we have that this second level of Comprehension is tied up with the theory of arbitrarily long finite Σ_1-elementary chains through the L_α-hierarchy: indeed the first level L_α which is an infinite tower of such models, is the first whose reals form a β-model of Π^1_2-CA$_0$. The same occurs for Π^1_3-CA$_0$: the first L_γ whose reals form a β-model of Π^1_3-CA$_0$ is the union of an infinite tower of models $L_{\zeta_n} \prec_{\Sigma_2} L_{\zeta_{n+1}}$. Presumably one will need to analyse finite chains of such models to get at an ordinal notation system for this theory. Seeing that AQI is connected with levels L_ζ of the Gödel's L-hierarchy, and with Σ_2-end extensions (albeit only chains of length 1), analysing the proof theoretic strength of AQI would be a natural stepping stone.

Other notions of inductive definition have been tied to determinacy. Positive monotone arithmetical operators have fixed points bounded by the first admissible ordinal ω^{ck}_1, and in turn strategies for recursive open (that is Σ^0_1) games are either in $L_{\omega^{ck}_1}$ (for player I) or definable over it (for the "closed" player II). Solovay (*cf* [11]) showed that for Σ^0_2-games, strategies for player I in such games occur in L_σ where σ is the closure ordinal of Σ^1_1 monotone inductive definitions (and for Player II they lie in the next admissible set beyond it). Tanaka [19] formulated a subsystem of analysis related to Σ^1_1 monotone inductive definitions, Σ^1_1-MI$_0$, and showed that over RCA$_0$ it was equivalent to Σ^0_2-Determinacy. Our original intention had been to tie in AQI with some level of determinacy. Is there anything at all analogous for AQI?

Turning naturally to Σ^0_3-Determinacy, the location of strategies for such games in the constructible L-hierarchy seems to be unknown. There is little published on this question; John does state some results on this in [10]. However this does not really yet reveal (at least to us) where such strategies lie. A closer reading of Davis's proof of Σ^0_3-Determinacy showed that it was provable in KP $+\Sigma_2$-Comprehension, and thus that winning strategies appear in the least model which is an infinite tower of the form of a union of a chain of submodels $L_{\zeta_n} \prec_{\Sigma_2} L_{\zeta_{n+1}}$. Whilst we always thought it would be a happy coincidence if Σ^0_3-Determinacy matched up exactly with AQI we never really believed it would be so, and the theorems here show this. However they are *extremely* close. We have not located the exact ordinal where the winning strategies (for either player) appear, but we have shown that for some games they must appear after the first Σ_2-extendible ξ_0, *i.e.* ξ_0 initiates a proper Σ_2-chain of length 1: $L_{\xi_0} \prec_{\Sigma_2} L_{\xi_1}$ (indeed after the first Σ_2-admissible μ so that the reals of L_μ are closed under boldface AQI). However for all Σ^0_3 games they are strictly bounded below the first γ where the reals of L_γ are a model of " $\Pi^1_3(\Pi^1_3)$-CA$_0$", *i.e.* they are closed under instances of Π^1_3-Comprehension, with only Π^1_3 (lightface) definable set parameters allowed (a more precise definition of a better bound is below - see also the final corollary). Thus our β-models of interest in Section 4, occur well before the beginning of the least Σ_2-chain of length 2: $L_{\zeta_0} \prec_{\Sigma_2} L_{\zeta_1} \prec_{\Sigma_2} L_{\zeta_2}$. It thus seemed to us that even with extra effort, finding exactly at which level each Σ^0_3 game had a strategy might not be very much more illuminating: at least not in the way that Solovay's was for Σ^0_2 *vis à vis* levels of the L-hierarchy: there are seemingly no well-established theories or ordinals here in this narrow gap to aim for.

§**2. Preliminaries.** The structure of the paper is as follows: we outline some of the theories and subsystems of second order number theory that we use. Then, in the next

subsection we do the same for the constructible hierarchy L and prove a result on uniformly definable Skolem functions. In Section 2.2 we introduce arithmetical quasi-inductive definitions, list some of the basic facts concerning them, and formulate the subsystem AQI. We state there also how it is situated with regards to known theories such as $\Delta^1_3\text{-CA}_0$. In Section 2.3 we introduce mainly notation that we are going to use concerning game trees, strategies and the like. We are much indebted to D.A. Martin's (regrettably unpublished) account [14], of the basics of determinacy and in particular also for his account of Davis's theorem on Σ^0_3-Determinacy which we follow closely in Section 4. We classify the complexity of some of these notions in terms of the Levy hierarchy and the theory KPI_0. In Sections 3 and 4 we prove the Theorems 2 and 1 of the abstract respectively.

The reader may consult [18] on the mutual interpretability between theories in subsystems of second order number theory, such as $\Delta^1_3\text{-CA}_0$ for example, and set theoretical counterparts $\Delta^1_3\text{-CA}_0^{\text{set}}$, or $\Sigma_2\text{-KP}$. We do not have to do any fine analysis of provability in any such subsystems, as we are typically showing that a rather strong theory A proves the existence of (many) β-models of theory B.

By Σ_2-Comprehension we mean the usual Axiom of Comprehension but restricted to formulae that are Σ_2 in the Levy hierarchy. By "KP" we shall mean the usual axioms of Kripke-Platek set theory, but we shall assume these include the Axiom of Infinity. By "KPI_0" we shall mean the conjunction of the axiom of extensionality together with the assertion "For every set x there is a set y with $x \in y \wedge (\text{KP})_y$." (By $(\text{KP})_y$ we mean the axioms of KP relativised to y.) By "KPI" we shall mean "$\text{KP} + \text{KPI}_0$." By "Σ_2-KP" we shall mean KP with the Comprehension and Replacement Axiom schemes reinforced to allow for Δ_2 and Σ_2 formulae respectively.

We call a set *admissible* if it is a model of the KP axioms. An ordinal α is admissible if $L_\alpha \models \text{KP}$. Note that KP alone does not prove Σ_1-Comprehension. The Comprehension axioms are themselves essentially "boldface" axioms, as they allow parameters into the axiom schemes. The class of all admissible ordinals, other than ω shall be denoted by ADM. For information on admissibility theory the reader may consult [1]. The reals of a model of $\text{KP} + \Sigma_i$-Comprehension form a model of $\Pi^1_{i+1}\text{-CA}_0$ for $i \in \{1, 2\}$. We follow the definitions and development of the theories $\Pi^1_{i+3}\text{-CA}_0$ (the latter we take to include the set induction axiom) of [18]. The set of reals belonging to L_α, where α is least such that L_α is a model of $\text{KP} + \Sigma_i$-Comprehension, are those of the minimum β-model of $\Pi^1_{i+1}\text{-CA}_0$, $(i > 0)$ (*cf.* [18],VII.5.17). It is well known, and easy to see, that if L_α is a countable model of $\text{KP} + \Sigma_i$-Comprehension, then L_α is a union of an infinite Σ_i-elementary chain of submodels: $L_{\zeta_k} \prec_{\Sigma_i} L_{\zeta_{k+1}} \cdots \prec_{\Sigma_i} L_\alpha$. Hence the least such α is the union of an ω-length such chain. As a consequence $\text{KP} + \Sigma_i$-Comprehension proves the existence of β-models whose reals code Σ_i-elementary chains of length, say, 2: $L_{\zeta_0} \prec_{\Sigma_i} L_{\zeta_1} \prec_{\Sigma_i} L_{\zeta_2}$. In Section 4 we shall use something much weaker: we shall assume only that there is a trio of ordinals $\gamma_0 < \gamma_1 < \gamma_2$ with $L_{\gamma_0} \prec_{\Sigma_2} L_{\gamma_1}$ and $L_{\gamma_0} \prec_{\Sigma_1} L_{\gamma_2}$. Obviously such occur well below such Σ_2-elementary chains of length 2.

Our set theoretical notation is standard. If $z = \langle u, v \rangle$ is an ordered pair, then we use the functions $(z)_0 = u$, $(z)_1 = v$ to denote the unpairing functions. For D a class of ordinals we let D^* be the closed class of its limit points. By HC we denote the class of sets that are hereditarily countable.

2.1. On the constructible hierarchy. We shall use some further facts about the L-hierarchy. (These can be found in [4].) In particular that if $\omega\alpha = \alpha$ then L_α is equal to the level J_a of the Jensen hierarchy. (We shall write L_α rather than J_α when we know that they are equal, and for the ordinal heights of the models of our theories, this will always be the case.)

We shall write $\Sigma_n(L_\alpha)$ for the relations Σ_n (for $n \leq \omega$) over $\langle L_\alpha, \in \rangle$ possibly with parameters from L_α. If we wish to display the parameters $p, q, r \ldots$ we shall write $\Sigma_n^{L_\alpha}(\{p, q, r\})$ (and similarly for J_α if needed in place of L_α.) We shall not make much use of the fine-structure theory but we shall use the Σ_1-Skolem function $h_\alpha^1(v_0, v_1)$ which is uniformly definable over such levels. Thus if $\langle \varphi_k \mid k < \omega \rangle$ is a recursive listing of the Σ_1 formulae of $\mathcal{L}_{\dot{\in}}$, with say φ_k with free variables amongst those displayed as $\varphi_k(v_0, \ldots, v_{n(k)})$, and if $L_\alpha \models \exists v_0 \varphi_k(v_0, x_1 \ldots, x_{n(k)})$ then $L_\alpha \models \varphi_k(h_\alpha^1(k, \langle p_\alpha^1, \overrightarrow{x_i} \rangle), x_1 \ldots, x_{n(k)})$ where p_α^1 is the (possibly empty) Σ_1-*standard parameter.* The reader should remark that we are working very low down in the L-hierarchy: any model that occurs will have at most a single proper Σ_2-elementary substructure; hence any level will have at most some Σ_2 definable map from ω onto the whole structure; every L_α of interest will have only ω as the single infinite cardinal, and any standard parameters that occur can be taken to be single ordinals. Also for any limit α there will be always a $\Delta_1(L_\alpha)$ definable map of α onto L_α (although for α not closed under Gödel pairing this is not necessarily a parameter free definition; for those α which are Gödel closed however, the definition is uniform in α.)

