On the consistency strength of the inner model hypothesis

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The *inner model hypothesis (IMH)* and the *strong inner model hypothesis (SIMH)* were introduced in [5]. In this article we establish some upper and lower bounds for their consistency strength.

We repeat the statement of the IMH. A sentence in the language of set theory is *internally consistent* iff it holds in some (not necessarily proper) inner model. The meaning of internal consistency depends on what inner models exist: If we enlarge the universe, it is possible that more statements become internally consistent. The *inner model hypothesis* asserts that the universe has been maximised with respect to internal consistency:

The Inner model hypothesis (IMH): If a statement φ without parameters holds in an inner model of some outer model of V (i.e., in some model compatible with V), then it already holds in some inner model of V.

Equivalently: If φ is internally consistent in some outer model of V then it is already internally consistent in V. This is formalised as follows. Regard V as a model of Gödel-Bernays class theory, endowed with countably many sets and classes. Suppose that V^* is another such model, with the same ordinals as V. Then V^* is an *outer model of* V (V is an *inner model of* V^*) iff the sets of V^* include the sets of V and the classes of V^* include the classes of V. V^* is *compatible with* V iff V and V^* have a common outer model.

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The IMH implies absoluteness for sentences which are Σ_2 over $H(\omega_1)$ (equivalently, for sentences which are Σ^1_3 in the sense of descriptive set theory). This is because by Lévy-Shoenfield absoluteness, such a sentence is true iff it is true in some inner model.

Theorem 1 *([5]) The inner model hypothesis implies that for some real R, ZFC fails in $L_\alpha[R]$ for all ordinals α. In particular, there are no inaccessible cardinals, the reals are not closed under # and the singular cardinal hypothesis holds.*

Theorem 2 *The IMH implies that there is an inner model with measurable cardinals of arbitrarily large Mitchell order.*

Proof. Assume not and let K denote Mitchell's core model for sequences of measures (see [6]). Let δ be the maximum of ω_1 and the supremum of the Mitchell orders of measurable cardinals in K. By Mitchell's Covering Theorem for K we have:

$(*)$ *cof* $(\alpha) \geq \delta$, α *regular in* $K \rightarrow$ *cof* $(\alpha) =$ *card* (α).

Now normally iterate K by applying each measure of order 0 exactly once, i.e., if K_i is the i-th iterate of K, K_{i+1} is formed by applying the measure of order 0 in K_i at κ_i, the i-th measurable cardinal of K. Let $\sigma : K \rightarrow K'$ be the resulting iteration map. Then:

Lemma 3 $(*)$ *holds with K replaced by K'.*

Proof. It suffices to show by induction on i that $(*)_i$ holds, where $(*)_i$ is $(*)$ with K replaced by K_i.

Base case: $(*)_0$ is just $(*)$.

Successor case: Suppose that $(*)_i$ holds and that α is K_{i+1}-regular with cofinality at least δ. Let $\pi_{i,i+1} : K_i \rightarrow K_{i+1}$ be the ultrapower map resulting from applying the measure of order 0 at κ_i.

We may assume that α is greater than κ_i, else α is K_i-regular and we are done by induction. If α is at most $\pi_{i,i+1}(\kappa_i)$ then α has the same cardinality as κ_i^+ of K_i, and, as K_{i+1} and K_i contain the same κ_i-sequences of ordinals, the same cofinality as κ_i^+ of K_i. So we are again done by induction.

Now suppose that α is greater than $\pi_{i,i+1}(\kappa_i)$. Represent α in K_{i+1}, the ultrapower of K_i, by $[f]$ where $f : \kappa_i \to \mathrm{Ord}$. We may assume that f is either constant or increasing, and also that $f(\gamma)$ is K_i-regular and greater than κ_i for all $\gamma < \kappa_i$. If f is constant then $\alpha = \pi_{i,i+1}(\bar{\alpha})$ for some $\bar{\alpha}$ which is regular in K_i and greater than κ_i; but then $\bar{\alpha}$ is a fixed point of $\pi_{i,i+1}$ so $\alpha = \bar{\alpha}$ and we are done by induction. So assume that f is increasing.

