On a question of Deolalikar, Hamkins & Schindler

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We consider some queries arising from the paper [1]. These concerned various complexity pointclasses defined using halting times of computations on Infinite Time Turing machines, with or without existential 'non-determinacy' witnesses. We refer the reader for all our definitions etc. to the introduction of that paper, which we shall assume the reader has to hand. Besides their question immediately below, we make some further comments and improvements on several of their other theorems.

They pose the following question:

Question 6 *Suppose an algorithm halts on each input x in fewer than $\omega_{1\mathrm{ck}}^x$ steps. Then does it halt uniformly before $\omega_{1\mathrm{ck}}$?*

As they say an affirmative answer explains some of the phenomena observed in their paper. This is the case (we drop the subscript ck and write ω_1^x for the first ordinal not recursive in x etc.).

Proposition 1. *Let F: $\mathbb{R} \longrightarrow \mathbb{R}$ be ITTM-computable and total as witnessed by φ_e. If $\forall x \varphi_e(x)\downarrow^{<\omega_1^x}$ (meaning halts in $<\omega_1^x$ steps) then $\exists\gamma<\omega_1\ \forall x\,\varphi_e(x)\downarrow^{<\gamma}$.*

Proof Let $y_x \in \mathrm{WF}$ be a code for the computation sequence of $\varphi_e(x)\downarrow$ witnessing that it halts. Then such a y exists in $L_{\omega_1^x}[x]$. So let y_x be the $L[x]$-least code for such a sequence. Let $\Phi(y, e, x)$ abbreviate:

" y is the $L[x]$-least code for a wellfounded halting computation sequence witnessing $\varphi_e(x)\downarrow$"

But then $\Phi(y,e,x) \Longleftrightarrow L_{\omega_1^x}[x] \models \Phi(y,e,x)$.

$\Phi(y,e,x)$ can be expressed as a Σ_1 statement over $L_{\omega_1^x}[x]$. Moreover

$$\neg\Phi(y,e,x) \Longleftrightarrow L_{\omega_1^x}[x] \models \exists z(\Phi(z,e,x) \wedge z \neq y) .$$

Hence $\Phi(y,e,x)$ is Δ_1^1. Hence

$$B = \{y_0 \mid \exists x\,(\,\Phi(y,e,x) \wedge y_0 = \mathrm{Field}(y))\ \} \in \Sigma_1^1 \cap \mathrm{WO}.$$

By (lightface) Σ_1^1-boundedness, (see, *e.g.*, [3] 4A.6) $\{\|y_0\| : y_0 \in B\}$ is bounded in ω_1. □

At the close of Section 4 of their paper they consider an analysis of the concept of NP^f for various suitable f, and say that "one should be able to show that $\mathrm{NP}^f = \Gamma^f$-dual". We give a counterexample to this as Proposition 3. If $L_{f(x)}[x]$ is sufficiently closed then $\mathrm{NP}^f = \Gamma^f$. The point here is to ask the question "where does the existential witness y to x being in some NP^f set live?"

So let f be suitable such that $L_{f(x)}[x] \models \mathrm{KPI}$ (KPI is the theory asserting that the universe is an admissible set which is a union of such.) Let:

$$\Gamma^f = \{A \subseteq \mathbb{R} : \exists\Sigma_1\varphi\forall x\,[x \in A \longleftrightarrow L_{f(x)}[x] \models \varphi[x]\ ]\ \}.$$

Proposition 2. $\mathrm{NP}^f = \Gamma^f$; $P^f = \Delta(\Gamma^f) = \mathrm{NP}^f \cap \mathrm{coNP}^f$.

Proof. It suffices to show that if $A \in \mathrm{NP}^f$ then $A \in \Gamma^f$. The other inclusions follow by general arguments. φ_e be such that $\forall x, y\, \varphi_e(x, y)\downarrow^{<f(x)}$ and $A = \{x : \exists y\; \varphi_e(x, y)\downarrow 1\}$. So where can we find a y if x is in A?

