Global Reflection Principles

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Abstract

Reflection Principles are commonly thought to produce only strong axioms of infinity consistent with $V = L$. It would be desirable to have some notion of strong reflection to remedy this, and we have proposed *Global Reflection Principles* based on a somewhat Cantorian view of the universe. It is the main contention of this paper that such a reflection principle when taken as an integral part of the point of view of our universe of sets enables important, or even necessary, principles needed for current set theory, and indeed the modern view of that universe. Such principles justify the kind of cardinals needed for, *inter alia*, Woodin's Ω-Logic and definable determinacy.[1]

1 Reflection Principles in Set Theory

Historically reflection principles are associated with attempts to say that no one notion, idea, or statement can capture our whole view of the universe of sets $V = \bigcup_{\alpha \in \mathrm{On}} V_\alpha$ where On is the class of all ordinals. That no one idea, expressed usually in some formal fashion, can pin down the universe of all sets has firm historical roots (see the quotation from Cantor later or the following):

> *The Universe of sets cannot be uniquely characterized (i.e. distinguished from all its initial segments) by any internal structural property of the membership relation in it, which is expressible in any logic of finite or transfinite type, including infinitary logics of any cardinal number.*
>
> Gödel (attrib.), in Wang [19]

Indeed once set theory was formalized by the (first order version of) the axioms and schemata of Zermelo with the additions of Skolem and Fraenkel, it was seen that reflection of *first order* formulae $\varphi(v_0, \ldots, v_n)$ in the language of set theory $\mathcal{L}_{\dot\in}$ could actually be proven:

(Montague-Levy: First order Reflection)

(R_0) : *For any* $\varphi(v_0, \ldots, v_n) \in \mathcal{L}_{\dot\in}$

$$\mathrm{ZF} \vdash \forall\alpha\exists\beta > \alpha\forall\vec{x} \in V_\beta[\varphi(\vec{x}\,) \leftrightarrow \varphi(\vec{x}\,)^{V_\beta}].$$

1. With deep thanks to Leon Horsten for conversations on the matters herein. This paper was delivered as a lecture in the *Exploring the Frontiers of the Infinite* series in Harvard, Feb. 2012, whilst the author was a Fellow of the Isaac Newton Institute in Cambridge to which he offers gratitude for their support.

This is a scheme involving as it does one theorem for each φ. (Here $\varphi(\vec{x})^{V_\beta}$ is the relativization of φ to V_β obtained by restricting all quantifiers in φ to V_β.) However far from being an epiphenomenon of the ZF axiomatisation, one may interpret the fact that the axioms of infinity and the axiom scheme of replacement are provable from the remaining axioms by means of (R_0) as indicating that such reflection is an integral part of our view of the universe of sets, at least as far as we view that universe with only our ZF-spectacles on. It is the main contention of this paper that a *global reflection principle* when taken as an integral part of the point of view of our universe of sets enables important, or even necessary, principles needed for current set theory, and indeed the modern view of that universe.

Indeed by formalising a Σ_n-Satisfaction predicate for an appropriate class of Σ_n-formulae, call it Fml_{Σ_n}, we have (still within ZF):

For each $n\in\omega$:
$\mathrm{ZF}\vdash\exists C_n[C_n\subseteq\mathrm{On}$ *and is a closed and unbounded class, so that for any* $\varphi\in\mathrm{Fml}_{\Sigma_n}$:

$$\forall\beta\in C_n\forall\vec{x}\in V_\beta[\varphi(\vec{x})\leftrightarrow\varphi(\vec{x})^{V_\beta}]].$$

For the existence claim of the first quantifier, one shows that such a proper class C_n is definable by a formula of $\mathcal{L}_{\dot{\in}}$. However, as is well known, there is no formula that works uniformly for all n. Informally we may however write $\forall\beta\in C_n$: $(V_\alpha,\in)\prec_{\Sigma_n}(V,\in)$.

The story of how one arrives at the series of cardinals *inaccessible, Mahlo, indescribable* by reflecting on normal functions $F\colon\mathrm{On}\to\mathrm{On}$ as having regular fixed points, (in the work of Levy and Bernays) and thence to indescribability properties expressed as sentences in higher order languages is well known and we do not re-tell that here (see Bernays [2].) However the *Global Reflection Property* (GRP) we had proposed (whilst we had thought originally of something coming from weakening the third order notion of *sub-compact* cardinal) was actually much closer in consistency strength in the final analysis to the notion of *'1-extendible'* cardinal introduced by Reinhardt. Indeed it has been said that the motivation of GRP is "just the same" as Reinhardt's 1-extendible. We briefly survey some of Reinhardt's ideas to argue that in fact the justification of our GRP is rather different. A thread uniting his work on set theoretical foundations is that he was often dealing with, or proposing, differing conceptions of set from the standard Zermelo-Fraenkel one; on those occasions when he was considering classes of sets, or in particular "the class of all sets" ,V, he was usually thinking of V as some flexible concept, either as undetermined (in his discussion of Ackermann's set theory - the latter seems to have been particularly influential on his later conceptions) or else as being some initial part of some "virtual realm" of sets somehow "beyond" the usual V, thus with V as somehow an "initial rank" V_{On} of this realm. We enter in to this discussion partly to contrast this view (or these views) with the reflection property here proposed: we do not think of V in either of these fashions: just as it is determinate as to whether a given set together with an ordering is an ordinal, without the whole class On having a length fixed by some expression, so it is with V. The world of our V is that of the ZF-set theorists, or arguably leaning towards that of Cantor. We seek to distance ourselves thus from the Reinhardtian viewpoint(s).[2]

2. It is interesting to see where this viewpoint ended up. In [17], Reinhardt's final published paper in this area (Intro.): Extendability is briefly motivated by asserting that there is 'resemblance' between different ranks V_α, V_β within the V hierarchy. However in Sect 5, extendible cardinals are motivated by reflecting upon a transfinite theory of types over and above the universe V. "With the natural reflection down into the world of sets we have the concept of an extendible cardinal. (As Reinhardt [15] points out, however this sort of internalisation within V rather begs the question if we want to discuss fundamental issues about the nature of V and Ω.)"

We were spurred to think about these questions by reading a preprint of Peter Koellner's [8], where he considered the general problem of reflection, and the particular suggestions of Tait [18] for strengthened higher order languages which admitted limited forms of higher order parameters. These limitations were known since Reinhardt's investigations, as the latter provided a simple counterexample to third order reflection with straightforward third order parameters. Tait questioned whether any of his own principles would yield a cardinal beyond L (which we call here *extra-constructible*), but Koellner showed that the principles divide along a line of either being weaker than the existence of an ω-Erdős cardinal and so consistent with $V = L$ (which we may call *intra-constructible* rather than the more usual, but somewhat oxymoronic, "small large cardinal") or else inconsistent. Koellner argued for the conclusion that this was in fact a threshold: general reflection principles were either weaker than the existence of $\kappa(\omega)$ - the first such Erdos cardinal or else inconsistent.

However as is now well known, there are many questions (besides CH the continuum hypothesis) which can be resolved if one assumes large cardinals, and indeed much of Woodin's work is predicated on the assumption of the existence of, *e.g.* a proper class of Woodin cardinals, which far outstrip the consistency strength of $V = L$. Martin-Steel showed that from infinitely many Woodin cardinals it follows outright that Projective Determinacy (PD) holds. It therefore follows that if a proper class of such cardinals exists then in no Boolean valued set-sized forcing extension of the universe can PD actually fail, as any set sized forcing notion can only destroy boundedly many Woodin cardinals. An attractive and easily stated principle not involving (directly) determinacy is that of *Projective Uniformization* (PU): every projectively definable relation may be uniformised by a projectively defined function. Co-analytic uniformisation (even with co-analytic functions) is a theorem of ZF, but by Gödel's results on the constructible universe, nothing further can be proven in ZFC alone. However PU becomes a theorem if we add to ZFC the assumpion of the existence of infinitely many Woodin cardinals.