Now the K_i-cofinality of α is at least the supremum μ of the $f(\gamma)$'s, as we can everywhere-dominate any set in K_i of $f(\gamma)$-many functions from κ_i into $\prod_{\gamma' > \gamma} f(\gamma')$ by a single such function in K_i. As μ is K_i-singular, the K_i-cofinality of α is in fact greater than μ. And the K_i-cardinality of α is $\mu^\kappa = \mu^+$ of K_i. It follows that α and μ^+ of K_i have the same cofinality and the same cardinality, so we are done by induction.

Limit case: Suppose that i is a limit and α is K_i-regular with cofinality at least δ. For large enough $j < i$ we can write α as $\pi_{j,i}(\alpha_j)$, where $\pi_{j,i}$ is the natural embedding of K_j into K_i. Let κ^* denote the supremum of the κ_j, $j < i$.

We may assume that each α_j is at least κ^*, as if α_j is less than κ_k, $j < k < i$, it follows from the fact that κ_k is a fixed point of $\pi_{j,k}$ that α equals $\pi_{j,k}(\alpha_j)$, which is regular in K_k, and so we are done by induction. But if α_j is at least κ^* then α_j is a fixed point of $\pi_{j,i}$ and therefore $\alpha = \alpha_j$; so we are again done by induction. □

Lemma 4 *If λ is a cardinal then $\mathit{cof}^{\,K'}(\lambda)$ is not measurable in K'.*

Proof. If $\kappa = \mathrm{cof}^{\,K}(\lambda)$ is not measurable in K then λ is a fixed point of the map σ and therefore the result follows by elementarity. Otherwise, let $\sigma_0 : K \to K'_0$ be the iteration map that results from applying only the order 0 measures at cardinals less than κ. Then $\lambda = \sigma_0(\lambda)$ has cofinality $\sigma_0(\kappa) = \kappa$ in K'_0. As the ultrapower K'_1 of K'_0 by the order 0 measure at κ contains all κ sequences of ordinals that belong to K'_0, it follows that λ has cofinality κ in K'_1 and therefore also in K'. As κ is not measurable in K' we are done. □

Now after [2] define a function $d : \mathrm{Ord} \to \omega$ as follows. Fix a lightface K'-definable global $\square$-sequence $\langle C_\alpha \mid \alpha \text{ singular in } K' \rangle$: C_α is closed unbounded in α with ordertype less than α for each K'-singular α and $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$ whenever $\bar{\alpha}$ is a limit point of C_α. If α is not K'-singular then $d(\alpha) = 0$. Otherwise define:

$\alpha_0 = \alpha$
$\alpha_1 = \text{ot } (C_{\alpha_0})$
$\alpha_2 = \text{ot } (C_{\alpha_1})$
$\cdots$
$\alpha_{n+1} = \text{ot } (C_{\alpha_n})$,

as long as α_n is K'-singular, and let $d(\alpha)$ be the least n such that α_n is not K'-singular. $\alpha_{d(\alpha)}$ is the K'-cofinality of α.

Lemma 5 *(Main Lemma, after [2]) For each n there is a ZFC-preserving class forcing P_n that adds a CUB class C_n of singular cardinals such that for all α in C_n of cofinality at least δ, $d(\alpha)$ is at least n.*

Proof. We use the following.

Lemma 6 *Suppose $k < m$, $\alpha \geq \delta$, α is regular and C is a closed set of ordertype $\alpha^{+m}+1$, consisting of ordinals $\geq \alpha^{+m}$ (where $\alpha^{+0} = \alpha$, $\alpha^{+(p+1)} = (\alpha^{+p})^+$). Then $(C \cap \{\beta \mid d(\beta) \geq k+1\}) \cup \mathit{Cof}\ (< \delta)$ has a closed subset of ordertype $\alpha^{+(m-k-1)} + 1$.*

Proof. The proof is by induction on k, using Lemma 3.