Indeed we are so far down in the L-hierarchy that we always have *uniformly Σ_n-definable Skolem functions.* For this recall that β_0 is defined as the least β such that $L_{\beta_0} \models \text{ZF}^-$. (It is well known that the reals of L_{β_0} are those of the minimal model of full comprehension, the minimal model of full second order analysis.) Sy Friedman showed in [6] that if $J_\alpha \models \text{``}V = \text{HC''}$ (that is, every set is hereditarily countable), then for any n there is g_α^n, a Σ_n-Skolem function for J_α, which is Σ_n-definable without parameters over J_α. (The point here is the phrase "without parameters".)

Remark: It is well known that for $n > 1$ there is no uniform Σ_n-definable Σ_n-Skolem function for all the levels of the J_α hierarchy. It is perhaps less well known that down low, there are in fact such, as we now show.

THEOREM 3. (Uniform Σ_n-Skolem Functions) *For every $n < \omega$ there is a single Σ_n-definition of a partial function h^n, which defines a Σ_n-Skolem function over any $\langle J_\alpha, \in \rangle$ with $\alpha < \beta_0$.*

Proof. For a fixed countable α_0 with $J_{\alpha_0} \models \text{``}V = \text{HC''}$, it is proven in [6], by using an induction on $n < \omega$, that Σ_n-Skolem functions exist for J_{α_0} which are definable without parameters. For the theorem under discussion here, any J_α for $\alpha \leq \beta_0$ satisfies the requirement $J_\alpha \models \text{``}V = \text{HC''}$. For an α_0 as given and starting with $n = 2$ Friedman's definition for a Skolem function $g_{\alpha_0}^2$, is in terms of the Σ_1-Skolem function which is indeed uniformly parameter free Σ_1 over all α. An inspection of the definition (on p3328 of [6]) shows however that it does not depend on α_0 in any way. We thus have that for any $\alpha \leq \beta_0$ there is indeed a Σ_2-Skolem function uniformly parameter free Σ_2-definable over J_α, which we shall call h_α^2. Turning to $n = 3$ his definition again for the fixed α_0, is in terms of this Σ_2-definable Skolem function $g = g_{\alpha_0}^2$ (as defined at the bottom of p3328) for J_{α_0}, but is otherwise independent of α_0. However we

have just argued that the Σ_2-Skolem functions h_α^2 may be assumed to be here uniformly definable. Hence so will be the Σ_3-Skolem functions, defined using them, that we may call h_α^3 for all $\alpha \leq \beta_0$. Similarly for other $n < \omega$.   QED

Consequently, every $x \in J_\alpha$ (for $\alpha < \beta_0$) is of the form $h_\alpha^n(i,n)$ some $i,n < \omega$. In fact the argument in the above proof readily works *uniformly* beyond β_0 up until the least $\alpha + 1$ such that $J_{\alpha+1} \models$"*there is an uncountable cardinal.*"

We shall also use the fact that if T is a recursive theory of Σ_1 sentences, and α is the least such that $L_\alpha \models \mathrm{KP} + T + V = \mathrm{HC}$ then every $x \in L_\alpha$ is definable by some $\Sigma_1^{L_\alpha}$ term. This is because, by using an elementary Σ_1-Skolem hull argument together with the uniform Σ_1-definable Skolem function h_α^1, there will be a $\Sigma_1^{L_\alpha}$ map, perhaps partial, of ω onto L_α: hence for every $x \in L_\alpha$ there is $i,n < \omega$ with $x = h_\alpha^1(i,n)$.

2.2. Arithmetical quasi-inductive definitions. Let $\Gamma : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ be any arithmetic operator (that is "$n \in \Gamma(X)$" is arithmetic; we emphasise that Γ need be neither monotone nor progressive). We define the following iterates of $\Gamma : \Gamma_0(X) = X; \Gamma_{\alpha+1}(X) = \Gamma(\Gamma_\alpha(X))$;

$$\Gamma_\lambda(X) = \liminf_{\alpha \to \lambda} \Gamma_\alpha(X) = \bigcup_{\alpha < \lambda} \bigcap_{\alpha < \beta < \lambda} \Gamma_\beta(X).$$

Following Burgess we say that $Y \subseteq \omega$ is *arithmetically quasi-inductive* if for some such Γ, Y is (1-1) reducible to $\Gamma_{\mathrm{On}}(\varnothing)$. Any such definition has a least countable $\xi = \xi(\Gamma)$ with $\Gamma_\xi(\varnothing) = \Gamma_{\mathrm{On}}(\varnothing)$. If we let ζ denote the supremum of all such $\xi(\Gamma)$, then we have:

PROPOSITION 1. (Burgess [2] Sect.14) *(i) ζ is the least Σ_2-extendible ordinal; that is the least ζ so that there is a $\Sigma > \zeta$ with $L_\zeta \prec_{\Sigma_2} L_\Sigma$.*
(ii) A set Y is arithmetically quasi-inductive iff $Y \in \Sigma_2(L_\zeta)$.

In general we shall stay with the notation that Σ denotes the ordinal height of the least proper Σ_2-end extension of L_ζ. It can be shown:

PROPOSITION 2. *There is a recursive operator Γ with $\xi(\Gamma) = \zeta$.*

Proof: The universal Infinite Time Turing machine of Hamkins and Kidder ([7]) is in effect such a recursive operator Γ. That $\xi(\Gamma) = \zeta$ is shown in [20].   QED

Some quasi-inductive definitions may reach a fixed point.

DEFINITION 1. *We say that Γ reaches a fixed point on X, if there is α so that $\Gamma_\alpha(X) = \Gamma_{\alpha+1}(X)$; and if so we call $\Gamma_\alpha(X)$ the* fixed point.

The following can be shown.

PROPOSITION 3. *For any arithmetical operator Γ, either $\xi(\Gamma) = \zeta$, or else there is an equivalent recursive operator $\tilde{\Gamma}$ in the sense that it reaches $\Gamma_{\xi(\Gamma)}(\varnothing)$ as a fixed point; specifically there is a recursive operator $\tilde{\Gamma}$ and an $\alpha < \zeta$ with $\Gamma_{\xi(\Gamma)}(\varnothing)$ Turing (1-1) reducible to $\tilde{\Gamma}_\alpha(\varnothing) = \tilde{\Gamma}_{\alpha+1}(\varnothing)$.*

The quasi-inductive definitions that reach fixed points (on $\varnothing$, or on some particular input X) form an interesting subclass. Investigation of such is an appealing combination of admissibility theory and reflection properties of ordinals.

Propositions 2 and 3 indicate that in one sense, to study recursive operators, is to study all arithmetical ones: if one has a Σ_n-definable operator, one seemingly only needs to look instead at Π_{n+1}-reflecting ordinals. For such an arithmetical operator Γ, it is easy to see from the definition of (ζ, Σ) that $\Gamma_\zeta(\varnothing) = \Gamma_\Sigma(\varnothing)$ and thus we might call (ζ, Σ) a "repeat pair" for Γ and $\varnothing$. Again one may show for such a Γ, that (ζ, Σ) is the lexicographic least such repeat pair. We use this to formulate a definition allowing parameters x as starting inputs.

DEFINITION 2. AQI *is the sentence: "For every arithmetic operator* Γ, *for every* $x \subseteq \omega$, *there is a wellordering* W *with a repeat pair* $(\zeta(\Gamma, x), \Sigma(\Gamma, x))$ *in* $\mathrm{Field}(W)$*". If an arithmetic operator* Γ *acting on* x *has a repeat pair (in* $\mathrm{Field}(W)$*), we say that* Γ *stabilizes along* W *(with input x).*

Clearly a certain amount of set theory (or analysis) is needed to show that every operator stabilizes. Reformulated using the relativisation of Proposition 1, this is thus:

LEMMA 1. $\mathrm{KP} \vDash \mathsf{AQI} \longleftrightarrow \forall x \subseteq \omega \exists \xi, \sigma(L_\xi[x] \prec_{\Sigma_2} L_\sigma[x])$.

We note some facts concerning the pair (ζ, Σ) in L:

PROPOSITION 4. *(i) ([20] Thm. 2.1)* L_ζ *is a model of* Σ_2-KP *(and is a union of such).*
(ii) ([21] Cor.3.4) L_Σ *is not a model of* KP, *but is a model of* KPI_0 *plus the scheme of parameter-free* Σ_2-Comprehension.

In the language of subsystems of second order number theory, the reals of L_Σ form a β-model of Π_1^1-CA_0 together with a β_3-submodel (provided by the reals of L_ζ), and in fact it is the minimum such β-model. (See [18] VII.7.1 for the notion of a $\beta = \beta_1$ and β_k-submodels.) This β-model thus gives an example of a model of parameter free, so lightface, Π_3^1-CA_0, together with (boldface) Π_1^1-$\mathsf{CA}_0 + \neg\Delta_2^1$-$\mathsf{CA}_0$.