Why be concerned about PD? Because (Woodin again, [21]) PD is *something* of a 'complete theory' of countable sets, much as PA is *something* of a 'complete theory' of the finite natural numbers, in the sense that we have no examples of sentences σ about HC (the hereditarily countable sets) that are not decided by $\mathrm{ZFC}^- + \mathrm{PD} + V = \mathrm{HC}$ *other than* Gödel-style diagonal sentences. (ZFC^- is the theory of ZFC with the power set axiom removed - usually with a scheme of collection substituting for replacement.)

An exactly similar consideration shows Woodin's result that from infinitely many Woodin cardinals, we deduce that the full Axiom of Determinacy holds in $L(\mathbb{R})$, (the least inner model built by closing the reals under the Gödel operations), written as "$\mathrm{AD}^{L(\mathbb{R})}$". This too cannot be falsified in set generic forcing models if there are a proper class of such cardinals. Such absoluteness or fixity results are manifestations of very general facts due again to Woodin:

1) (Woodin, *cf* [20]) *Assume there is a proper class of Woodin cardinals. Then* $\mathrm{Th}(L(\mathbb{R}))$ *is fixed: no set forcing notion can change* $\mathrm{Th}(L(\mathbb{R}))$*, and in particular the truth value of any sentence about reals in the language of analysis, including such statements as* PU.

2) (Woodin, *cf* [20]) *If, additionally, there is a proper class of measurable Woodin cardinals, and* CH *holds in V, then we have (in the language of third order number theory)* Σ^2_1*-elementary equivalence between V and any set forcing extension in which* CH *also holds.*

Furthermore the assumption of unboundedly many Woodin cardinals is a cornerstone of many of the results on Woodin's Ω-logic (which we do not discuss here).

Our GRP_0 will demonstrate that there are unboundedly many measurable Woodin cardinals, and hence proves PD, PU and the absoluteness results of Woodin just stated. (We want to emphasise that there are absolutely *no* new mathematical or set-theoretical results here: the derivation of the proper class of measurable Woodin cardinals from the principle is an exercise that can be done by a reader of [7]. The point is that we wish to obtain a reasonable, and proper class, principle in the form of a reflection.)

2 Reinhardt's Aims

We first set the scene with the briefest of discussions of Ackermann's set theory, which Levy, Vaught and Reinhardt investigated. Reinhardt's mathematical work on this theory appeared in his thesis which was for the most part then published as [14].

Briefly: Ackermann's set theory A provided for a universe with extensionally determined entities (classes) and a predicate $\dot{V}$ for set-hood: "$x \in V$". Besides axioms for extensionality, a class construction scheme, and set completeness ("all classes that are subclasses of sets are sets"), it contained the following crucial principle:

- (Ackermann's Main Principle) *If $X \subseteq V$ is definable using only set parameters, and* not *using the predicate $\dot{V}$, then $X \in V$. Thus if θ does not contain V:*

$$x \in V \wedge \forall t(\theta(x,t) \longrightarrow t \in V) \quad \longrightarrow \quad \exists z \in V \forall t(t \in z \leftrightarrow \theta(x,t))$$

- (Levy, Vaught) [11] Let A^* be A with the addition of Foundation. Then A^* is consistent relative to A, and proves the existence of the classes: $\{V\}, \mathcal{P}(V), \mathcal{PP}(V)$...

- Levy ([10]): A^* is $\mathcal{L}_{\dot{\in}}$- conservative over ZF: $A^* \vdash \sigma^V \Longrightarrow \mathrm{ZF} \vdash \sigma$

Levy considered models of A^* of the form $\langle V_\alpha, \in, V_\beta \rangle$.

Reinhardt in [14] proved the converse implication of this last result also; hence putting these together the set-theoretical content of A^* had always just been that of ZF. It is thus somewhat remarkable that two rather different conceptualisations - the one leading to the ZF formalisation, and Ackermann's - have the same set-theoretical content. Whilst Ackermann interpreted Cantor's "By a set we understand any collection of definite distinct objects ... into a whole" as saying "we must require from already defined sets that they are determined and well-differentiated, thus the [foregoing] conditions for a collection [to be a set] only turn on that it must be sufficiently sharply delimited what belongs to a collection and what does not belong to it. However now the concept of set is thoroughly open." (Ackermann [1] p.337)[3], Indeed Reinhardt, whilst working from the premise that Ackermann considered the concept of set itself as not sharply delimited

3. "... wir von den schon definierten Mengen verlangen müssen, dass sie bestimmt und wohlunterschieden sind, so kann es sich bei der obigen Bedingung für eine Gesamtheit nur darum handeln, dass genügend scharf abgegrenzt sein muss, was zu der Gesamtheit gehört und was nicht zu ihr gehört. Nun ist aber der Mengenbegriff durchaus offen."

([14] p190-1), surmises that the intuition behind Ackermann's Main Principle is that a sharply defined collection of sets is a set, and that, given the set x, the property '*t is a set such that* $\theta(x,t)$' is independent of the (extension of) the concept of set, but gives a sufficient condition for a collection to be sharply delimited. The Ackermann quotation continues (in paraphrase) that in the Cantorian definition it is intended that a collection should be investigated only on a case by case basis as to whether it represents a set, and it is not meant that it is determined in one fell swoop for all classes whether they are sets or not. We therefore see that a collection is not sufficiently well-differentiatied if it is defined through its relationship to the concept of set.

In his final Chap. 5 Reinhardt expands on this, and says that if that is so, then the collection of all sets V is not sharply defined, and that there could be other V' which could also serve as the collection of all sets, but differ in extension from V. In the closing sections he extends A^* to an A^+ which includes suitable schemes and an axiom that with $V \subseteq V'$ would together prove that $V \in V'$. Referring back to systems of Ackermann (in particular A_∞ from [1]) that entertained precisely this possibility of a hierarchy of such $V_n \in V_{n+1}$ for $n < \omega$, he considers the case of a triple $V_0 \in V_1 \in V_2$. Reinhardt then introduces an operator G to act as a 're-interpretation functor' for the constants c_x, c_X he has introduced as a suitable language: when $x \in V_0 : Gc_x$ will be interpreted as x, whilst for $X \subseteq V_0$, Gc_X will be interpreted as some $\tilde{X} \subseteq V_1$. For restricted set-examples taken from our everyday ranks then, he essentially sets up something such as an elementary embedding of $j\colon (V_\alpha, \in, \mathcal{P}(V_\alpha)) \longrightarrow (V_\beta, \in, \mathcal{P}(V_\beta))$. As Koellner dubs it (in the final part of [8] but actually when discussing Reinhardt's later paper [15]) there is a problem: there are not enough intensional notions to go around; there is moreover a problem of justifying the underlying modal conception of necessary mathematical truth in 'legitimate candidates' for V; finally there is the problem of what Koellner calls 'the extension to inconsistency': Reinhardt talks of generalising the above by removing the restriction that X be restricted to $\mathcal{P}(V_0)$ and we know now where that will lead; however the point that Koellner makes is that the steps all seem as well justified as each other, but one ends up with Kunen's inconsistency result ([9]) of an elementary embedding of the whole universe to itself. We agree with Koellner here, as elsewhere in his analysis of Reinhardt, and rather than reiterating his points refer the reader to the final part of his [8].

We turn to the later appearing papers [15] and [16]. In [15] Reinhardt again imagines the possibility of "getting outside" of Cantor's $V = V_{\mathrm{On}}$ and so thinks of "ordinals" such as "On + 1" and further "sets" such as $V_{\mathrm{On}+\mathrm{On}}$ *etc.* He thinks of this as akin to "virtual displacements" in physics, and as a not inconsistent imaginative foray. He thus considers a realm of "*projected objects*" (hence the use of the scare quotes around 'sets' and 'On + 1' above) and adopts Cantor's theory of $V = V_{\mathrm{On}}$ to this extended realm. He takes the axioms of ZF but in a language with the letters V, Ω added as syntactic constants for the Cantorian universe and the class of ordinals On. Assuming that the theory of V_{On} is applicable to this projected realm he arrives at the schema:

$$\text{(S2)} \qquad (\forall x, y \in V)(\theta^V(x,y) \leftrightarrow \theta(x,y))$$

This schema is similar to Ackermann's schema and asserts that any first order sentence of the theory of V_{On} is true in the projected universe. The theory $\mathrm{ZF} + (S2) + V = V_{\mathrm{On}}$ is then consistent relative to ZF.