Suppose $k = 0$. Let β be the $\alpha^{+(m-1)}$-st element of C. Then β is K'-singular since its cofinality ($= \alpha^{+(m-1)}$) is at least δ and less than its cardinality ($\geq \alpha^{+m}$). Similarly, each element of Lim $(C \cap \beta)$ of cofinality $\geq \delta$ is K'-singular and therefore Lim $(C \cap \beta)$ is a closed subset of $(C \cap \{\beta \mid d(\beta) \geq 1\}) \cup \text{Cof } (< \delta)$ of ordertype $\alpha^{+(m-1)} + 1$, as desired.

Suppose that the Lemma holds for k and let $m+1 > k+1$, C a closed set of ordertype $\alpha^{+(m+1)}+1$ consisting of ordinals $\geq \alpha^{+(m+1)}$. Then $\mu = \max C$ is K'-singular, as its cofinality is at least δ and less than its cardinality. Let β be the $(\alpha^{+m} + \alpha^{+m})$-th element of $C \cap C_\mu$. β is K'-singular as its cofinality is at least δ and less than its cardinality. Let $\bar{\beta}$ be the α^{+m}-th element of C. Then $\bar{C} = \{\text{ot } C_\gamma \mid \gamma \in C \cap \text{Lim } C_\beta \cap [\bar{\beta}, \beta]\}$ is a closed set of ordertype $\alpha^{+m} + 1$ consisting of ordinals $\geq \alpha^{+m}$. By induction there is a closed $\bar{D}$ contained in $(\bar{C} \cap \{\gamma \mid d(\gamma) \geq k+1\}) \cup \text{Cof } (< \delta)$ of ordertype $\alpha^{+(m-k-1)} + 1$. But then $D = \{\gamma \in C \cap \text{Lim } C_\beta \mid \text{ot } C_\gamma \in \bar{D}\}$ is a closed subset of $(C \cap \{\gamma \mid d(\gamma) \geq k+2\}) \cup \text{Cof } (< \delta)$ of ordertype $\alpha^{+(m-k-1)} + 1$. As $m - k - 1 = (m+1) - (k+1) - 1$, we are done. $\square$ (Lemma 6)

Lemma 5 now follows: Let P_n consist of closed sets c of singular cardinals such that

$\alpha \in c$, cof $(\alpha) \geq \delta \to d(\alpha) \geq n$,

ordered by end-extension. Lemma 6 implies that this forcing is κ-distributive for every cardinal κ. $\square$

Now for each n there is an outer model of V containing a real R_n such that in $L[R_n]$:

$(*)_{R_n}$ R_n codes a CUB class C_{R_n} of singular cardinals and an iterable, universal extender model K'_{R_n} such that
a. $d_{R_n}(\alpha) \geq n$ for α in C_{R_n} of sufficiently large cofinality, where $d_{R_n}(\alpha)$ is defined in K'_{R_n} just like $d(\alpha)$ is defined in K'.
b. $\alpha \in C_{R_n} \to$ cof (α) in K'_{R_n} is not measurable in K'_{R_n}.

This is because we can use Lemma 5 to force a CUB class C_n of singular cardinals such that $d(\alpha) \geq n$ for all α in C_n of sufficiently large cofinality, and then L-code the model $\langle V, C_n, K' \rangle$ by a real R_n. The extender model K' is universal in the extension as successors of strong limit cardinals are not collapsed and therefore weak covering holds relative to K' in the extension at all such cardinals of sufficiently large cofinality.