Note also that immediately from the definition of Σ_2-extendability and (i) of the last lemma, L_Σ is also a union of models of Σ_2-KP.

Prop. 4(i) then already shows that AQI is stronger than Δ_3^1-CA_0 since any Σ_2-extendible is already a (union of) models of Σ_2-KP. Somewhat more formally:

COROLLARY 1. Δ_3^1-$\mathsf{CA}_0 + \mathsf{AQI} \vdash \forall x (\exists \beta$-*model of* Δ_3^1-CA_0 *containing* x).

Proof: In fact we have: Δ_3^1-$\mathsf{CA}_0 + \mathsf{AQI} \vdash$"*There is a countably coded β-model of* Δ_3^1-CA_0". This is because the former proves "$\forall x \exists W \in \mathrm{WO}(\exists z, s \in \mathrm{Field}(W) \wedge L_{\|z\|}[x] \prec_{\Sigma_2} L_{\|s\|}[x])$." By Prop. 4(i), $\mathcal{P}(\mathbb{N})$ of the model $L_{\|z\|}[x]$ form a model of Δ_3^1-CA_0.   QED

2.3. Strategies and game trees. We assume familiarity with the basic notions of two person perfect information games played using integers. We shall follow Martin and shall assume that such games are played on *game trees* $T \subseteq {}^{<\omega}\omega$ although we disallow terminal nodes. We let $G(A; T)$ denote the game with payoff set $A \cap \lceil T \rceil$ where $\lceil T \rceil$ denotes the set of all plays in T. A *position* in the game is simply a finite sequence $r \in T$. For $q \in T$ we let T_q denote the set of all positions $r \in T$ where $r \supseteq q$.

LEMMA 2. *Let A be arithmetic; let M be a transitive model of* KPI_0 *with $T \in M$. Then (i) "$G(A; T)$ is not a win for I" is Π_1^M; (ii) if this holds then "p is a position in II's non-losing quasi-strategy for $G(A; T)$" is Π_1^M.*

Proof: "$G(A; T)$ is not a win for I" is equivalent to "$\forall \sigma \in {}^\omega\omega(\sigma$ is not a winning strategy for I in $G(A; T)$)"; which in turn is equivalent to : "$\forall\sigma$(if σ is a strategy for I in $G(A; T)$ then $\exists r \in {}^\omega\omega \; \sigma * r \notin A \cap \lceil T \rceil)$". The set $\{r \mid \sigma * r \notin A \cap \lceil T \rceil\}$ is then $\Delta_1^1(\sigma, T)$, and hence, if non-empty with $\sigma \in M$, has an element definable over the least admissible set containing σ and T, and hence has an element in M. This completes (i). For (ii): let T' be II's non-losing quasi-strategy for $G(A; T)$. Then

$$p \in T' \Leftrightarrow \mathrm{lh}(p) = k \rightarrow \forall n \leq k$$
$$p \upharpoonright n \in T \wedge \forall 2n + 1 < k(q = (p_0, p_1, \ldots, p_{2n+1}) \rightarrow \text{``}G(A; T_q)\text{ is not a win for }I\text{''}.$$

<div align="right">QED</div>

One should note that a quasi-strategy for II in $G(A; T)$ is then a subtree of T that does not restrict I's moves in any way, but potentially only II's moves.

§3. AQI is weaker than Σ_3^0-Determinacy.

DEFINITION 3. $T_\alpha^j(X)$ *denotes the set of Σ_j formulae, true of parameters from X, in the structure $\langle J_\alpha, \in \rangle$; T_α^j abbreviates $T_\alpha^j(\varnothing)$, the Σ_j-theory of $\langle J_\alpha, \in \rangle$.*

DEFINITION 4. *(i) Let E_0 be the class of Σ_2-extendible ordinals. If E_α is defined, let $E_{\alpha+1}$ be the class $E_0 \cap E_\alpha^*$. If $Lim(\lambda)$ let $E_\lambda = E_0 \cap \bigcap_{\alpha < \lambda} E_\alpha$;*
(ii) Let $F_0 = E_0 \cap \{\mu \mid \forall \alpha < \mu(\mu \in E_\alpha)\}$; let F_{n+1} be the class $E_0 \cap F_n^*$; let $F = F_\omega = E_0 \cap \bigcap_{n < \omega} F_n$;

Part (ii) of the above defines the E_α hierarchy through a diagonalisation, which one might have called a "hyperextendible". Clearly these hierarchies continue, but we had to stop somewhere. For $k < \omega$ the classes E_k we can think of as having depth k in the "Σ_2-extendible limits of Σ_2-extendible ..." hierarchy: if $\gamma \in E_k$ then there are ordinals $\gamma = \mu_k \leq \mu_{k-1} \leq \ldots \leq \mu_0 < \nu_0 < \nu_1 < \ldots < \nu_k$ satisfying $L_{\mu_j} \prec_{\Sigma_2} L_{\nu_j}$ for $j \leq k$.

The ordinals in the F-hierarchy also have made their appearance on the stage: Lubarsky has formulated a notion of iterated Infinite Time Turing Machines: *ITTM's with feedback* [13]. For these machines the least "eventually writable" ordinal (to take over the terminology from ITTM theory, but which is the analogue of our least Σ_2-extendible ζ here), is actually the least element in F_0. We could appropriately formulate a theory of *quasi-inductive* using these notions, and again we could prove by our methods that a complete such quasi-inductive set is also $\supseteq \Sigma_3^0$.

THEOREM 4. Σ_2-KP $+\forall \alpha \exists \beta (\alpha < \beta \wedge \beta \in F) \nvdash \Sigma_3^0$-Determinacy.

Proof: We shall effect this by showing that the least level of the L-hierarchy that is a model of the antecedent theory is a model $M_0 = L_\delta$ in which Σ_3^0-Determinacy fails. The reals of this model will *a fortiori* form a β-model of Δ_3^1-CA$_0+$ AQI. The rest of this section is taken up with proving this theorem (which is Theorem 2 of the Abstract).

We do this, using a technique that goes back to H. Friedman, by defining certain games G_ψ so that codes for initial segments of the L-hierarchy are recursive in any winning strategy for the game. So henceforth, let $M = L_\delta$ be the least level of the antecedent theory in the statement of the theorem.

Let $\Psi = \{\psi | \psi \in \Sigma_1 \cap Sent \wedge L_\delta \models \psi\} = T_\delta^1$ be the Σ_1-theory of L_δ. (Essentially we show that Ψ is a $\supseteq \Sigma_3^0$ set of integers: see the Corollary 2 below.)

(1) "Σ^0_3-Determinacy" is Σ^{KPI}_1.

Proof: Σ^0_3-Determinacy is equivalent to

$\forall n \in \omega[$*if* A_n *is the n'th* Σ^0_3 *set then* $\exists\sigma(\sigma$ *is a winning strategy for a player in* $G(A_n; {}^{<\omega}\omega))]$

The statement that σ is a winning strategy for Player I is equivalent to saying: "The $\Delta^1_1(\sigma)$ set $\{x \mid \sigma * x \notin A\}$ is empty". If it were non-empty then it would have a member $\Sigma^1_1(\sigma)$-definable and thus definable over the least admissible set containing σ. (The latter by Kleene's Basis Theorem relativised to x *cf* [17] III.1.3.) Thus "σ is a winning strategy" is Σ^{KPI}_1. We thus have a numerical quantifier in front of a Σ^{KPI}_1 predicate, and is thus overall Σ^{KPI}_1.   QED (1)

Hence, were Σ^0_3-Determinacy to hold in L_δ it would be equivalent to some $\psi \in \Psi$. For any $\psi \in \Psi$ we define: $\alpha_\psi =$ the least β so that $L_\beta \models \mathrm{KP} + \psi$.

Note: The minimality of β ensures that every $x \in L_{\alpha_\psi}$ is Σ_1-definable by some parameter free Σ_1 term t_x. (In other words the Σ_1-Skolem hull inside $\langle L_{\alpha_\psi}, \in \rangle$ of $\varnothing$ is all of L_{α_ψ} itself.)

The following is straightforward:

(2) Let $\overline{\alpha} = \sup\{\alpha_\psi \mid \psi \in \Psi\}$. Let $\alpha' =$ the least $\beta(L_\beta \prec_{\Sigma_1} L_\delta)$. Then $\alpha' = \overline{\alpha}$.

We shall show for every $\psi \in \Psi$ there is a game G_ψ with a Π^0_3 payoff set, but without a winning strategy in L_{α_ψ}. In view of the comment just after the proof of (1), this will suffice.

Fix for the rest of the argument $\psi \in \Psi$. Let α denote α_ψ. We consider the following game $G = G_\psi$.

I	plays	$m_0, m_1, \ldots, m_i$		$x = (m_0, m_1, \ldots, m_i, \ldots)$
II	plays	$n_0, n_1, \ldots, n_i$		$y = (n_0, n_1, \ldots, n_i, \ldots)$

in the usual way, playing in the i'th round integers (m_i, n_i). Let $z = x \oplus y$.

Rules for I.

Let T be the theory $\mathrm{KP} + V = L + \psi$. I's play x must be a set of Gödel numbers for the complete Σ_1-theory of an ω-model of $T +$"*there is no set model of T*".

Using the Note 3 we denote by $\langle M, E \rangle$ the model I essentially constructs if he obeys this rule. We may regard also as part of the rule that x as given by I should be specified simply by I stating "$k \in T^1_M$" or "$k \notin T^1_M$" where $T_M = T^1_M$ is the standard Σ_1-code or 'truth-set' for his model. Also, as in L_{α_ψ}, in $\langle M, E \rangle$, every set is given by a Σ_1 parameter free Skolem term.