After a discussion of Bernay's second order reflection property he remarks on the failure of this to carry up to third order (as we have alluded to above: the schema is inconsistent if third order parameters are allowed). If we are to consider class-set theory we have classes distinct from sets. However "'if we are to conceive of them merely as collections then this looks like a distinction without a difference." He then remarks on the view that once one allows further ranks of classes of classes *etc.* beyond "$V_{\mathrm{On}+1}$" it looks simply as if one has stopped too soon whilst collecting together all sets. *"Moreover the classes threaten the universality of set theory."* However he remarks that a counterfactual difference between a set x and a class P is: were there to be more ordinals, then a class P would contain more elements, whereas the extension of a mere set such as x would be fixed. He imagines the extension of P in this projected universe to be some jP. This brings him from (S2) to a *projection schema*:

$$\text{(S4)}\quad (\forall x,y\in V)(\forall P\subseteq V)(\theta^{\mathcal{P}V}(x,y,P)\leftrightarrow\theta(x,y,jP)).$$

with the relativisation $\theta^{\mathcal{P}V}$ indicating that all quantifiers are relativised to "$x\subseteq V$". In (S2) the original universe V is a set; he introduces here classes and sets and (S4) asserts that "the true sentences of the theory of sets, whilst allowing quantifiers over classes, and both sets and classes as parameters, are precisely those true in the projected universe." When he turns to consideration of interpretation of the formal scheme (S4) he writes "... having now introduced proper classes of sets as distinct from sets (To be sure the distinction is drawn only by considering "imaginary" sets and classes.)". He thus now considers, or imagines, classes of proper classes, and then classes of these *etc.*, thus building up some hierarchy of the form $V_{\mathrm{On}'}$ with On' beyond On. These new classes are the 'Ω-classes' (or we should say in this paper's notation 'On-classes'). He then considers an axiom closest to extendability. If $\lambda>\mathrm{On}$ so that V_λ is now a collection of On-classes, he wishes to consider a realm $V_{\mathrm{On}'}$ of imaginary sets and a corresponding realm $V_{\lambda'}$ of imaginary On-classes, in which V_{On} is an imaginary set. In order to have this and in addition a correspondence between classes of sets and imaginary classes of sets, and further to have other On-classes x to correspond to other imaginary classes jx, he sets up the relation $E_0(\mathrm{On},\lambda;\mathrm{On}',\lambda')$ of, what we might now call "λ-extendability beyond On", namely:

(i) $\mathrm{On}<j\,\mathrm{On}=\mathrm{On}'<\lambda'$.

(ii) $\forall x\in V_{\mathrm{On}}\,jx=x$;

(iii) $j\colon(V_\lambda,\in)\to_e(V_{\lambda'},\in)$.

This leads into a discussion of whether the On-classes are all the *possible* On-classes; in that case we might want λ to equal λ', and in turn this leads to a discussion of Kunen's theorem on the impossibility of a 'Reinhardt cardinal.'

We thus see that the motivations for introduction of 'extendability' as laid out here, are tied up with possible *extensions* of $(V,\in)$ into other imaginary realms. (In later terms it is On that is the 'extendable cardinal' not On'; the emphasis is on the extendability of the whole domain.) The difference between this treatment and that of [13] is that the latter took the alternatives to the actual universe of sets to be in terms of representations of V in which it was itself a set.

In [16] he discussed a number of then current set theories due to Ackermann (that we have just seen), again, and to Shoenfield and Powell.

Shoenfield's intuitive system $\mathcal{S}$ gets a formalisation as S: not a dynamic vision of V (bottom of p9) where the extension of V is variable, but growing with the number of stages constructed - we might say an 'actualist vision'. However he thinks this is suggestive of regarding V as the least V_κ compatible with what can be truly expressed about the Cantorian universe by means of a language L. Since the language L is not fixed, but nevertheless at any one moment we are caught using a particular L "it seems impossible to distinguish between V_κ and the Cantorian universe $V = V_{\mathrm{On}}$ by any absolute means." ([16] p9). (The consistency proof of Levy for Ackermann's A yields a proof of the consistency of S.)

Our interest here is that Reinhardt then introduces a strengthening to S^+ by means of an enriched notion of property: there is a predicate P with the interpretation that $P(x)$ is to mean that "x is an existing property of sets." One may construct classes by quantifiers relativised to the existing properties of sets, and there is a comprehension principle stated in such a language; there is further a non-constructive class existence principle. If On represents the class of ordinals of the theory then it can be deduced that there is a non-principle On-complete normal measure on On, thus making On a 'measurable cardinal' in a larger imaginary universe. This is because we may for each $X \subseteq \mathrm{On}$ inspect the property Q corresponding to X:

$$\forall x \in V(x \in X \longleftrightarrow x \in Q)$$

(such an X exists by a formalisation of a Shoenfieldian principle), and then put X into the ultrafilter iff $\mathrm{On} \in Q$. He concludes (p33) by saying that he regards the paper as a justification for measurable cardinals and that "he knows no more natural way to introduce [them] than via properties... ." Again there are two points to be made: the formal property theory is building some considerable superstructure or hierarchy again beyond On; secondly current thinking would introduce measurable cardinals as at the bottom of a hierarchy of axioms of strong infinity based on elementary embedding properties of the universe V to some inner model M, but that is a more modern perspective. (Although of course it is precisely to an appeal of such embeddings' existence that we do *not* want to resort to in this paper.)

Gödel again:

> "All the principles for setting up the axioms of set theory should be reducible to Ackermann's principle: The Absolute is unknowable. The strength of this principle increases as we get stronger and stronger systems of set theory. The other principles are only heuristic principles. Hence, the central principle is the reflection principle, which presumably will be understood better as our experience increases. Meanwhile, it helps to separate out more specific principles which either give some additional information or are not yet seen clearly to be derivable from the reflection principle as we understand it now." (Wang *ibid.*) .

The conclusion that we wish to draw from this section is that the differing views of the universe of sets studies or proposed by Reinhard, are ones of a flexible notion of V, usually kitted-out with some extensions into a virtual or imaginary realm of further sets or classes. On occasion he proceeds then to show that On is itself a large cardinal, such as measurable, in that virtual realm. However on each of these viewpoints, (and they are derived from some quite different set theories which we have only sketched in the above) whether or not he would have agreed to the term 'potentialist' for these views of sets or not, they are not about a fixed, or 'actual' realm or even a fixed concept of set.

3 Strengthening Reflection Principles

Having discussed some of Reinhardt's formalisation of property theories, we come back to the more usual style of reflection principle posited in the language of set theory or using higher-order languages about V. As mentioned above Tait introduced in [18] some sophisticated languages with positive occurrences of parameters to skirt around Reinhardt's observation that third order parameters cannot be substituted consistently into third order reflection principles.

Tait argues from an iterative conception of set, and as Koellner analyses in [8][4], seems to argue that reflection principles based on this conception exhaust *intrinsic justifications*. (We here refer to the well-known division of possible justifications for set theoretic axioms discussed by Gödel, into the *intrinsic* and the *extrinsic* where the latter are ultimately justified by the richness, clarity and desirability of their consequences. The former on the other hand are supposed to derive their justification from the conception of set, or the universe of sets and classes considered as a whole.) Tait further appears to rule out extrinsic justifications.

Koellner gave a careful investigation of the consistency, or otherwise of such principles. He thinks they give the best current candidates for axioms that admit such intrinsic justifications based on the iterative conception of set ([8], p208).