Suppose $x \in A$ and y witnesses this. Suppose $\varphi_e(x, y)\downarrow^{\gamma} 1$ with $\gamma < f(x)$. Let $u_\gamma \in \mathcal{M}_x =_{\mathrm{df}} L_{f(x)}[x]$, $u_\gamma \in \mathrm{WO}$, with $\|u_\gamma\| = \gamma$. Let

$$B = \{\, y : \exists z\, (\, z \text{ codes a wellfounded comp. witnessing } \varphi_e(x, y)\downarrow^{\|u_\gamma\|} 1 \;) \quad \}$$

$B \neq \varnothing$, and $B \in \Sigma^1_1(x, u_\gamma)$. Hence $\exists y_0 \in B\; y_0 \leq_T \mathcal{O}^{x, u_\gamma}$ by the Kleene Basis Theorem (see, *e.g.*, [5] Theorem III.1.3, $\mathcal{O}$ here is Kleene's $\mathcal{O}$ notation.) However then there is such a $y_0 \in \mathcal{M}_x$, as $\mathcal{O}^{x, u_\gamma} \in \mathcal{M}_x$ by our KPI assumption. So now

$$\forall x\, [\, x \in A \Longleftrightarrow \mathcal{M}_x \vDash \text{“}\exists y_0\, \varphi_e(x, y_0)\downarrow 1\text{ ”}\,]$$

and this yields a defining Σ_1 formula for A, putting A into Γ^f. □

Corollary 3. *If* $f(x) = \lambda^x$, *then* $\mathrm{NP}^f =$ *Semi-decidable.*

The previous Proposition shows that Γ^f can be NP^f; (not of course, for $f(x) = \omega_1^x$, or indeed for functions whose values are uniformly other successor x-admissibles). We can get another Bounding Lemma:

Proposition 4. *(Bounding Lemma) Suppose* β *be admissible. Let* F *be ITTM-computable, total so that* $\forall x\; \varphi_e(x)\downarrow^{\leq \beta}$ *where* φ_e *computes* F*. Then* $\exists \gamma < \beta \quad \forall x\; \varphi_e(x)\downarrow^{<\gamma}$.

We first recall Definition 25 from [1]: an ordinal α is *non-deterministically clockable* if there is some algorithm that halts on some input in exactly α steps, and on any input in $\leq \alpha$ steps. The following Corollary is then a restatement of the last proposition.

Corollary 5. *No admissible* β *is non-deterministically clockable.*

Proof. (of Proposition 4)

Suppose the proposition false as witnessed by the total function $\varphi = \varphi_e$. Then β is obviously countable. By a theorem of H. Friedman and Jensen any countable admissible β is ω_1^r for some $r \subseteq \omega$ (*cf.* [4]). Let T be the following theory consisting of the following sets of sentences in the language $\mathcal{L}_{\in, \dot{r}}$ augmented by a new constant c:

(i) $\mathrm{KP} + \dot{r} \subseteq \omega$; (ii) the diagram of $\langle L_\beta[r], \in \rangle$;

(iii) “$\forall x[x \in \dot{y} \longrightarrow \bigvee\!\!\!\bigvee_{z \in y}\; x = \dot{z}\,]$” for all $y \in L_\beta[r]$.

(iv) “$\gamma \in c \wedge \mathrm{c}$ is an ordinal” for all $\gamma < \beta$.

(v) “$\forall a \leq c\; L_a[\dot{r}] \;\nvDash \mathrm{KP}$ ”

(vi) “$\exists x \exists f\, [\, f$ maps c order preserving into $\mathrm{Field}(y)$ where y codes a halted course of computation of the form $\varphi(x)\downarrow$.”

Claim If $T_0 \subseteq T$, $T_0 \in L_\beta[r]$, then T_0 has a model.