Note: If $\langle M, E \rangle$ is wellfounded then it is isomorphic to $\langle L_{\alpha_\psi}, \in \rangle$. We have built into these Rules for I of the game *inter alia* the 'smallness' requirement on the models played, that no level $(L_\kappa)_M$ has an ordinal $\kappa \in F$ which is also Σ_2-admissible.

Amongst the codes for sentences that I plays are those of the form

$$\ulcorner t_m \in \mathrm{On} \wedge \ t_n \in \mathrm{On} \wedge \ t_m < t_n \urcorner$$

These we shall use to formulate rules for player II. So far the Rules for I amount to a Π_2^0 condition on x and so on z. (We may take a recursive listing of Σ_1-sentences $\langle \psi_k | k \in \omega \rangle$ and we then require $\forall k \exists k'(m_{k'} = \ulcorner \psi_k \urcorner \vee m_{k'} = \ulcorner \neg \psi_k \urcorner)$, thus the theory I constructs will be Σ_1-complete; we obtain that M has at least the integers as standard also by a Π_2^0 condition.) Let $r : \omega \to \omega \times \omega$ be a recursive enumeration of ω^2 in which each (i, j) appears infinitely often.

Rules for II.
At round k: if $(i, j) = r(k)$ and $n_k \neq 0$, then we shall say that "*II makes the entry n_k on list $L_{i,j}$.*". These 'Listing' *Rules* here require her to list terms in a correct order. She may make an entry on list $L_{i,j}$ in round k if:
Either $L_{i,j}$ is empty at the current round, in which case n_k can be any term t_s *as long as I* has asserted at an earlier round $\ulcorner t_s \in \mathrm{On} \urcorner \in T_M$;
or $L_{i,j} \neq \varnothing$, and if t_s was the last entry *II* made on this list, then n_k can be any term t_r, again *provided that I* has at an earlier round $k' < k$ asserted $m_{k'} = \ulcorner t_r \in \mathrm{On} \wedge t_r < t_s \urcorner \in T_M$.

The winning conditions. I wins immediately at a finite round if *II* breaks one of her *Listing Rules* just enumerated. *II* wins if I fails to obey his conditions on x, or both players obey their respective rules and additionally
$$\exists (i, j)[II \text{ makes infinitely many entries on list } L_{i,j}].$$
This is a Σ_3^0 winning condition for *II* on z. Hence G_ψ has a Π_3^0 payoff set.

In other words, if I obeys his rules, *II* can win if for some (i, j), $r^{-1}"(i, j)$ in effect picks out an infinite descending chain through the ordinals of the model M that I reveals *via* the gödel numbers of the Σ_1 sentences true in M.

II is not allowed to make an entry indicating that $t_s < t_r$ until I has asserted this at some earlier stage. *II* is thus not predicting what the model will look like below t_r; by making an entry on a list she is merely adverting to the fact that I has already revealed that $t_s < t_r$.

(3) I has a winning strategy.
Proof: I plays out all "$k \in T_M$" for all $k \in T_{\alpha_\psi}^1$, and "$k \notin T_M$" for all $k \notin T_{\alpha_\psi}^1$. Obviously then, $\langle M, E \rangle \simeq \langle L_{\alpha_\psi}, \in \rangle$ and *II* has no chance to pick out any infinite descending chains.   Q.E.D. (3)

The point is the following:

(4) Let τ be any winning strategy for I. Let $x = T_{\alpha_\psi}^1$; then $x \leq_T \tau$.
From this the theorem then follows as $x \notin L_{\alpha_\psi}$, being essentially the latter's Σ_1-truth set.

Proof of (4) We argue that, with *II* only playing constantly $n_k = 0$ for all k, that τ forces *I* to play for x a list of all and only the correct facts "$k \in / \notin T$" for $T_M = T^1_{\alpha_\psi}$. The point is to show that if at any time *I* deviates from this course of action, then he will lose - and hence the purported strategy τ is not a winning one.

II plays "*Pass*" (*ie* plays $n_k = 0$) until such a point, if ever, when *I* asserts '$k \in T_M$' or '$k \notin T_M$' whereas in reality $k \notin T^1_{\alpha_\psi}$ or $k \in T^1_{\alpha_\psi}$. At this point *II* knows that *I* 's eventual model $\langle M, E \rangle$ will be illfounded, and so she must act to discover a descending chain. In this case we shall set $\beta = \beta_M =_{df} \mathrm{On} \cap \mathrm{WFP}(M)$. However she will not yet know, and in fact will not at any move know, where β_M lies. All she will know is that as $(\mathrm{KP})_M$ (if *I* plays correctly) then by the Truncation Lemma (*cf.* [1]) $\beta_M \in \mathrm{ADM}$. By our requirements on the theory $T^1_{\alpha_\psi}$, and upwards persistence of Σ_1 formulae, we must have $\beta_M \leq \alpha_\psi$.

DEFINITION 5. *Let $F : \omega \twoheadrightarrow \mathrm{ADM} \cap \alpha_\psi + 1$ be some fixed surjection.*

The idea is that at rounds k where $r(k) = (i, j)$ *II* will be making the working assumption that the ordinal height of the wellfounded part of M, β_M, is precisely $F(i)$, and will be trying to find an illfounded chain through On^M above β_M. She will be working simultaneously on all such possible β_M. We shall prove that if *I* deviates from enumerating $T^1_{\alpha_\psi}$, then *II*, knowing only that one of them is the correct assumption, can nevertheless be successful and win the game G_ψ; thus *I* is forced to play only the truth concerning $T^1_{\alpha_\psi}$.

We assume then that *I* has played an untruth. We concentrate on a fixed i and hence on $\beta = \beta_M = F(i)$, and describe how *II* can move in rounds k with $r(k) = (i, j)$.

(5) *Claim* $\exists \bar{a} \notin \mathrm{WFP}(M) \forall b < \bar{a}(b \notin \mathrm{WFP}(M) \to T^2_b \not\subset \widetilde{T} =_{df} T^2_\beta$.

Proof. Supposed this failed, then $\forall \bar{a} \notin \mathrm{WFP}(M) \exists b < \bar{a}(b \notin \mathrm{WFP}(M) \wedge T^2_b \subset \widetilde{T})$. For such b we shall show that $J_{\gamma_b} \prec_{\Sigma_2} J_b$ for a $\gamma_b < \beta$. To do this we could employ a version of Theorem 3 on uniformly definable Skolem functions, to apply to the non-wellfounded model M. To do this one would simply apply h^2_b (the uniformly defined Σ_2-Skolem function, as defined over the illfounded model J_b), and look at $h^2_b{}^{``}\omega \times \omega$, and argue that this is a Σ_2 Skolem hull of J_b which is transitive, and in fact wellfounded, and is then a J_{γ_b} for a $\gamma_b < \beta$. However we prefer to argue for this directly as follows.

Let $\eta = \eta_b =_{df} \sup\{c < b \mid \exists f \in \Sigma^{J_b}_2, f : \omega \twoheadrightarrow c, f \text{ partial, onto}\}$. Note also, for use in a moment, that if $\{c\}$ is any $\Sigma^{J_b}_2$ definable ordinal (that is, defined without parameters) then $c < \eta$, as the L-least onto map $f : \omega \to c$ is then also $\Sigma^{J_b}_2$ definable (it lies in J_{c+1}). We first claim that $\eta < \beta$. Clearly equality fails, as otherwise that would make β definable inside M from b. If however $c \notin \mathrm{WFP}(M)$, with $f \in \Sigma^{J_b}_2$, f partial, but onto c, then the sentences "$f(n) \downarrow, f(m) \downarrow \wedge f(n) < f(m) \in \mathrm{On}$" are all in T^2_b and so in $\widetilde{T}$. This is absurd as β is wellfounded! Hence $\eta < \beta$. Note that this somewhat trivially implies that b is an $<_M$-limit ordinal: were $b = b_0 + 1$ then b_0 itself is $\Sigma^{J_b}_2$ and by the above reasoning we'd have the absurdity $b_0 \in \mathrm{WFP(M)}$!

It is not hard to see that η is closed under the Gödel pairing function and this implies that there is a parameter free $\Delta^{J_\eta}_1$ bijection $\eta \leftrightarrow J_\eta$ (*cf* [4]). Suppose $J_b \models \exists u \psi(u, \xi)$ where $\xi < \eta$ and $\psi \in \Pi_1$. (It suffices to verify Σ_2-elementarity just on formulae with single ordinal parameters ξ by the above remarks.)

Let δ be the least ordinal such that $J_b \models \text{``}\forall \delta' > \delta J_{\delta'} \models \exists u \psi(u, \xi).\text{''}$ Then $\{\delta\} \in \Pi_1^{J_b}(\{\xi\})$. There is thus a $\Sigma_2^{J_b}(\{\xi\})$ partial map $f_\delta : \omega \twoheadrightarrow \delta$ given by some formula: $f_\delta(m) = \tau \leftrightarrow \exists w \chi(w, m, \tau, \xi)$ for a Π_1 χ.

As $\xi = f_0(n)$ for some $\Sigma_2^{J_b}$ f_0, we have

$$f_\delta(m) = \tau \leftrightarrow \exists x [x = f_0(n) \wedge \exists w \chi(w, m, \tau, x)].$$

Replacing "$x = f_0(n)$" with its Σ_2 definition, this yields a parameter free $\Sigma_2^{J_b}$ definition of f_δ. Hence $\delta < \eta$. By the definition of δ we shall have $J_\eta \models \exists u \psi(u, \xi)$ as required.