Tait had speculated whether some of his principles might lead beyond the usual kinds of inaccessible cardinals and their strengthenings in L and might lead to measurable cardinals. Koellner showed that in fact the principles were, when consistent, weaker than the existence of $\kappa(\omega)$ (the first ω-*Erdős* cardinal). Indeed some of Tait's candidates in this regard turned out to be inconsistent. Koellner, and we agree, draws the conclusion that these principles on first blush look little different from their consistent (but weak brethren), and this should alone give us pause to reconsider these *kinds* of principles as being 'intrinsically justified' - how could they be when some of them turn out to be inconsistent? Where we fooled into thinking them intrinsically justified because they were similar to others? Koellner looks at the *Relativised Cantorian Principles* which Tait used to justify various reflection principles. One difficulty is that of the burden of identifying the 'good' *existence conditions* C and arguing each time for a condition C that gives rise to a reflection principle, that the condition is justified on the basis of the iterative conception. And Koellner identifies a broad class of such conditions that simply yield up inconsistent principles.

4. Koellner states that he would rather use the phrase "conception of set" rather than "concept of set" as nothing in his article turns on adopting a robust realism of the kind usually attributed to Gödel. We shall follow his example, but leave it for the reader to judge to what extent our principles require such a realism. We take our actualism to be an attitude differing from, and not dependent on, realism.

The first ω-Erdős cardinal (if it exists) can exist in L and thus no extra-constructible large cardinal could follow from the consistent reflection principles identified to date. Again: all ZF-informed reflection principles based solely on the iterative conception, and even those such stated in higher type logics, for example, seem to be consistent with $V = L$.

Koellner (Sect. 4 *op.cit.*) posits $\kappa = \kappa(\omega)$ as a threshold cardinal in this respect: this is a good choice. Firstly, the definition of the cardinal yields an ω-sequence of indiscernibles for $\langle V_\kappa, \in, \vec{R} \rangle$ for any finitary relations $\vec{R}$ on V_κ; each such indiscernible γ will enforce that $\langle V_\gamma, \in \rangle \prec \langle V_\kappa, \in \rangle$ and will itself enjoy very extensive reflection properties (because it is is indiscernible to a very great height, namely all the way up to V_κ). Secondly, from such a cardinal one can find a countable transitive model M of ZFC (namely the transitivisation of the skolem hull of such an indiscernible set) and a non-trivial j with $j\colon M \to_e M$. He then concludes that as regards consistency proofs, any *internal* reflection principle which is provable from $j\colon V \to_e V$ (now in choiceless ZF) would then be provable from such a $j\colon M \to_e M$ (my italics) and therefore will not break the '$\kappa(\omega)$-barrier'. Indeed as current reflection principles are usually motivated as reflection of formulae at the syntactic level of some possible partial but higher order logical language it would be hard to argue against this. For Koellner "Since reflection would appear to be an entirely internal matter, this is a reason for thinking any reflection principle must have consistency strength below that of $\kappa(\omega)$" (p212). Whether or not second order reflection principles are taken as "*internal matters*" or not, which is perhaps debatable, one might go further and state that just as classical analysis takes place within the Borel (or occasionally analytic) realm (why? because analysts have naturally developed tools and methodologies of proof which they can work with; as such tools are definable within ZF it is with hindsight clear that the field would evolve within the confines of, or only expand into the volume or "proof-space" available, which comprises that which was mathematically attainable: namely results on classes of sets simpler than Δ^1_2 - a ZFC-limitation imposed by Gödel's results), so also the early researchers in foundations of set theory would have been led to posit reflection principles based on the notions or concepts then available and seen to be internal in Koellner's sense; it thus might be seen as inevitable, in retrospect, that they would have not grounded any principles using those means, that would would turn out to be extra-constructible.

Whilst this may indeed be convincing for such languages or for internally justified principles, ('internally' taken here to include those derived from the iterative conception or otherwise), these arguments show we must look beyond such strictly internal and logico-syntactic principles. The GRP we propose is not an internal-to-V principle or a formula-by-formula reflection principle for formulae in a higher-order language. It does not rest on the iterative conception alone.

Koellner sets the terms of the debate concerning extrinsic justifications as being about a "reduction in incompleteness." This goes roughly as follows: ZFC alone does not decide many important questions, the regularity properties of the projective sets, PU, or CH. We therefore should look for axioms, hypotheses, postulates that yield a rich picture of, say the real continuum, with features or properties desirable to analysts or set theorists. Whilst it is hard to give a metric on the quantity of incompleteness removed by assuming a particular strong axiom of infinity, we may agree, or at least agree to go along with, the motivating principle. Of course, Gödel again advocated stronger principles, these strengthenings are to obtained through linguistic or logical means.

We hold then that the moral of the foregoing is the following: We need stronger Reflection Principles: those that generalise Montague-Levy are not up to the task, they remain intra-constructible, and if we agree with Koellner, they will always be destined to be weaker than the existence of $\kappa(\omega)$.

We shall therefore define such a *Global Reflection* principle (GRP). In doing so we are mindful of the Gödel quotation above: we wish to adopt as central a new reflection principle, which is suggested by a Cantorian viewpoint, but one viewed in the light of principles and the wealth of "additional information" about the universe of sets, V, developed since those earlier times.

4 A Global Reflection Principle

> *Das Absolute kann nur anerkannt, aber nie erkannt, auch nicht annähernd erkannt werden.*
> (Cantor: Über unendliche, lineare Punktmannigfaltigkeiten. Mathematische Annalen 1883, Anmerk. 2, [3] p.587.)

Instead of 'formally projecting V' *à la Reinhardt*, let us turn the whole argument all around and generalise to obtain a stronger reflection principle. Let us take at first a somewhat naïve Cantorian (and non-Zermelian) stance.

We have seen that the apparatus of linguistic reflection falters at the third order level once one allows parameters into the formulae. Parameter-less reflection principles on the other hand are unproblematic, but again lead only to notions of Π^1_n-indescribability and the like. Such again are weaker than the existence of $\kappa(\omega)$ and so will not enlarge our conception of V beyond L. We should still be intra-constructible. Attempts to strengthen the content of the languages which can consistently reflect, such as those of Tait already mentioned also, by Koellner's arguments, cannot produce ω-Erdos cardinals. We adopt an *actualist* stance that views the totality of sets, V, as an actual (as opposed to a *potential*) totality. We do not wish to consider interpretations of V as temporarily some V_α, or as some 'intended' or 'suitable' domain of all sets. Nor do we wish to project V into some imaginary realm. We abandon hope of a strong internally defined reflection principle, meaning one defined using formula-by-formula reflection in some higher-type language using the iterative conception of set alone. We seek a more global principle of reflection that is not purely motivated by logical or syntactic considerations, but by a more holistic view of the nature of set *and* class-theoretic activity. The view exposited does not contradict the iterative-conception: it can be seen to extend it.

We take the viewpoint that the *mathematical* objects of our discourse are all *sets*: thus V and On, or the class Card of cardinals, as they are not sets, so they are not mathematical objects: they are the *absolute infinities* or *inconsistent multiplicities* of Cantor (depending on when he was writing (see [6]). We swallow the Cantorian pill that there are two types of objects: the mathematical-discourse or set objects, and the absolute infinities.

Let us imagine that $\mathcal{C}$ is the collection of all such absolute infinities. The nature of $\mathcal{C}$ is admittedly somewhat ineffable, but we can agree that as absolute and final knowledge cannot be obtained about the realm of mathematical discourse V alone neither can we find such for $(V, \in, \mathcal{C})$. Initially at least we presuppose very little about $\mathcal{C}$ and its members.

We may, if we wish, tell a mereological story about the whole mathematical universe (which is told in more detail in [5]). As above we think of sets as the sole *mathematical* objects. However sets together with the absolute infinities are *parts* of the whole realm of mathematical discourse V (V itself is, again, not a mathematical object, nor, in general, are its parts.) We denote by $\mathcal{C}$ the collection of all possible parts of V; it thus contains the collection of all absolute infinities (or proper classes in more traditional terms).

We then consider the whole of this constellation or global realm $(V, \in, \mathcal{C})$ using membership symbol $\in$ in the usual sense: " $x \in y$" for x, y in V, but also "$x \in Y$" for any Y in $\mathcal{C}$. (That is, we do not distinguish membership in classes in the Bernaysian terminology of "$x \eta Y$" and we try to reserve upper case letters X, Y, Z for parts of V, and loosely use $\in$ again without danger of confusion in "$X \in \mathcal{C}$" *etc.*) This then is the realm of Cantorian discourse that properly encompassed V (again: the realm of mathematical object discourse and the collection of its absolute infinities $\mathcal{C}$).

