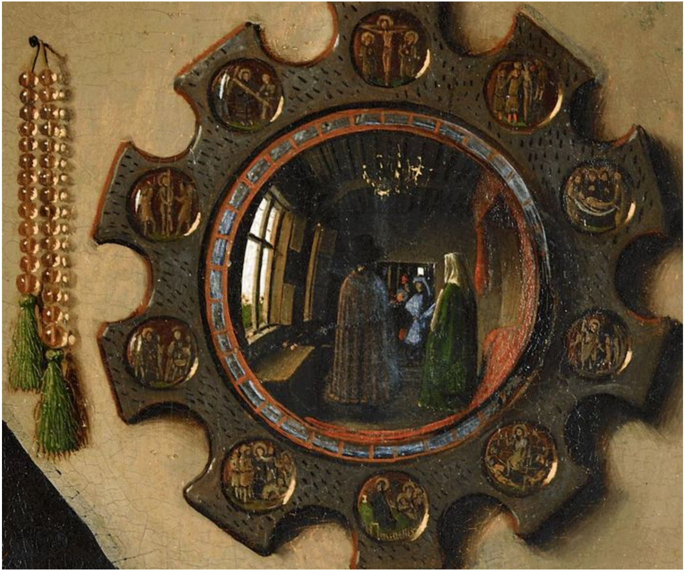
Conceptualism: sets and absolute infinities

P.D.Welch, Cambridge, June 10 2012



(I)

• We have no need to ‘perceive’ in a Gödelian sense, or otherwise ‘locate’ any mathematical objects in order to understand and communicate that understanding of the concepts involved.

• Thus *instantiation* of our concepts is not necessary, but what is needed is uniqueness of the concept up to isomorphism.

(II)

• We seek to extend arguments of Martin’s along the above lines which he advocated for the ‘concept of ω-sequence’ and the ‘concept of set of x’s’ to a ‘concept of set with absolute infinities’.

(III)

• We then formulate within this framework *reflection principles* that establish large cardinals beyond those consistent with Gödel’s L.

(I) Conceptualism: Sets and Classes

• Martin[1] has questioned the level of realism that Gödel, although on occasion expressing this with talk of 'perception of mathematical objects' *etc.* , needs in order to make some of his arguments, *e.g.* , of the analyticity of mathematical truths work.

• Martin identifies two kinds of sense to 'concept of set'. (i) That more nearly akin to pure platonism - the whatever it is that falls under the extension of 'concept of set (of x's)' - that is sets (of x's), and
(ii) a more general sense of 'concept of set' under which falls concepts of sets, or 'set-structures'.

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[1] *"Gödel's Conceptual Realism"*, BSL, 2005.

Martin: concepts of sets

> *My sense differs from the straightforward sense in that instances of a concept of set in the straightforward sense - the objects that fall under the concept - are sets (or, at least, what the concepts are count as sets). The instances of a concept of set in my sense are not sets. There are two versions of my sense. In one version the instances are concepts: straightforward-sense concepts of set. In the other version the instances might be described as set structures or universes of sets.*

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However first:

> *A concept of set expressed by axioms such as comprehension axioms cannot put any constraint on which objects count as sets and which do not. Such axioms put constraints on the isomorphic type of set theoretic structure . . . a concept of set could count as concept of set in my [indirect] sense even if it determined completely what objects count as sets and what counts as the membership relation. A concept of this sort would have at most one instance: it would allow at most one structure to count as a set-theoretic universe . . .*

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Basic concepts and their properties

He[2] discusses in the following terms the *concept of an ω-sequence* and the *concept of set*.
identifies three properties a basic concept may have:

(i) First order completeness: the concept determines truth values for all first order statements.

(ii) Full determinateness: the concept fully determines what any instantiation would be like.

(iii) Categoricity.

• For the ω-sequence case he asserts that the concept of natural number yields IPA: Informal Peano Axioms, (not in the usual 1st order sense) which in turn yields categoricity of $\mathbb{N}$.

> *I believe that full determinateness of the concept* [of ω-sequence] *is the only legitimate justification for the assertion that the concept is instantiate or that natural numbers exist.*

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The concept of set

The modern iterative concept has four important components:
(1) concept of natural number

(2) concept of ‘set of x’s’

(3) concept of transfinite iteration

(4) concept of absolute infinity.

Informal Comprehension

- Which informal axioms are implied by the concept of set?

(i) If a and b have the same members, then $a = b$. (Extensionality)

(ii) For any property P, there is a set whose members are those x's that have P (Informal Comprehension).

Theorem

(Essentially Zermelo) Informal Axioms (i) and (ii) are categorical: if $(\mathfrak{V}_1, \in_1)$, $(\mathfrak{V}_2, \in_2)$ *are two structures satisfying (i) and (ii) with the same x's, then with each set* $b \in_1 \mathfrak{V}_1$ *we associate a* $\pi(b) \in_2 \mathfrak{V}_2$.

Proof: Let P be the property of being an x such that $x \in_1 b$. By the Informal Comprehension Scheme there is a $c \in_2 \mathfrak{V}_2$ such that

$$\forall x[x \in_2 c \leftrightarrow P(x)] \qquad \textit{Q.E.D.}$$

• Similarly for any two $\mathfrak{V}_1 = (V_1, \in_1)$, $\mathfrak{V}_2 = (V_2, \in_2)$ obtained by iterating the V_α function throughout all the absolute infinity of ordinals, we have an isomorphism

$$\pi : (V_1, \in_1) \cong (V_2, \in_2)$$

• Then we see that $\pi \restriction \mathrm{On}^{\mathfrak{V}_1} : \mathrm{On}^{\mathfrak{V}_1} \cong \mathrm{On}^{\mathfrak{V}_2}$

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(II) Absolute Infinities

• We take a naive, pre-theoretic Cantorian stance on absolute infinities.