Proof. Let $\delta < \beta$ be the least ordinal not “mentioned” in T_0. Find a (wellfounded) KP model $\mathcal{N}$, with $r \in \mathcal{N}$, and with an $x \in \mathcal{N}$, with $\mathrm{On}^{\mathcal{N}} > \delta$, and so that $\mathcal{N} \vDash \varphi(x)\downarrow^{\not\leq \delta}$. Then $\exists f \in \mathcal{N}$ with $f : \delta \longrightarrow \mathrm{Field}(y)$ where $y \in \mathcal{N}$ codes the course of computation. Let δ interpret c. □

By Barwise Compactness T has a model $\mathcal{M}$. By (i)-(iii) this is a KP model whose $L[r]$-part end-extends $L_\beta[r]$, and moreover $\mathrm{WFP}(\mathcal{M}) \cap \mathrm{On} = \beta$ (by virtue of (v).) Let $x_0 \in \mathcal{M}$ witness (vi). Then we shall have that for every $\delta < \beta$, $\mathcal{M} \vDash$ " $\varphi(x_0)\not\downarrow^{<\delta}$ ". However in V we have then that $\varphi(x_0)\downarrow^\beta$. Moreover note that β is x_0-admissible (otherwise we could Σ_1-define inside $\mathcal{M}$, β from x_0 and ordinal parameters less than β). However we have just argued that β is x_0-clockable! This contradicts [2] Theorem 8.8. □

Hence in the terminology of [1] "$P_\beta = P_{\beta+1}$" and "$\mathrm{NP}_\beta = \mathrm{NP}_{\beta+1}$" so this shows that the requirement on β not being a limit of non-clockables, can be lifted from their Theorem 17.

In section 6 they consider the P^f/NP^f classes restricted to sets of integers. The above arguments show that for many of them $P^f = \mathrm{NP}^f$! We shall use the following result which is cited in their paper as Lemma 15.

Lemma 6. *([6] Lemma 2.5) If α is a clockable ordinal, then every ordinal up to the next admissible ordinal beyond α is writable in time $\alpha + \omega$.*

Proposition 7. *Let $\beta \leq \lambda$ be such that β is an admissible limit of admissibles but is not interior to any gap in the clockables (i.e., it is a limit of clockables). Then*

$$P^\beta \cap \mathcal{P}(\omega) = NP^\beta \cap \mathcal{P}(\omega).$$

Proof. Let $A \in \mathrm{NP}^\beta \cap \mathcal{P}(\omega)$. Let φ_e witness this: $\forall n, y\ \varphi_e(n, y)\downarrow^{<\beta}$ and $\forall n[n \in A \Longleftrightarrow \exists y\, \varphi_e(n, y)\downarrow 1]$. The argument of Prop. 4 shows that there is a smaller bound $\gamma_0 < \beta$ for the lengths of all these computations. Hence if $n \in A$ then there is a y witnessing this, with $\varphi_e(n, y)\downarrow 1$ and converging in $\leq \gamma_0$. steps. Let $u \in L_\beta \cap \mathrm{WO}$ have rank γ_0. Set:

$$B_n = \{z : \exists y\,(z \text{ codes a wellfounded comp. witnessing } \varphi_e(n, y)\downarrow^{\|u\|} 1\ )\ \}$$

Again $\varnothing \neq B_n \in \Sigma^1_1(u)$. As above there are witnessing $z, y_0 \in L_{\gamma_0^+ + 1}$ if $n \in A$. (α^+ = next admissible $> \alpha$.) In other words to test for membership in A all we have to do is search through potential NP-witnesses y in $L_{\gamma_0^+ + 1} \in L_\beta$. But this puts $A \in \Delta_1^{L_\beta}(\{\gamma_0\})$. By our assumption on β, by Lemma 6, γ_0 is itself writable by some program φ_f in time $< \gamma_0^+$. Putting this together $A \in \Delta_1^{L_\beta}$, so $A \in P_\beta$.

□

Bibliography

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