In the preformalised stage of the evolution of set theory, or say its Cantorian period, $\mathcal{C}$ would have some 'potentialist' flavour: the examples of absolute infinities in $\mathcal{C}$ to hand would have been restricted to those definable as extensions of predicates that were provably not sets, as Cantor had realised, or Burali-Forti demonstrated. Initially we do not have to assume that $\mathcal{C}$ is closed under first-order manipulations, which it would if we were to assume $(V, \in, \mathcal{C})$ to be a model of NBG from the outset. If we consider the finite axiomatisation of NBG by Bernays, and in particular the finite collection of operations on classes which it embodies, then these operations seem unproblematically to lead from absolute infinities to absolute infinities, thus from elements of $\mathcal{C}$ to elements of $\mathcal{C}$. To summarise, initially we assume that we have some notion of 'absolute infinity' and 'part' of an actually given V, and $\mathcal{C}$ is comprised of them. We do not have to assume that every element of $\mathcal{C}$ is the extension of some predicate (in some yet unspecified formal language) or some linguistically specified property. $(V, \in, \mathcal{C})$ is our informal embodiment of that notion.

4.1 GRP

We then assert a first Global Reflection Principle (GRP_0) that reflects the whole realm $(V, \in, \mathcal{C})$ down to some initial piece: some V_α, together with the collection of all its parts which we may identify with classes over V_α - and as thus collectively forming $V_{\alpha+1}$. (In Koellner, an isomorphic copy of the classes (here called parts) of V_α is used to avoid confusion between when an x is considered a set element of V_α, and when a class of V_α but still of course a set of V. We don't seem to need this subtlety at this point so we ignore it.) The elements of $V_{\alpha+1}$ are of course mathematical objects from the point of view of V - but play the role of parts, or classes for V_α.

(GRP_0): *There is* $\kappa \in \mathrm{On}$, *there is* $j \neq \mathrm{id}$, $\mathrm{crit}(j) = \kappa$,

$$j \colon (V_\kappa, \in, V_{\kappa+1}) \longrightarrow_e (V, \in, \mathcal{C})$$

Here the subscript e denotes elementarity with respect to the usual first order language $\mathcal{L}_{\dot{\in}}$ but augmented with second order free variable symbols $\dot{A}$, $\dot{B}$..., (let us call it $\mathcal{L}_{\dot{\in},\dot{A}}$, and in order to explicitly distinguish languages, we shall write, for example, 'Σ_n^0' for formulae at that level of complexity in $\mathcal{L}_{\dot{\in},\dot{A}}$) with the interpretation of the second order variables to range over the collection $\mathcal{C}$ of parts. That $\text{crit}(j)=\kappa$ ensures that $j(\beta)=\beta$ for any $\beta<\kappa$ but that κ (as a member of $V_{\kappa+1}$) is sent to On (as a member of $\mathcal{C}$): $j(\kappa)=\text{On}$. More generally that $j\restriction V_\kappa=\text{id}\restriction V_\kappa$ and the assumed elementarity will require that j must preserve as follows for φ in $\mathcal{L}_{\dot{\in},\dot{A}}$, $x\in V_\kappa$, $X\subseteq V_\kappa$:

$$\varphi(x,X)^{(V_\kappa,\in,V_{\kappa+1})}\leftrightarrow\varphi(j(x),j(X))^{(V,\in,\mathcal{C})} \qquad (1)$$

but $j(x)=x$ so: $$\leftrightarrow\varphi(x,j(X))^{(V,\in,\mathcal{C})}.$$

On a first pass, we say no more than this: we do not yet require that $(V,\in,\mathcal{C})$ be closed under predicative comprehension, *i.e.*, is not yet closed under all instances of *elementary* comprehension with the formulae in the Zermelo scheme from $\mathcal{L}_{\dot{\in},\dot{A}}$. (Let us call this comprehension scheme ECA.) Later we shall say that it is reasonable for $(V,\in,\mathcal{C})$ to be so closed and thus an NBG model. Although we have required j to be fully elementary in the language $\mathcal{L}_{\dot{\in},\dot{A}}$ we shall see later that we could simply have asked that it be Σ_1^0-elementary.

At this stage a number of points emerge. This says a lot more than V_κ is an initial segment of V that satisfies the same first order sentences (this would be just to say $(V_\kappa,\in)\prec(V,\in)$). The (GRP_0) asserts that the whole universe is so rich in its connections to its initial segments that there is a κ with a 'reflecting connection' j satisfying (1). The map j then is between the parts V_κ and those of V in $\mathcal{C}$. It does not move sets, and we may think of it as derived from a reflection scheme on those parts.

(A) κ is a regular cardinal, indeed strongly inaccessible. We may either assume this directly as an extra requirement on j, or else as follows: it is natural to assume that any instances of replacement with functions from $\mathcal{C}$ hold true: our conception that the whole of V with its parts reflects down to some $(V_\kappa,\in,V_{\kappa+1})$ rules out the existence of some absolutely infinite function class with a set domain in V and range unbounded in V. However any counterexample to the regularity of κ lying in $V_{\kappa+1}$, $G\colon\alpha\to\kappa$ say, would be carried up to a $j(G)$ with domain $j(\alpha)=\alpha$ negating this. (B) Moreover if we allow the axiom of choice, then there is some $F\subseteq\kappa\times V_\kappa$ which is a bijection between κ and the universe of V_κ. Hence we should conclude (again an easy exercise in the elementarity requirements of (1) above) that some proper class $\tilde{F}=j(F)$ is such a bijection of On with V. We thus have that $(V_\kappa,\in,V_{\kappa+1})$ can be considered even as a Zermelian 'normal domain' with its classes. It is a 'typical' such domain since nothing that we may assert about it in the given language, differentiates it from the whole realm meaning V together with its parts.

Thus in particular we have the *consequence* that if (GRP_0) holds for any j, κ, then for any $X\in V_{\kappa+1}$ if $j(X)=\tilde{X}$, X *must* equal $j(X)\cap\kappa$. (It could not be that $j(X)$ is some $\tilde{Y}\in\mathcal{C}$ differing from X below κ, as we should have the absurdity that for some $z\in V_\kappa$ we'd have $z\in X\triangle\tilde{Y}=j(X)$.)

By asserting (GRP_0) we thus are also asserting the property that anything satisfied in $(V_\kappa,\in,V_{\kappa+1})$ by a set $x\in V_\kappa$ and one of its parts $X\subset V_\kappa$ say, is the reflection of something satisfied in the realm $(V,\in,\mathcal{C})$ by x and by one of the parts, or classes, of the whole realm, $\tilde{X}$ say, as $j(X)$. And that part should be a part of the whole (V) that extends X, *i.e.* $\tilde{X}\cap V_\kappa$ should be the same as X.

As an alternative thought experiment (which the author does not particularly like) one may imagine $(V, \in, \mathcal{C})$ and for varying κ performing the truncations of $X \in \mathcal{C}$ to V_κ, until one finds a κ where this process allows one to see that every element of $V_{\kappa+1}$ is such a truncate, and moreover that there are sufficient witnesses to Σ_1-statements in V_κ itself, that we may obtain the relationship of Σ^0_1-elementarity as at (1) above for some associating map j which chooses images for the parts $X \in V_{\kappa+1}$. The richness of the structure $(V, \in, \mathcal{C})$ inherent in the principle (GRP) below, then asserts that this will reveal unboundedly many such κ in On where this will be successful.