Hence for such a b we have $(J_{\eta_b} \prec_{\Sigma_2} J_b)_M$. However the supposition implies there is an infinite descending chain of such b in the illfounded part of M. This implies that we have an infinite nested sequence of Σ_2 reflecting intervals: there exists $\langle b_n | n < \omega \rangle$, $\langle \eta_n | n < \omega \rangle$ with $(\eta_n \leq \eta_{n+1} \leq \ldots < b_{n+1} < b_n)$, and with $(J_{\eta_n} \prec_{\Sigma_2} J_{b_n})_M$, for $n < \omega$. Elementary reasoning using this Σ_2 reflection shows that each $\eta_n \in F$, and in fact is in $F^*, (F^*)^*, \ldots$; moreover such η_n are Σ_2-admissible. This contradicts our smallness hypothesis on our L hierarchies: more specifically then, I has broken a basic *Rule*.   Q.E.D. (5)

Let $\langle t_k | k \in \omega \rangle$ be our priorly fixed recursive enumeration of the Σ_1-Skolem terms (using the standard Σ_1-Skolem function, this could simply be an enumeration of $\langle h^1(i, n) \mid i, n < \omega \rangle$). *II* makes the additional working assumption, or guess if you will, that $t_j^M = a_0$, where a_0 is a witness for $\bar{a}$ to the truth of the last Claim. (Again the point is that *II* does not know in advance which term in M will denote such a_0.) As I reveals more and more facts about his model, he must, if M is not going to be isomorphic to L_{α_ψ}, at some point reveal a Σ_1-fact which is true in M but false in L_{α_ψ}. There really is then such an M-ordinal a_0. *II* will, in effect, place her 'guess' $a_0 = t_j^M$ at the head of her putative descending chain, on list $L_{i,j}$. In order to choose the next element of the chain *II* considers the set $\widetilde{T} = T_\beta^2$. Set $T_0 = (T_{t_j}^2)_M$.

II now waits until I asserts that some σ_0 is in T_0, (this itself being one of the Σ_1 facts about M she must enumerate) but *II* sees is not in $\widetilde{T}$. (If *II* is wrong in her guess about t_j of course, then she may fruitlessly wait for ever.)

(6) *Suppose* $M \models \text{``}a_1 < a_0$ *is least so that* $\forall b \leq a_0 (b \geq a_1 \rightarrow (\sigma_0)_{L_b}).\text{''}$ *Then* $a_1 \notin \text{WFP}(M)$.

Proof: Were $a_1 \in L_\beta$ then we should have $\sigma_0 \in \widetilde{T}$.   Q.E.D. (6)

II may thus wait until I asserts that some such $\sigma_0 \in T_0 \backslash \widetilde{T}$ and additionally, perhaps later, the Σ_1 fact that some term t_{j_1} names the ordinal a_1 defined in (4) above. And at some round l then, I must play the number $m_l = \ulcorner t_{j_1} \in \text{On} \wedge t_j \in \text{On} \wedge t_{j_1} < t_j \urcorner$; once all these facts have been gathered together, *II* may at the next appropriate round k with $r(k) = (i, j)$, set $n_k = t_{j_1}$.

II now has two elements of a descending chain in the illfounded part of M. Now she watches out for assertions that I makes about $T_1 = (T_2^{t_{j_1}})_M$, waiting for some σ_1 asserted by him to be in T_1 but which does not lie in $\widetilde{T}$. By exactly the same considerations that held at (4) some a_2, t_{j_2}, are definable, and so she can continue. By the end of the game, *if* this working assumption about β_M and t_j was the correct one,

the chain so defined by continuation of this process will be infinite, and she will have won.

If *I* deviates from playing the correct truth set, then at least one of *II*'s assumptions will turn out to be a correct one, and hence she will be assured of winning.

Q.E.D. (**(4)& Theorem 4**)

Remark: One might wonder why Σ_2 theories play the role they do here. However in this kind of game they are necessary. For suppose we tried to do without considering Σ_2 sentences but *II* tried to divine a descending chain using only Σ_1 sentences. Any M-ordinal is defined by a Σ_1 term t_k say. However 'new' Σ_1 facts about standard ordinals less than β_M may be true in M. Hence *II* cannot use the terms alone to pick out a chain without the danger of falling into the wellfounded part. Even if she waits to hear that a term t_l specifies a level hopefully in the ill-founded part where the Σ_1 term t_k (i.e. one not mentioned in $\widetilde{T}$) becomes true (which may be at a much higher level than the ordinal named by t_k) then this could also fail. For suppose that in M there is a $\gamma < b$ with γ in the wellfounded part, and b in the illfounded part, and with $J_\gamma \prec_{\Sigma_1} J_b$. No new Σ_1 facts become true within the interval $[\gamma, b)_M$. *I* may artfully play terms so that *II* ends up choosing a finite chain of terms naming ordinals where new Σ_1 facts are witnessed, with bottom element b. If she drops any further using such Σ_1 terms, she will be in the wellfounded part. Hence the use of the device above of looking where Σ_2-sentences become true.

Remark: One might further wonder whether the use of infinitely many lists only indicates our poor ability to devise a better game that used only one list. However the use here of infinitely many lists is necessary: for suppose it could be effected with a single list say, then the game would be $\Pi_2^0 \backslash \Pi_2^0$ (at least in the version described here) and it is known (see [8]) that the least level L_δ with $L_\delta \models \Sigma_1$-Comprehension is a model of Boolean(Σ_2^0)-Determinacy, and this is a very much smaller ordinal than the first Σ_2-extendible ζ above. Thus AQI proves there are β-models of Boolean(Σ_2^0)-Determinacy (indeed $\mathbf{\Delta}_3^0$-Determinacy).

Recall the following definition.

DEFINITION 6. *Let Γ be a pointclass. A set $Z \subseteq \mathbb{N}$ is said to be in $\partial \Gamma$ if there is a set $X \subseteq \mathbb{N} \times \mathbb{N}^{\mathbb{N}}$ in Γ so that*

$$Z = \{n \mid I \text{ has a winning strategy in } G(X_n; \omega^{<\omega}) \text{ where } X_n = \{y \mid \langle n, y \rangle \in X\}.$$

The next Corollary is in a sense merely a restatement of the result above. Let σ_3 denote the least σ so that every Σ_3^0 game that is a win for *I* , has a winning strategy in L_σ.

COROLLARY 2. *$\bar{\alpha} \leq \sigma_3$. Indeed each $T_{\alpha_\psi}^1$ is a $\partial \Sigma_3^0$ set of integers, as is Ψ.*

Proof: Let α_ψ *etc.* be defined as above. We switch roles in the games. *I* will try and find descending chains through *II* 's model M. (This is only to make the payoff set Σ_3^0 rather than Π_3^0.) For $\varphi \in \Sigma_1$ let $G_{\psi,\varphi}$ be the game described in the last theorem, except that *II* must now play a code x for a model of $T+$"there is no set model of T" $+\neg\varphi$. Everything else remains the same *mutatis mutandis*: *I* 's task is to find an infinite descending chain through the ordinals of *II* 's model. Note that if $\varphi \in T_{\alpha_\psi}^1$

I now has a winning strategy: for if *II* obeys her rules, and x codes an ω-model M of this theory, then M is not wellfounded, and has $\mathrm{WFP}(M) \cap \mathrm{On} < \alpha_\varphi$, where the latter is the least admissible α where φ is true in L_α. However *I* playing (just as *II* did in the last theorem) can find a descending chain and win. On the other hand if $\varphi \notin T^1_{\alpha_\psi}$ *II* may just play a code for the true wellfounded L_{α_ψ} and so win. This shows that $T^1_{\alpha_\psi}$ is a $\supseteq\Sigma^0_3$ set of integers.

Suppose now that $\bar{\alpha} > \sigma_3$. Let ψ be such that α_ψ is the second least admissible greater than σ_3. There is thus a set $H \in L_{\alpha_\psi}$ (definable over the next admissible $\gamma > \sigma_3$) containing winning strategies for all Σ^0_3-games that are a win for player *I* and in particular a set H_0, definable at the same level, of those winning strategies for *I* in games of the form $G_{\psi,\varphi}$. Hence membership of φ in $T^1_{\alpha_\psi}$ is determined by searching through H_0 for a winning strategy for *I* ; this is a bounded search. Hence $T^1_{\alpha_\psi} \in \Delta_1^{L_{\alpha_\psi}}(\{H_0\})$. Hence $T^1_{\alpha_\psi} \in L_{\alpha_\psi}$ which is a contradiction.

The result for Ψ can be obtained similarly; recall that we had spare room in our smallness assumption, in that we could rerun all our arguments for $\Psi' =_{df} T^1_{\delta'}$ where δ' is least in F^*. Q.E.D.

COROLLARY 3. *The complete* AQI *set is a* $\supseteq\Sigma^0_3$ *(but not* $\supseteq\Sigma^0_2$*) set of integers.*

§**4. Boldface Σ^0_3-Determinacy is strictly weaker than Π^1_3-CA$_0$.** We shall closely follow Martin's account of Davis's proof ([3]) of Σ^0_3-Determinacy. That account is performed within $\mathrm{ZC}^- + \Sigma_1$-Replacement, but we shall pay attention to definability considerations. It is fairly easy to see that the proof can be effected in the weaker theory of KP augmented by Σ_2-Replacement and Σ_2-Comprehension, but we want to do better than that. As remarked above KP $+\Sigma_2$-Comprehension (or equivalently Π^1_3-CA$_0$) proves the existence of β-models M of the theory S comprising: $V = L+$"*There are ordinals $\gamma_0 < \gamma_1 \in \mathrm{ON}^M$ with $(L_{\gamma_0} \prec_{\Sigma_1} V \wedge L_{\gamma_0} \prec_{\Sigma_2} L_{\gamma_1})$*"$+\forall x \exists y(\mathrm{Trans}(y) \wedge y$ is admissible).