• We take a view that absolute infinities are *parts of* V rather than some dramatically new entity or object. As such they are necessary to our iterative concept of set (where the modality is logical necessity).

• V is then the universe of mathematical discourse, but the absolute infinities are not mathematical or formal set theoretical structures.

Let $\mathcal{C}$ denote the collection of the parts of the domain of the universe $\mathfrak{V}$.

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Let $\mathcal{C}$ denote the collection of the parts of the domain of the universe $\mathfrak{V}$.

What is the character of $\mathcal{C}$?

- What informal axioms follow from the concept of 'set/part of'?

(i) $\{(x,x) \mid x \in V\}$ and $\{(y,x) \mid y \in x \in V\}$ are both absolute infinities.

(ii)

$\forall X, Y \exists Z (Z = X \cap Y)$

$\forall X \exists Y (Y = V \backslash X)$

$\forall X, Y \exists Z (Z = X \times Y)$

$\forall X \exists Y (Y = \mathrm{dom}(X))$

$\forall X \exists Y \forall xyz ((x,y,z) \in X \leftrightarrow (z,x,y) \in Y)$

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When talking more formally about a structure with its parts as a predicate such as $\mathfrak{V} = (V, \mathcal{C}, \in)$ we are thinking of a two sorted language $\mathcal{L}^+$ with variables $x, y, z, \ldots$ for sets in V, and $X, Y, Z, \ldots$ for the parts in $\mathcal{C}$, and no quantification over the latter.

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Isomorphism again

Theorem

If we have two structures of sets $\mathfrak{V}_i = (V_i, \in_i)$ $(i = 1, 2)$ *satisfying (i) and (ii) above, with collections of parts* $\mathcal{C}_i$*, we may define an isomorphism* $\pi : (V_1, \in_1) \to (V_2, \in_2)$ *as before.* π *then extends to an isomorphism:*

$$\pi : (V_1, \mathcal{C}_1, \in_1) \cong (V_2, \mathcal{C}_2, \in_2).$$

(III) Reflection

We let $\mathcal{L}^+$ be the usual first order language, augmented with second order variables $X_1, X_2, \ldots$ but no second order quantification. The X_i are interpreted as ranging over the parts in $\mathcal{C}$.
Formula-by-formula reflection now is unexceptional: Fix an $i \leq \omega$: then for any $\varphi \in \Sigma_i$:

$$\forall \vec{x}_i \in V \forall \overrightarrow{X}_j \in \mathcal{C} : \forall \alpha \exists \beta > \alpha \;\; \varphi(\vec{x}_i, \vec{X}_j)^{(V,\mathcal{C},\in)} \leftrightarrow \varphi(\vec{x}_i, \overrightarrow{X_j \cap V_\beta}))^{(V_\beta, V_{\beta+1}, \in)}.$$

Or the whole existential part of $\mathcal{L}^+$:

$$\forall \vec{x}_i \in V \forall \overrightarrow{X}_j \in \mathcal{C} \forall \alpha \exists \beta > \alpha \forall \varphi \in \Sigma_1$$

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GRP - a Global Reflection Principle

We take the whole $(V, \mathcal{C}, \in)$ and reflect it down to some initial segment.

We assert the explicit existence of a connection, or reflecting map j as follows:

$$\forall\alpha\exists\beta > \alpha\exists j_\beta : (V_\beta, V_{\beta+1}, \in) \longrightarrow_{\Sigma_1} (V, \mathcal{C}, \in) \qquad \text{(GRP)}$$

where $j \restriction V_\beta = \mathrm{id} \restriction V_\beta$, and the elementarity is Σ_1 in the language $\mathcal{L}^+$.

1) Notice that $j_\beta(\beta) = \mathrm{On}$ where β is a 'part' of V_β and so is in $V_{\beta+1}$ and similarly $\mathrm{On} \in \mathcal{C}$.

2) More generally for $X \in V_{\beta+1}$ $j_\beta(X) = X \cap V_\beta$.

3) Crucially $dom(j_\beta) \supseteq V_{\beta+1} \supset \mathcal{P}(\beta)$.

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Theorem

$(\mathrm{GRP}_0) \Longrightarrow$ *There is an absolute infinity of measurable Woodin cardinals.*

Together with the work of Martin & Steel[3], and Woodin[4], we have:

Corollary

$(\mathrm{GRP}_0) \Longrightarrow$ Projective Determinacy, $\mathrm{AD}^{L(\mathbb{R})}$, *and no statement of analysis can be forced to change its truth value by Cohen style set forcing.*

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[3] *A Proof of Projective Determinacy*, J. of the AMS,**2**, 1989

[4] *cf. The Axiom of Determinacy, Forcing Axioms and the Non-stationary Ideal*

Principles at a threshold

A comment here:
Such principles seem to sit at a watershed between those weaker large cardinals and those that imply there are $\mathcal{L}$-elementary embeddings $j : V \longrightarrow M$ with critical point some κ so that $j(\kappa^+) > \sup j``\kappa^+$. (All weaker large cardinals have equality here.)