In summary:

- (A) Assuming AC for sets, $(V, \in, \mathcal{C})$ will be a model of global choice.
- (B) $(V_\kappa, \in) \prec_e (V, \in)$ (here meaning elementary in the usual language of first order set theory, $\mathcal{L}_{\dot\in}$.
- (C) *Sufficient elementarity* (indeed Σ^0_1-elementarity), rather than the full elementarity of j in the language $\mathcal{L}_{\dot\in,\dot A}$, is all that will be required to derive desirable large cardinal consequences here. Initially we are not bound to require of $(V, \in, \mathcal{C})$ that it be a model of ECA.
- (D) We can equally as well ask that there be unboundedly many such κ in On for which there is such a j:

(GRP): *For any* α*, there is* $\kappa \in \mathrm{On}, \alpha < \kappa$*, and there is* $j \neq \mathrm{id}$, $\mathrm{crit}(j) = \kappa$,

$$j\colon (V_\kappa, \in, V_{\kappa+1}) \longrightarrow_e (V, \in, \mathcal{C})$$

The principle (GRP) then asserts that for any set $x \in V$ there is the possibility of reflecting the whole realm to some normal domain containing x as an element.

4.2 Origins

As an aside we remark on the original motivations for these principles; in fact these came not from extendibility at all, but from a (literally) completely different direction: *weak subcompactness.*

Definition 1. *A cardinal* λ *is* subcompact *if for any* $A \subseteq H(\lambda^+)$ *there is a* $\mu < \lambda$*, a set* $B \subseteq H(\mu^+)$*, and an elementary embedding (in* $\mathcal{L}_{\dot\in,\dot A}$*)*

$$j\colon (H(\mu^+), \in, B) \longrightarrow_e (H(\lambda^+), \in, A).$$

Call κ weakly subcompact *if the above is true but requiring only* $A \subseteq H(\lambda)$.

Notice the obvious point here that λ is the large cardinal and it determines the range structure: it is not the 'large cardinal' of the side of the domain structure which is to be projected (as for extendability). Since $H(\lambda^+)$ is nothing other than the transitivisation of the extensional wellfounded relations in $V_{\lambda+1}$ we may identify these two structures.

Now subcompactness is straightforwardly a third order property over V_λ. Moreover taking $A = H(\lambda) = V_\lambda$ it is easy to see (by throwing in some arbitrarily large ordinal δ into the predicate A for V_λ, *eg* by substituting $V_\lambda \times \{\delta\}$, - which has the effect of forcing the critical point of the embedding j to be above δ) that the various embeddings j_δ witnessing the weak subcompactness of $(V_{\lambda+1}, \in, V_\lambda \times \{\delta\})$ (so with A as the various $V_\lambda \times \{\delta\}$) yields the strong (GRP). Thus the existence of a weakly subcompact cardinal establishes the existence of a transitive set model of (GRP) (and hence its consistency).

4.3 Consequences of $\mathbf{GRP_0}$

Suppose GRP_0 holds as witnessed by a j with critical point κ. Define a field of classes U on $\mathcal{P}(\kappa)$ by

$$X \in U \leftrightarrow \kappa \in j(X)$$

As $\mathcal{P}(\kappa) \subseteq V_{\kappa+1} \subseteq \mathrm{dom}(j)$ by Σ_1^0 -elementarity, this is an ultrafilter.

• The strong inaccessibility of κ yields the δ-additivity of U for any $\delta < \kappa$, and non-principality of U trivially follows from $j \restriction \kappa = \mathrm{id} \restriction \kappa$. Thus U establishes κ is a 'measurable cardinal' (and thus we have a strongly extra-constructible principle). However then:

For any $\alpha < \kappa$: "$\exists \kappa > \alpha(\kappa$ *a measurable cardinal*" $^{\langle V, \in\rangle} \Longrightarrow$

$\Longrightarrow$ "$\forall \alpha \exists \lambda > \alpha(\lambda$ *a measurable cardinal*)" $^{\langle V_\kappa, \in\rangle} \Longrightarrow$

$\Longrightarrow$ "*There is a proper class of measurable cardinals*"$^{\langle V, \in\rangle}$

(We have purposefully dropped the predicate of parts/classes here since the elementarity needed is just Σ_3-preserving in the language $\mathcal{L}_{\dot{\in}}$.) More pertinently (but similarly; *cf.* [7] for a definition of Woodin cardinal):

Lemma 2. $(\forall \alpha \exists \lambda > \alpha(\lambda$ *a measurable Woodin cardinal*$))^{(V, \in)}$

Proof: We first recall the definition of "κ is Woodin": this holds iff for any $f \in {}^{\kappa}\kappa$, there is an $\alpha < \kappa$ with $f``\alpha \subseteq \alpha$, and there is a $j: V \to_e N$ with $\mathrm{crit}(j) = \alpha$ and $V_{j(f)(\alpha)} \subseteq N$.

So let $f \in {}^{\kappa}\kappa \subseteq V_{\kappa+1}$, be arbitrary and consider $j(f)$. Then $j(f)$: On $\longrightarrow$ On; $j(f)``\kappa \subseteq \kappa$. Take $\lambda > \kappa$ a sufficiently large inaccessible, so that $j(f)(\kappa) < \lambda$, and consider the "λ-strong" extender derived from j:

For $a \in [\lambda]^{<\omega}$: $\quad E_a =_{\mathrm{df}} \{z \in \mathcal{P}([\kappa]^{|a|}) : a \in j(z)\}$.

This has the following properties:

(1) $\mathcal{E} = \langle E_a : a \in [\lambda]^{<\omega}\rangle$ *is a* (κ, λ)*-extender with* $j(f)(\kappa) = j_{\mathcal{E}}(f)(\kappa) < \lambda$, *and such that* $\mathrm{Ult}((V, \in), \mathcal{E})$ *is wellfounded, and if* $k: V \longrightarrow N \cong \mathrm{Ult}((V, \in), \mathcal{E})$, *is the unique transitivisation collapse map, then* $V_\lambda = (V_\lambda)^N$.

This may be formalised as a first order property and we abbreviate it as $\Phi(\kappa, \lambda, j(f), \mathcal{E})$ about the displayed objects. Then:

$$(\exists \alpha[\exists \lambda \exists \mathcal{E}(j(f)``\alpha \subseteq \alpha \wedge \Phi(\alpha, \lambda, j(f), \mathcal{E})]^{(V, \in, \mathcal{C})}$$

We may further abbreviate this as

$$\exists \alpha \varphi(j(f), \alpha)^{(V, \in, \mathcal{C})}$$

and this is a first order statement about $j(f)$. By GRP_0:

$$\exists\alpha\varphi(f,\alpha)^{(V_\kappa,\in,V_{\kappa+1})}.$$

Thus α witnesses the Woodiness property of κ at least as regards the case of f. Now let f vary over ${}^\kappa\kappa$ and we see that κ is indeed Woodin.

Hence:

$$\text{“}\kappa \text{ is Woodin (and measurable)”}^{(V,\in)}$$

and thus as for measurables, such measurable Woodin cardinals are unbounded in both κ and On. Q.E.D.(Lemma 2)

We finally remark here, that the strength of this theory, when formalised as below, is weaker than any embedding property of V into a universe M where, if the critical point of that embedding, k say, is κ, then $k``\kappa^+$ is bounded $k(\kappa)$. This may seem merely a technical observation, but such embeddings $k$ where $k(\kappa^+) > \sup k``\kappa^+$ typify supercompactness embedding and similar such and play a decisive, and novel, role, in *e.g.* forcing argument proofs from such cardinal assumptions. Hence our embedding j is in consistency strength below strong axioms of infinity positing such cardinals: we cannot deduce that our critical point κ is a cardinal of that kind, or that such exist in V from (GRP) alone.

4.4 Formalisation of $\mathbf{GRP_0}$.

If we wish to discuss this viewpoint within a formal theory we may proceed as follows. The assertion that there exists such a j is *prima facie* a third order statement; however such a j (as a function) may be *coded* by a class $J = \{\langle x, y\rangle : y \in j(x)\} \subseteq V_{\kappa+1} \times V$ and then $j(x)$ is just $J^{(x)} =_{\mathrm{df}} \{y \mid \langle x, y\rangle \in J\}$. (Note then, for example, that the measure U above is then $\{X \mid \langle X, \kappa\rangle \in J\}$.) We then have the possibility of asserting the existence of a second order class J with our desired properties.