Applying the IMH, there are such reals R_n in V. As each R_n codes a CUB class of singular cardinals, the K of $L[R_n]$ is universal and therefore so is the K_{R_n} arising from $(*)_{R_n}$. Now co-iterate the K_{R_n}'s to a single K^*, resulting in embeddings $\pi_n : K_{R_n} \to K^*$. As singular cardinals in C_{R_n} of sufficiently large cofinality are fixed by π_n (as their K_{R_n}-cofinality is not measurable in K_{R_n}), it follows that there is a single γ belonging to all of the C_{R_n}'s which is fixed by all of the π_n's. But then $d^*(\gamma) \geq n$ for each n, where $d^*(\gamma)$ is defined relative to K^* just like $d(\gamma)$ was defined relative to K'. This is a contradiction. $\square$

For each real x let M_x, if it exists, be the minimum transitive set model of ZFC containing x. Thus M_x has the form $L_\mu[x]$ for some countable ordinal $\mu = \mu(x)$. If d is a Turing degree we write M_d, $\mu(d)$ for M_x, $\mu(x)$ (x in d).

Theorem 7 *Assume the existence of a Woodin cardinal with an inaccessible above. Then the IMH is consistent. Moreover for all d in a cone of Turing degrees, M_d exists and satisfies the IMH.*

Proof. First we prove the consistency of the IMH by showing that M_d satisfies the IMH for some Turing degree d in a forcing extension of V.

Let κ be Woodin with an inaccessible above in V. Let G be generic over V for the Lévy collapse of κ to ω. Work now in $V[G]$. Σ^1_2 determinacy holds and, as there is still an inaccessible, M_d exists for each Turing degree d. It follows that the theory of $(M_d, \in)$ is constant on a cone of Turing degrees d. Let d be a Turing degree such that the theory of $(M_e, \in)$ is constant for Turing degrees e at least that of d.

We claim that M_d, endowed with its definable classes, witnesses the IMH. Indeed, suppose that φ is a sentence true in some model M of height $\mu(d)$ compatible with M_d. By Jensen coding there is a real y such that d is recursive in y, $\mu(y) = \mu(d)$ and M is a definable inner model of M_y. Let e be the Turing degree of y. Then for some formula ψ, M_e satisfies the sentence

The inner model defined by ψ *(with some choice of parameters) satisfies* φ.

It follows that there is an inner model of M_d which satisfies φ, as desired. This proves the consistency of the IMH.

To say that a countable M, together with its countable collection of definable classes, satisfies IMH is simply a Π^1_1-statement with a real coding M as parameter, since one only needs to quantify over outer models of M of height $M \cap \text{Ord}$. Thus the assertion that there exists a Turing degree d such that M_d (with its definable classes) satisfies IMH is a Σ^1_2-statement and hence absolute. So the existence of a Woodin cardinal with an inaccessible above implies that such an M_d exists in V (and indeed in L).

To prove the stronger statement that in V, M_d satisfies the IMH for a cone of d's, one argues as follows. Say that a set of reals X is *absolutely* Δ^1_2 iff there is a pair of Σ^1_2 formulas $\varphi(x)$, $\psi(x)$ such that X consists of all solutions to $\varphi(x)$ in V and φ is equivalent to the negation of ψ both in V and all of its forcing extensions.

Claim. Assume that there is a Woodin cardinal. Then determinacy holds for absolutely Δ^1_2 sets.

Proof of Claim. As before let G be generic for the Lévy collapse of the Woodin cardinal to ω. Then Σ^1_2 determinacy holds in $V[G]$. By the Moschovakis Third Periodicity theorem, ([7] Theorem 6E.1), if X is Σ^1_2 in $V[G]$ there is a definable winning strategy in $V[G]$ for one of the players in the game G_X. By the homogeneity of the Lévy collapse, it follows that absolutely Δ^1_2 sets are determined in V. This proves the Claim.