Notice that if M is a model as just described, and we fix for definiteness, $\gamma_0 < \gamma_1$ the lexicographically least such pair fulfilling S, then for parameters $t \in L_{\gamma_0}$, $\Sigma_2^{L_{\gamma_1}}(\{t\})$-definable subsets of ω are all in L_{γ_1}, and are in fact so-definable over $(L_{\gamma_0})_M$. We set $\gamma_2 = \mathrm{ON}^M$. The reader should be warned however that L_{γ_1} in general is not admissible: indeed requiring the existence of $L_{\delta_0} \prec_{\Sigma_2} L_{\delta_1}$ with the latter admissible, is much stronger than the assumptions here, and proves the existence of many triples of ordinals $\gamma_i(i < 3)$ as above.

We shall show for such an M :

THEOREM 5. $L_{\gamma_0}^M \models \boldsymbol{\Sigma^0_3}$-Determinacy.

By "$N \models \boldsymbol{\Sigma^0_3}$-Determinacy" we shall mean that for game trees $T \subseteq {}^{<\omega}\omega$ with $T \in N$ and for any $A \in \Sigma^0_3(T)$, that a winning strategy σ for player *I* or *II* exists in N. As $L_{\gamma_0}, L_{\gamma_1}$ are both models of KPI$_0$, the property of any $\sigma \in L_{\gamma_0}$ being a winning strategy for such a game $G(A; T)$ is absolute between L_{γ_0} and the universe. This will complete the first theorem of the abstract (as well as the non-reversibility of its implication, since by taking $\gamma_0, \gamma_1, \gamma_2$ least with such properties we have that then Σ_2-Separation fails in L_{γ_0}).

Proof: We shall assume that $V = M$ where M is a model with the above properties. We shall thus drop the subscript M throughout the proof. Let A be $\Sigma_3^0(T)$ for some game tree $T \in L_{\gamma_0}$. We shall show $G(A;T)$ is determined, with either a winning strategy for I in L_{γ_0} or a winning strategy for II definable over L_{γ_0}. We suppose that player I does not have a winning strategy in L_{γ_0} and shall then proceed to prove that the latter case for II does hold. The key lemma is the following.

LEMMA 3. *Let* $B \subseteq A \subseteq \lceil T \rceil$ *with* $B \in \Pi_2^0$. *If* $(G(A;T)$ *is not a win for* $I)_{L_{\gamma_0}}$, *then there is a quasi-strategy* $T^* \in L_{\gamma_0}$ *for* II *with the following properties:*
(i) $\lceil T^* \rceil \cap B = \varnothing$;
(ii) $(G(A;T^*)$ *is not a win for* $I)_{L_{\gamma_0}}$.

Remark: We recall from Lemma 2 that "$G(A;T)$ is not a win for I" is a $\Pi_1^{\mathrm{KPI_0}}$ statement about A, T and is absolute between $L_{\gamma_0}, L_{\gamma_1}$, and $V = L_{\gamma_2}$ by our elementarity properties assumed of these models; similarly if there is such a T^* as claimed by the conclusion, then $(G(A;T^*)$ is not a win for $I)_{L_{\gamma_i}}$ for $i < 3$.

Proof of Lemma 3. By Lemma 2(ii) "p is in II's non-losing quasi-strategy for $G(A;T)$" is not only $\Pi_1^{L_{\gamma_0}}$, but also $\Pi_1^{L_{\gamma_i}}$ for $i = 1, 2$, due to the assumed Σ_1 elementarity. We let T' denote then this non-losing quasi-strategy as defined, equivalently, over any of the above models. As L_{γ_0} is a model of Σ_1-Comprehension, $T' \in L_{\gamma_0}$, and we thus have, summarising, that for every $p \in T$:

(1) $(p \in T')_{L_{\gamma_0}} \longleftrightarrow (p \in T')_{L_{\gamma_i}}$ *for* $i = 1, 2$. *Hence* T' *is* II *'s non-losing quasi-strategy for* $G(A;T)$ *in* V.

Likewise for any $p \in T'$ "$G(A, T'_p)$ is not a win for I" is absolute between L_{γ_i} for $i < 3$.

Following closely the original argument, we call a position $p \in T'$ *good* if there is a quasi-strategy T^* for II which is a sub-tree of T'_p so that the following hold:
(i) $\lceil T^* \rceil \cap B = \varnothing$;
(ii) $G(A;T^*)$ is not a win for I.

Here (ii), again, is a $\Pi_1^{\mathrm{KPI_0}}(T^*)$ statement by Lemma 2. Thus the existence of such a T^* becomes a $\Sigma_2^{\mathrm{KPI_0}}$ fact about T (as (i) is simply $\Delta_1^{\mathrm{KPI_0}}$). Hence by our assumed Σ_2 reflection properties it makes no difference whether we defined 'good' relative to L_{γ_0} or L_{γ_1}:

(2) "p is good" *is* $\Sigma_2^{\mathrm{KPI_0}}(T)$ *and hence* $(p$ is good$)_{L_{\gamma_0}} \longleftrightarrow (p$ is good$)_{L_{\gamma_1}}$.

We set $H \subseteq {}^{<\omega}\omega$ to be the class $\{p \in T' \mid (p$ *is good*$)_{L_{\gamma_0}}\}$. Then

(3) $H \in \Sigma_2^{L_{\gamma_0}}(T)$ *and hence* H *is a set in* L_{γ_1}.

We define the function $t : H \longrightarrow L_{\gamma_0}$ by:

$t(p) = L$-least quasistrategy $\widehat{T}(p)$ witnessing (i) and (ii) that $(p$ is good $)_{L_{\gamma_0}})$.

Then t is definable over L_{γ_0} in T', but is also $\Sigma_2^{L_{\gamma_0}}(\{T'\})$ as a relation.

We are thus trying to prove that the starting position $\varnothing$ is good in L_{γ_0}, and and we have seen that such a quasi-strategy T^* exists satisfying (i) and (ii) in L_{γ_1} if and only if such exists in L_{γ_0}.

Let $B = \bigcap_{n \in \omega} D_n$ with each D_n recursively open. Define

$$E_n = A \cup \{ x \in \lceil T' \rceil \mid (\exists p \subseteq x (\lceil T'_p \rceil \subseteq D_n \wedge \text{``}p \text{ is not good''}) \}.$$

Then, by (2), E_n has a $\Sigma_3^{\mathrm{KPI_0}}$ definition (in the parameter T'). By the Σ_2-admissibility of L_{γ_0} the bounded integer quantifier can be absorbed and ($\text{``}x \in E_n\text{''})_{L_{\gamma_0}}$ is $\Pi_2^{L_{\gamma_0}}$. In any case by our elementarity assumptions we have:

(4) $x \in L_{\gamma_0} \longrightarrow ((x \in E_n)_{L_{\gamma_0}} \longleftrightarrow (x \in E_n)_{L_{\gamma_1}})$.

Note also that E_n, when defined over L_{γ_1}, can be considered a $\Sigma_3^0(H)$ set of reals, fixing the good parameter set H as above, that was defined over L_{γ_0}. Hence, in order to differentiate E_n when defined over L_{γ_1} (or L_{γ_0}) and V we set:

$$\tilde{E}_n = A \cup \{ x \in \lceil T' \rceil \mid (\exists p \subseteq x (\lceil T'_p \rceil \subseteq D_n \wedge p \notin H) \}.$$

Then $(E_n)_{L_{\gamma_1}} = \tilde{E}_n \cap L_{\gamma_1}$, and $(E_n)_{L_{\gamma_0}} = \tilde{E}_n \cap L_{\gamma_0}$. The proof now proceeds by showing:

(+) $\exists n \in \omega \forall \sigma \in \boldsymbol{\Sigma}_\omega(L_{\gamma_1})(\sigma \text{ is not winning strategy for } I \text{ in } G(\tilde{E}_n; T'))_V$

We first suppose that $(+)$ can be proven, and work towards showing how this yields the Lemma:

(5) $(+) \Rightarrow \exists T^* \in L_{\gamma_0}(L_{\gamma_0} \models \text{``}T^* \text{ witnesses that } \varnothing \text{ is good ''}).$

We assume then $(+)$.

DEFINITION 7. *For $i = 0, 1$:*
  $T_i'' =_{\mathrm{df}} \{ q \in T' \mid \forall p \subseteq q (G(E_n, T'_p) \text{ is not a win for } I)_{L_{\gamma_i}} \}.$

In short, T_i'' is *II* 's non-losing quasi-strategy for $G(E_n; T')$, as defined over L_{γ_i}. "$(G(E_n, T'_p)$ *is not a win for I)$_{L_{\gamma_i}}$*" means for all $\sigma \in L_{\gamma_i}$ which are strategies for *I* , there is $x \in L_{\gamma_i}$ with $\sigma * x \notin (E_n)_{L_{\gamma_i}}$. Neither T_i'' is *a priori* a set in L_{γ_i}, but are definable classes over the respective models: T_0'' is $\Pi_3^{L_{\gamma_0}}$ and T_1'' is $\Pi_1^{L_{\gamma_1}}(H)$. For $i = 0, 1$ we now further define $T_i^* \subseteq T_i''$ quasi-strategies for *II* as follows.