The elementarity of j may be expressed in several different versions according to taste. One may simply express it as a scheme involving $\mathcal{L}_{\dot\in,\dot A}$ formulae. Another option is to consider elementarity to be that in (a fragment of) the second order language $\mathcal{L}^2_{\dot\in}$ with second order quantifiers.

Under the first option we shall see that if we are only aiming at getting large cardinals, it would be sufficient to have a j that is Σ^0_1-preserving in the $\mathcal{L}_{\dot\in,\dot A}$ language (we emphasise again that this means Σ_1 with class parameters). Under this viewpoint we are only obliged to reflect Σ^0_1 existential assertions about parts of V in the range of j down to V_κ with its parts. There is no need to quantify over parts, or require them to be closed under certain operations *etc.*

It is not difficult to see that this level of preservation implies Σ_n-preserving in the first order language $\mathcal{L}_{\dot\in}$ for any n (and this delivers the large cardinal consequences we might be seeking).

[Sketch: Firstly, note that the function F_V: On $\longrightarrow V$ defined by $F_V(\alpha) = V_\alpha$ is Π_1^{ZF}-definable in $\mathcal{L}_{\dot{\in}}$, and thus for $\alpha < \kappa$ $(V_\alpha)^{V_\kappa} = F_V^{V_\kappa}(\alpha) = j(F_V^{V_\kappa})(\alpha) = F_V(\alpha) = V_\alpha$. Secondly formalising our semantics in a standard manner, thereby defining a relation of Σ_n-elementary substructurehood (and using F_V) and writing this as "$V_\alpha \prec_{\Sigma_n} V_\beta$", we let $C_n \subseteq \kappa$ be the class of α with $V_\alpha \prec_{\Sigma_n} V_\kappa$. Then $C_n \in \Delta_{n+1}^{\mathrm{ZF}}$ definable over V_κ (and is closed and unbounded in κ), and thus is in $V_{\kappa+1}$.

$$\forall\alpha < \beta < \gamma[(\alpha, \beta \in C_n))^{(V_\kappa, \in, V_{\kappa+1})} \longrightarrow (V_\alpha \prec_{\Sigma_n} V_\beta)^{V_\gamma}].$$

The consequent here is $\Delta_1(V_\gamma,\ \alpha,\ \beta)$; thus this Π_1^0 sentence about C_n carries up to $(V, \in, \mathcal{C})$ about $j(C_n)$ using Σ_1^0-elementarity. (Again by this elementarity, $j(C_n)$ is not a set and is c.u.b. in On.) Thus V is the union of the Σ_n-elementary tower of $V_\alpha \prec_{\Sigma_n} V$, for $\alpha \in j(C_n)$. So if now $\varphi(\vec{x})$ is some Σ_n formula of $\mathcal{L}_{\dot{\in}}$ and $\gamma < \kappa$ sufficiently large in C_n, with $\vec{x} \in V_\gamma$ then:

$$\varphi(\vec{x})^{V_\kappa} \leftrightarrow \varphi(\vec{x})^{V_\gamma} \leftrightarrow (\exists\gamma \in C_n(V_\gamma \vDash \varphi(\vec{x}))^{(V_\kappa, \in, V_{\kappa+1})} \leftrightarrow$$
$$\leftrightarrow (\exists\gamma \in j(C_n)[\, V_\gamma \vDash \varphi(\vec{x})])^{(V, \in, \mathcal{C})} \leftrightarrow \varphi(\vec{x})\ ].$$

This restriction to Σ_1-in-class-variables, then gives a satisfactory formalisation of GRP_0. Another point to also stress is that the full formulation GRP is preserved by set forcing extensions.

We discuss now the second option. In this case it would be natural to consider $(V, \in, \mathcal{C})$ as a potential NBG model. (We might in any case be well inclined to assume the latter, since the finitely many Bernays operations on classes with which NBG can be finitely axiomatised (two of which are given below), lead indubitably from absolute infinities to absolute infinities, and contain $[\in] = \{(x, y) | x \in y, x, y \in V\}$ and $[\mathrm{id}] = \{(x, x) | x \in V\}$. This suffices.) Then to infer that $(V, \in, \mathcal{C})$ is an NBG model, it suffices to only require that j be Σ_2^1-preserving in $\mathcal{L}_{\dot{\in}}^2$.

To see this note first that $(V_\kappa, \in, V_{\kappa+1})$ is a natural model of NBG (even of Kelley-Morse) with Global Choice as we have seen. The standard methods to finitely axiomatise NBG involve a series of simple axioms following on Extensionality for classes, as a finite set of axioms that can be expressed as Π_2^1 assertions of the form $\forall X, Y \exists Z \varphi(X, Y, Z)$. These all have a matrix φ in $\mathcal{L}_{\dot{\in}, \dot{A}}$ defining some simple operation on the classes X, Y which yields the class Z. Here are two typical such:

For classes A, B there is a class $C = A \times B$;

For any class R there is a class $S = \{((a, b), c) \mid (a, (b, c)) \in R\}$.

These Π_2^1 assertions claim that the model is closed under the Bernays operations mentioned above. Hence if j preserves these Π_2^1 assertions then $(V, \in, \mathcal{C})$ will necessarily be an NBG model.

That done, we may then define $\mathrm{Sat}_\in$ for the first order (*i.e.* the $\mathcal{L}_{\dot{\in}}$) part of the language within NBG (see *e.g.*, [12]); this may be extended to formulae in $\mathcal{L}_{\dot{\in}, \dot{A}}$ and obtain for any $n, m < \omega$ a Σ_n^0 formula $\mathrm{Sat}_n^1(z_o, z_1, z_2, Y)$ so that for any Σ_n^0 formula $\varphi(v_0, \ldots, v_{k-1}, Y_1, \ldots, Y_m)$ with the displayed free variables:

$$\forall h \in {}^{\omega}V \forall X_1, \cdots \forall X_m[\mathrm{Sat}_n^1(\ulcorner\varphi\urcorner, k, m, \langle h_0, \ldots, h_{k-1}\rangle, \langle X_1, \ldots, X_m\rangle) \leftrightarrow \varphi(\vec{h}, X_1, \ldots, X_m)].$$

(We have used $m-1$-fold iterations of a suitable pairing function to render the list $X_1,\dots,X_m$ as $\langle X_1,\dots,X_m\rangle$.)

Given Global Choice we may even define Skolem functions and obtain the above for fully second order Σ^1_n formulae (see, *e.g.*, [4]). We may thus for any $n\in\omega$ define a Σ^1_n formula $\mathrm{Sat}^2_n(v_0,v_1,\dots,v_k,Y_1,\dots,Y_m)$ so that, provably in NBG + Global Choice

$$\forall h\in{}^{\omega}V\forall X_1,\cdots\forall X_m[\mathrm{Sat}^2_n(\ulcorner\varphi\urcorner,k,m,\langle h_0,\dots,h_{k-1}\rangle,\langle X_1,\dots,X_m\rangle)\leftrightarrow\varphi(\vec{h},X_1,\dots,X_m)]$$

for any Σ^1_n formula $\varphi(v_1,\dots,v_k,Y_1,\dots,Y_m)$.

We may then schematically require that j be a fully Σ^1_ω-elementary preserving embedding by demanding it be Σ^1_n-preserving for each n.

Alternatively, we may wish to add to the language $\mathcal{L}^2_\in$ a quinary primitive relation symbol Sat^2 with the requirements that

$$(\mathrm{Sat}^2(\ulcorner\Theta\urcorner,k,m,\langle\vec{h}\,\rangle,\,\langle\vec{X}\,\rangle)\leftrightarrow\Theta(\vec{h}\,,\,\vec{X}\,))^{(V,\in,\mathcal{C})}\text{ for }\vec{h}\in{}^kV,\ \vec{X}\in{}^m\mathcal{C},\Theta\in\mathcal{L}^2_\in.$$

We could achieve the same effect by naturally requiring that J satisfy:

$$(\mathrm{Sat}^2(\ulcorner\Theta\urcorner,k,m,\langle\vec{h}\,\rangle,\,\langle\vec{X}\,\rangle))^{(V_\kappa,\in,V_{\kappa+1})}\ \Leftrightarrow\ (\mathrm{Sat}^2(\ulcorner\Theta\urcorner,k,m,\langle\vec{h}\,\rangle,J^{(\langle\vec{X}\rangle)}))^{(V,\in,\mathcal{C})}.\quad(2)$$

This then gives a stronger principle, GRP^+_0 say. We note that in both these cases more can now be determined about $\langle V,\in,\mathcal{C}\rangle$: since $(V_\kappa,\in,V_{\kappa+1})$ is a natural model of Kelley-Morse with the satisfaction predicate, the second order comprehension principle carries up to $(V,\in,\mathcal{C})$. We should then conclude that $(V,\in,\mathcal{C})$ also satisfied the KM-axioms. In this case we arguably have an even better agreement, or replication, of $(V,\in,\mathcal{C})$ with $(V_\kappa,\in,V_{\kappa+1})$. If j is Σ^1_ω-elementary more can be said about the range of j: for example Σ^1_1-elementarity shows that the class of measurable Woodin cardinals is stationary.

However there seems little in our Cantorian conception *prima facie*, that allows us to argue that $(V,\in,\mathcal{C})$ should be even an NBG model: the ineffability of the structure $(V,\in,\mathcal{C})$ would seem to gives us little to say about it mathematically. We have no guarantee that the parts of V are extensions of predicates in any particular language and we are not requiring such. We might, as remarked above, be inclined to allow nevertheless that reasoning about absolute infinities, or parts, of $\mathcal{C}$ may imply that it be closed under the kind of elementary operations required by Bernays's axiomatisation of NBG: the Bernays operations are indeed simple in form, and lead from absolute infinities to the same. Thus: even if we wish to consider V as our domain of mathematical objects and structures, and whilst the totality that is V, or in general its proper parts, are not, strictly speaking, to be regarded as mathematical objects, that should not imply they are *hors de combat* for any application of reasoning at all, and such reasoning as above may result in our conclusion that NBG is the right theory for $(V,\in,\mathcal{C})$.

In [5] we seek to give possible different interpretations to the members of $\mathcal{C}$: either a mereological one as alluded to above, or else one involving a plural interpretation of second order quantification to render the classes as plurals. (In that paper (GRP) is defined as requiring the stronger full second order elementarity.) We do not go into these matters here.

5 Conclusion: is GRP_0 a reflection principle?

Clearly the principle is not a reflection principle in the strict sense of a principle stating that any single sentence of a language (of any particular order or logical kind) does not pin down V: we have stated a principle that implies that a whole language may be simultaneously reflected to an initial segment of the V-hierarchy. The first order Montague-Levy Reflection theorem of ZF neither commits one to an actualist stance nor a potentialist one. We adopt the view that a universal second order, or so here, a universal quantifier (hence of the form 'all parts' or 'all classes') does tend towards an actualist viewpoint. We have an aversion, on grounds of coherence, to principles that require higher types over $(V, \in)$. However it would seem also that to formalise $(V, \in, \mathcal{C})$ as a KM model, and hence to be enriched with an impredicative comprehension schemes, does require a robust actualist viewpoint for the comprehension principle to work, and which would be difficult to formalise over moving potential domains. On the other hand, although we don't here see GRP_0, or GRP, at least initially, as endorsing KM for $(V, \in, \mathcal{C})$, or even its closure under, say comprehension expressed for $\mathcal{L}_{\dot{\in},\dot{A}}$-formulae, it would seem that the viewpoint behind it is an actualist one: as we have said, we take the whole of the (actual) universe with its parts and reflect upon it.

Nor is it an internally justified scheme in the sense above (meaning that it is derived from the iterative concept of set alone): the scheme is consistent with the iterative conception, but it has to go beyond it to endorse the richness of the reflection asserted by GRP_0 or GRP.

The structure $(V_\kappa, \in, V_{\kappa+1})$ must reflect all possible truths expressible in $(V, \in, \mathcal{C})$ (or at least the simplest existential Σ^0_1-truths about sets with parts as parameters under the first option sketched). Moreover any formula $\varphi(x, X)$ with any $x \in V_\kappa$, $X \in V_{\kappa+1}$ which is true in $(V_\kappa, \in, V_{\kappa+1})$ must be a reflection of a truth about x and the *extension* $j(X)$; and this lastly demands that the domain of j contain all parts of V_κ. This latter condition requires that our picture of this reflection is a *full* one when considering the parts of the normal domain of V_κ. It is not the weaker demand that $(V_\kappa, \in)$ be an elementary submodel of $(V, \in)$ nor is it that a particular NBG model M be elementary in some $(V, \in, \mathcal{C}')$; we demand the whole of V_κ *with all its parts* be elementary in $(V, \in, \mathcal{C})$; thus $(V_\kappa, \in, V_{\kappa+1})$ reflects the whole picture or conception of $(V, \in, \mathcal{C})$. This fullness is what gives the principle its large cardinal strength. We have to require it whether considering any of the formalisations. We leave open the nature of $\mathcal{C}$: we have seen that different formalisations may require $\mathcal{C}$ to have differing properties. (Our own predelictions are for the simplest possible, with minimal extra requirements on $\mathcal{C}$.) An objector may rightly say that their conception of "richness" of $(V, \in, \mathcal{C})$, does not lead them to posit a reflection of the kind described to some initial segment. However these objections could be turned against those willing to follow Bernays and admit second or third order reflection or higher to justify indescribables: such principles require one to reflect on a structure with $2, 3, \ldots, n$ more higher types of classes, classes of classes ... *etc.* over V. We may have in mind Gödel's dictum quoted in the first section about higher logics; but such are a disguised form of a typed class theory beyond On and as such are, we feel, also problematic. Getting an understanding of a 'rich' n'th order logic over V which is coherent seems no less problem-ridden than understanding and positing a reflection of this rich kind. By requiring only that we have reflection on $(V, \in, \mathcal{C})$, that is the universe of sets with its parts, then we may express this as a single Σ^1_1 existential second order statement.

We have avoided Reinhardt's difficulties about delineating the modal notion of *legitimate domains* which are perhaps to "model V", and we seek to avoid the difficulties (again pointed out by Koellner) of *tracking* the intensional aspect of classes over a domain V_α considered by Reinhardt perhaps as a legitimate domain, to some other extended realm V_β. We do not have to claim any kind of access, intensional or otherwise, to each and all the parts of V_κ or of V. The principle is thus one that asserts that the global realm $(V, \in, \mathcal{C})$ is *so rich both in its set-objects and its parts, and its initial segments with their parts, that there can be* such an association between the parts of some V_κ (or unboundedly many such) and those in $\mathcal{C}$, the parts of V. The association(s) j is (or are) such that first order existential sentential truths that we can formulate about the realm V with *all of* its parts (in the given language) reflect *via* j to truths about V_κ with *all of* its parts. More formally summarised: the relation that j embodies is that the junior realm $(V_\kappa, \in, V_{\kappa+1})$ is simply an initial segment of the full realm $(V, \in, \mathcal{C})$, and in such a way that any part $X \subseteq V_\kappa$ can be associated with an extension of it in $\mathcal{C}$, and in a sufficiently elementary, namely Σ^0_1, fashion. The role j plays is of exemplifying that association, both individually for each $X \in V_{\kappa+1}$ and, *via* elementarity, collectively for $(V, \in, \mathcal{C})$. Whilst conventionally the assertion of j's existence is a parameterless third-order one, as indicated we may if we wish think of (and reason about, and justify) j initially as a plurality, that of the parts $\tilde{X}$ in the range of j. Later we may formalise these ideas and the map j, as sketched above.

Gödel again:

> Generally I believe that, in the last analysis, every axiom of infinity should be derivable from the (extremely plausible) principle that V is indefinable, where definability is taken in more and more generalised and idealized sense. ([19], p285)

Clearly the (GRP) principles are taking that generalised or idealized sense liberally, when judged in the light of a strict iterative conception of sets alone, but in the light of the Cantorian inspired *gesamte Auffassung* we have sketched above, they are not unnatural and deliver a satisfyingly rich universe.

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