DEFINITION 8. $q \in T_i^*$ if either*:*
(a) $q \in T_i''$ and for all positions $p \subseteq q \lceil T'_p \rceil \nsubseteq D_n$; or
(b) let there be a shortest initial segment $p \subseteq q$ with $p \in T_i''$ and $\lceil T'_p \rceil \subseteq D_n$. In this latter case, by definition of T_i'' $p \in H$. (The latter must hold, since otherwise we should have that $(G(E_n; T'_p)$ is a win for $I)_{L_{\gamma_i}}$ from position p onwards, I making use of some arbitrarily defined but trivial strategy in L_{γ_i}, contradicting that $p \in T_i''$.) If the subsequent moves in q are consistent with $t(p) = \hat{T}(p)$, then we also put q into T_i^*. Otherwise $q \notin T_i^*$.

The following then hold:

(6) $T_1'' \subseteq T_0''$; $T_1^* \subseteq T_0^*$.

Proof: We show just the first, as the second is then straightforward: this would be simple absoluteness if H were an element of L_{γ_0}. Suppose this failed; let $q \in T_1'' \backslash T_0''$. So on one hand $L_{\gamma_0} \models$ "$\exists p \subseteq q \exists \sigma \forall x \; \sigma * x \in \lceil T_p' \rceil \cap E_n$." However on the other σ is not a win for I in L_{γ_1} thus $L_{\gamma_1} \models$ "$\exists z (\sigma * z \notin \lceil T_p' \rceil \cap E_n)$." Suppose there exists such an z satisfying

$$\sigma * z \in \lceil T_p' \rceil \backslash A \wedge \forall k < \omega \sigma * z \restriction k \in \{t \mid \lceil (T_p')_t \rceil \nsubseteq D_n\}.$$

Then by absoluteness this would go down to L_{γ_0} contradicting the assumption that σ is, in L_{γ_0}, a winning strategy for I in $G(E_n, T_p')$. Hence for all z witnessing that σ is not, in L_{γ_1} a winning strategy for I in $G(E_n, T_p')$ must, for some k, satisfy $\lceil (T_p')_{\sigma * z \restriction k} \rceil \subseteq D_n$. However, for such k also, $\sigma * z \restriction k$ is good. Then

$$\text{"}\exists z \big(\sigma * z \in \lceil T_p' \rceil \backslash A \wedge \lceil (T_p')_{\sigma * z \restriction k} \rceil \subseteq D_n \wedge \sigma * z \restriction k \text{ is good "}$$

is $\Sigma_2^{L_{\gamma_1}}(\sigma)$ and by elementarity goes down to L_{γ_0}. Again this would contradict our assumption on σ.   QED (6)

Note that T_i^* are still quasi-strategies for II in T'. The following then finishes the Lemma (still under the assumption $(+)$), since as remarked above, "$\varnothing$ is good" is $\Sigma_2^{\mathrm{KPI}_0}(T)$ so the existence of such a witnessing tree T_0^* goes down from L_{γ_1} to L_{γ_0}.

(7) $(T_0^*$ witnesses that $\varnothing$ is good$)_{L_{\gamma_1}}$.

Proof: If $x \in \lceil T_0^* \rceil$, then either $x \notin D_n$ or $x \in \lceil \hat{T}(p) \rceil$. In the latter case, as $\hat{T}(p)$ witnesses that p is good, (i) ensures that $x \notin B$. Thus in either case $\lceil T_0^* \rceil \cap B = \varnothing$. This gives part (i) of goodness. We next show that

Claim: $\forall \sigma \in L_{\gamma_1} (\sigma$ *is not a winning strategy for I in* $G(A; T_1^*))_V$.

Suppose for a contradiction that $\sigma \in L_{\gamma_1}$ were a winning strategy for him in this game.
Subclaim 1 $\forall y (\sigma * y \in [T_1''])$.
Proof: There cannot be a least position $p \in T_1''$ consistent with σ so that $\lceil T_p' \rceil \subseteq D_n$: for otherwise for this p we'd have $T_p^* = \hat{T}(p)$ for the same reason as in Def. 8 (b): if p were not in H we'd have
$L_{\gamma_1} \models$ "p is not good $\wedge \lceil T_p' \rceil \subseteq D_n \wedge G(E_n, T_p')$ is a win for I ."
contradicting that $p \in T_1''$. As there is no such position p like this, we must have that $\forall x \; \sigma * x \in \lceil T_1'' \rceil$.
                                                                        QED Subclaim 1

Subclaim 2 $\forall \sigma_0 \in L_{\gamma_1} (\sigma_0$ *is not a winning strategy for I in* $G(\tilde{E}_n; T_1''))_V$.
Proof: We reason in $V = L_{\gamma_2}$. Suppose for a contradiction $\tau_0 \in L_{\gamma_1}$ were such a winning strategy in $G(\tilde{E}_n; T_1'')$. Recall that T_1'' is potentially a proper class of L_{γ_1}. We convert this to a strategy τ for I in $G(\tilde{E}_n, T')$ which is winning in V.

We define τ by initially letting I play using τ_0. If II never departs from T_1'' and I uses τ_0 throughout, then this is a winning run of play for I in $G(\tilde{E}_n, T')$. On the other hand, assume II departs from T_1'' at some position p; then, as $p \notin T_1''$ we conclude (I has a winning strategy for $G(\tilde{E}_n, T_p'))_{L_{\gamma_1}}$ and, as the definition of the game is arithmetical in H, T', the latter statement is absolute to V. I may continue playing using the L_{γ_1}-least such winning strategy, which we may call σ_p. By continuing to play with σ_p he wins overall $G(\tilde{E}_n, T')$. The map $\pi \; p \rightarrowtail \sigma_p$ is definable over L_{γ_1}, and hence the overall cumulative strategy τ we have just implicitly described is also definable over L_{γ_1} from π and τ_0. It is thus an element of $\Sigma_\omega(L_{\gamma_1})$. However the existence of τ contradicts $(+)$.   QED Subclaim 2

In particular, using Subclaim 2: as $\sigma \in L_{\gamma_1}$, σ itself is not a winning strategy for I in $G(\tilde{E}_n; T_1''))$. Moreover by Subclaim 1 any play consistent with σ is in $[T_1'']$. Hence there is a play $x = \sigma * x_0$ consistent with σ satisfying $x \notin \tilde{E}_n$. As $\tilde{E}_n \supseteq A$, we thus have $x \notin A$. However it was originally assumed that σ was a winning strategy for I in $G(A; T_1^*)$. This is a contradiction!   QED *Claim*

(8) $(G(A; T_0^*)$ *is not a win for* $I)_{L_{\gamma_1}}$.

Proof: By the *Claim*, there is no $\sigma \in L_{\gamma_1}$ a winning strategy for I in $G(A; T_1^*)$. However $T_1^* \subseteq T_0^*$ and T_1^* does not restrict any of I 's moves. Hence (8) and thence (7) hold.

QED (7), (8)

We can finish (5):
By (7) we see that a T_0^* witnessing the requisite Σ_2 formula can be constructed definably over L_{γ_0} from T. Thus $T_0^* \in L_{\gamma_1}$. Hence by Σ_2-reflection of L_{γ_1} there is then such a $T^* \in L_{\gamma_0}$.   QED (5)

We thus have to show that $(+)$ above holds. We showed that:
$\varnothing$ *is not good in* $L_{\gamma_0} \longrightarrow$
$\forall n \in \omega$ (*There is a winning strategy for* I *in* $G(\tilde{E}_n; T')$ *which is* $\boldsymbol{\Sigma}_\omega(L_{\gamma_1})$-*definable*$)_V$.

If we define:
$$\tilde{E}_n^p = A \cup \{x \in \lceil T' \rceil \mid (\exists q \subseteq x(p \subseteq q \wedge \lceil T_q' \rceil \subseteq D_n \wedge q \notin H)\}$$

then the same argument shows that:

(9) $\forall p \in T'$ (*p is not good in* $L_{\gamma_0} \longrightarrow$
$\forall n \in \omega$(*There is a winning strategy for* I *in* $G(\tilde{E}_n^p; T_p')$ *which is* $\boldsymbol{\Sigma}_\omega(L_{\gamma_1})$-*definable*$)_V$).

We suppose the overall Lemma 3 false and obtain a contradiction by building a strategy σ for I for the game $G(A; T')$ which is definable over L_{γ_1} and winning in V.

We define the function $s : \overline{H} \times \omega \longrightarrow L_{\gamma_1+1}$ defined by: $s(p, n) = L_{\gamma_1+1}$-least winning strategy for I in $G(\tilde{E}_n^p; T_p')$. By (9) this function is well defined, and total, on $\overline{H} \times \omega$ and moreover is definable over the least admissible set containing L_{γ_1}, H, T'.

Let $\sigma_0 = s(\varnothing, 0)$. Then σ_0 is a winning strategy for I in $G(\tilde{E}_0; T')$. σ agrees with σ_0 until a first, if such occurs, position p_0 is reached with $\lceil T'_{p_0} \rceil \subseteq D_0$ but $p_0 \notin H$. If so, then we use the strategy $\sigma_1 = s(p_0, 1)$ for I in $G(\tilde{E}_1^{p_0}; T'_{p_0})$. σ now agrees with σ_1 until, if ever, a position p_2 is reached with $\lceil T'_{p_1} \rceil \subseteq D_1$ but p_1 is not good. The play continues using $\sigma_2 = s(p_2, 2)$. If $q = \bigcup_{n \in \omega} p_n$ is a non-terminal position, we let σ take some arbitrary but canonical choice on positions extending q. The strategy σ is then definable by a simple recursion involving s, H, T'.

(10) If $x \in V$ is any play consistent with σ then $x \in A$.

Proof: Suppose firstly we have that for some n $p_n \subseteq x$ is undefined. This implies that there is no initial position $p \subseteq x$ with (a) $p_{n-1} \subseteq p$ (if $n > 0$); (b) $\lceil T'_p \rceil \subseteq D_n$, and (c) $p \notin H$. On the other hand, if all the p_n are defined, then we shall have that $x \in \bigcap_{n \in \omega} D_n \subseteq B \subseteq A$. Either way we have shown that any play $x \in V$ arising from following the strategy σ lies in $A \cap [T']$. QED(10)

However T' is defined over L_{γ_0} to be II's non-losing quasi-strategy in $G(A; T)$ and at (1) it was shown that T' had this property in V. This contradicts (10)! This finishes the proof of the Lemma 3. QED Lemma 3

The proof of the theorem now follows Martin [14] pretty much verbatim but again paying attention to definability issues. We repeatedly apply the Lemma with $A = \bigcup_{n \in \omega} A_n$ and each $A_n \in \Pi_2^0$, acting in turn as an instance of B in the Lemma. This is a Σ_2-recursion defining a strategy τ for II over L_{γ_0} since all the relevant quasi-strategies given by the Lemma lie in this model. These details now follow.

One applies the lemma with $B = A_0$ obtaining a quasi-strategy for II : $T^*(\varnothing)$. By Σ_2-reflection the L-least such lies in L_{γ_0}, and we shall assume that $T^*(\varnothing)$ refers to it. For any position $p_1 \in T$ with $\mathrm{lh}(p_1) = 1$, let $\tau(p_1)$ be some arbitrary but fixed move in $T'(\varnothing)$, II 's non-losing quasi-strategy for the game $G(A, T^*(\varnothing))$. The relation "$p \in T'(\varnothing)$" is $\Pi_1^{L_{\gamma_0}}(\{T^*(\varnothing)\})$ and hence "$y = T'(\varnothing)$" $\in \Delta_2^{L_{\gamma_0}}(\{T^*(\varnothing)\})$ and thus $T'(\varnothing)$ also lies in L_{γ_0}. For definiteness we let $\tau(p_1)$ be the numerically least move. For any play, p_2 say, of length 2 consistent with the above definition of τ so far, we apply the lemma again with $B = A_1$ and with $(T^*(\varnothing))_{p_2}$ replacing T. This yields a quasi-strategy for II, call it $T^*(p_2)$, which is definable in a Σ_2 way over L_{γ_0}, in the parameter $(T^*(\varnothing))_{p_2}$. Let $T'(p_2) \in L_{\gamma_0}$ be II 's non-losing quasi-strategy for $G(A, T^*(p_2))$, this time with "$y = T'(p_2)$" $\in \Delta_2^{L_{\gamma_0}}(\{T^*(p_2)\})$. Again for $p_3 \in T^*(p_2)$ any position of length 3, let $\tau(p_3)$ be some arbitrary but fixed move in $T'(p_2)$. Now we consider appropriate moves p_4 of length 4, and reapply the lemma with $B = A_2$ and $(T^*(p_2))_{p_4}$. Continuing in this way we obtain a strategy τ for II, so that $\tau \restriction^{2k+1} \omega$, for $k < \omega$, is defined by a recursion that is $\Sigma_2^{L_{\gamma_0}}(\{T\})$. As $L_{\gamma_0} \models \Sigma_2$-KP, we have that $\tau \in L_{\gamma_0}$. If x is any play consistent with τ, then for every n, by the defining properties of $T^*(p_{2n})$ given by the relevant application of the lemma, $x \in \lceil T^*(x \restriction 2n) \rceil \subseteq \neg A_n$. Hence $x \notin A$, and τ is a winning strategy for II as required. QED(**Theorem 5**)

Since γ_0 is Σ_2-extendible, it is a model of Σ_2-KP. Since the existence of such a γ_0 was established in KP $+\Sigma_2$-Comprehension, the last theorem establishes Theorem 1 of the abstract.

We can reduce our assumptions an epsilon as infinitely many admissibles beyond γ_1 are not needed:

COROLLARY 4. *Let $A \in \Sigma_3^0$. Suppose T is a suitable game tree for A; then if (γ_0, γ_1) is the lexicographic least pair, so that setting γ_2 to be the third admissible ordinal above γ_1 we have: $\gamma_0 < \gamma_1 < \gamma_2$; with $T \in L_{\gamma_0}^1$; $L_{\gamma_0} \prec_{\Sigma_2} L_{\gamma_1}$ and $L_{\gamma_0} \prec_{\Sigma_1} L_{\gamma_2}$, then $L_{\gamma_0} \models$ "$G(A, T)$ is determined".*

Proof: That $V = L_{\gamma_2} \models \mathrm{KPI}_0$ was only used on the penultimate line of the proof of Lemma 3, to reason that T' as defined in L_{γ_0} as *II*'s non-losing quasi-strategy in $G(A, T)$ was still this in V. Let γ_1^+ denote the next admissible above γ_1. However all that is required at that point in the proof is that the strategies in $L_{\gamma_1^+ + 1}$, and in particular the σ being there defined over $L_{\gamma_1^+}$ using the function s, give no reason to eject any p from T' when defining it afresh in V. This can be ascertained definably over $L_{\gamma_1^{++}}$ the next admissible set containing $L_{\gamma_1^+}$. hence infinitely many more admissibles in V are not required.   QED

REFERENCES

[1] K.J. BARWISE, *Admissible sets and structures*, Perspectives in Mathematical Logic, Springer Verlag, 1975.

[2] J.P. BURGESS, *The truth is never simple*, this JOURNAL, vol. 51 (1986), no. 3, pp. 663–681.

[3] M. DAVIS, *Infinite games of perfect information*, *Annals of Mathematical Studies*, vol. 52 (1964), pp. 85–101.

[4] K. DEVLIN, *Constructibility*, Perspectives in Mathematical Logic, Springer Verlag, Berlin, Heidelberg, 1984.

[5] H. FIELD, *A revenge-immune solution to the semantic paradoxes*, *Journal of Philosophical Logic*, vol. 32 (2003), no. 3, pp. 139–177.

[6] S. D. FRIEDMAN, *Parameter free uniformisation*, *Proceedings of the American Mathematical Society*, vol. 136 (2008), pp. 3327–3330.

[7] J.D. HAMKINS and A. LEWIS, *Infinite time Turing machines*, this JOURNAL, vol. 65 (2000), no. 2, pp. 567–604.

[8] C. HEINATSCH and M. MÖLLERFELD, *Determinacy in second order arithmetic*, *Foundations of the formal sciences v* (London) (S. Bold, B.Löwe, Th. Räsch, and J. van Benthem, editors), Studies in Logic, College Publications, 2007, pp. 143–155.

[9] H.G. HERZBERGER, *Notes on naive semantics*, *Journal of Philosophical Logic*, vol. 11 (1982), pp. 61–102.

[10] T. JOHN, *Recursion in Kolmogoroff's R operator and the ordinal σ_3*, this JOURNAL, vol. 51 (1986), no. 1, pp. 1–11.

[11] A. S. KECHRIS, *On Spector classes*, *Cabal seminar 76-77* (A. S. Kechris and Y. N. Moschovakis, editors), Lecture Notes in Mathematics Series, vol. 689, Springer, 1978, pp. 245–278.

[12] S. KREUTZER, *Partial fixed point logic on infinite structures*, *Annual conference of the european association for computer science logic (csl)*, Lecture Notes in Computer Science, vol. 2471, Springer, 2002.

[13] R. LUBARSKY, *ITTMs with feedback*, *Ways of proof theory: Festschrift for W. Pohlers* (R-D. Schindler, editor), Ontos Series in Mathematical Logic, 2010.

[14] D. A. MARTIN, *Determinacy*, Unpublished book manuscript.

[15] M. RATHJEN, *An ordinal analysis of parameter-free Π_2^1 comprehension*, *Archive for Mathematical Logic*, vol. 44 (2005), no. 3, pp. 263–362.

[16] ———, *An ordinal analysis of stability*, *Archive for Mathematical Logic*, vol. 44 (2005), no. 1, pp. 1–62.

[17] G.E. SACKS, *Higher recursion theory*, Perspectives in Mathematical Logic, Springer Verlag, 1990.

[18] S. SIMPSON, *Subsystems of second order arithmetic*, Perspectives in Mathematical Logic, Springer, January 1999.

[19] K. TANAKA, *Weak axioms of determinacy and subsystems of analysis, II: Σ_2^0-games*, **Annals of Pure and Applied Logic**, vol. 52 (1991), pp. 181–193.

[20] P. D. WELCH, *Eventually Infinite Time Turing degrees: infinite time decidable reals*, this JOURNAL, vol. 65 (2000), no. 3, pp. 1193–1203.

[21] ——— , *The length of infinite time Turing machine computations*, **Bulletin of the London Mathematical Society**, vol. 32 (2000), pp. 129–136.

SCHOOL OF MATHEMATICS,
UNIVERSITY OF BRISTOL,
BRISTOL,
BS8 1TW,
ENGLAND
E-mail: p.welch@bristol.ac.uk