As there is an inaccessible in V, M_d exists for each Turing degree d in V. Now it follows from the Claim that in V, for any sentence φ, either for a cone of Turing degrees d,

$$M_d \models \varphi$$

or for a cone of Turing degrees d,

$$M_d \models \neg\varphi,$$

since the relevant games are absolutely Δ^1_2. Therefore in V the theory of $(M_d, \in)$ is constant for a cone of Turing degrees d. We can then apply the argument used earlier in $V[G]$ to conclude that also in V, M_d satisfies IMH for d in a cone of Turing degrees. $\square$

Parameters and the strong inner model hypothesis

How can we introduce parameters into the inner model hypothesis? The following result shows that inconsistencies arise without strong restrictions on the type of parameters allowed.

Proposition 8 *([5]) The inner model hypothesis with arbitrary ordinal parameters or with arbitrary real parameters is inconsistent.*

So instead we consider *absolute parameters*, as in [4]. For any set x, the *hereditary cardinality* of x, denoted hcard (x), is the cardinality of the transitive closure of x. If V^* is an outer model of V, then a parameter p is *absolute between* V *and* V^* iff V and V^* have the same cardinals $\leq$ hcard (p) and some parameter-free formula has p as its unique solution in both V and V^*.

Inner model hypothesis with locally absolute parameters Suppose that p is absolute between V and V^* and φ is a first-order sentence with parameter p which holds in an inner model of V^*. Then φ holds in an inner model of V.

For a singular cardinal κ, a $\square_\kappa$ *sequence* is a sequence of the form $\langle C_\alpha \mid \alpha < \kappa^+, \alpha \text{ limit}\rangle$ such that each C_α has ordertype less than κ and for $\bar\alpha$ in Lim C_α, $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$. *Definable* $\square_\kappa$ is the assertion that there exists a $\square_\kappa$ sequence which is definable over $H(\kappa^+)$ with parameter κ. We will be interested in the special case $\kappa = \beth_\omega$, in which case the parameter κ is superfluous.

Theorem 9 *The inner model hypothesis with locally absolute parameters is inconsistent.*

Proof. We first show that definable $\square_\kappa$ fails, where κ is $\beth_\omega$. Let $\langle C_\alpha \mid \alpha < \kappa^+,\ \alpha \text{ limit}\rangle$ be a $\square_\kappa$ sequence definable over $H(\kappa^+)$ without parameters. For each n let S_n consist of all limit $\alpha < \kappa^+$ such that the ordertype of C_α is greater than $\beth_n$.

Claim. Let P_n be the forcing that adds a CUB subset of S_n using closed bounded subsets of S_n as conditions, ordered by end extension. Then P_n is κ^+ distributive, i.e., does not add κ-sequences.

Proof of Claim. It is enough to show that P_n is $\beth_{m+1}^+$ distributive for each $m < \omega$. Assume that m is greater than n. Suppose that p is a condition and $\langle D_i \mid i < \beth_{m+1}^+\rangle$ are dense. Let $\langle M_i \mid i < \kappa^+\rangle$ be a chain of size κ elementary submodels of some large $H(\theta)$ such that M_0 contains $\kappa \cup \{\langle C_\alpha \mid \alpha < \kappa^+,\ \alpha$ limit$\rangle, p\}$ and for each $i < \kappa^+$, $\langle M_j \mid j \leq i\rangle$ is an element of M_{i+1}. Let κ_i be $M_i \cap \kappa^+$ and C the set of such κ_i's. Then a final segment D of $C \cap \mathrm{Lim}\ C_\gamma$ is contained in S_n, where $\gamma = \kappa_{\beth_{m+1}^+}$. Write D as $\langle \kappa_{\alpha_i} \mid i < \text{ordertype } D\rangle$. We can then choose a descending sequence $\langle p_i \mid i < \beth_{m+1}^+\rangle$ of conditions below p such that p_{i+1} meets D_i and belongs to $M_{\kappa_{\alpha_i}+1}$ for each i. Then the greatest lower bound of this sequence meets each D_i. This proves the Claim.

It follows that for each n the forcing P_n does not alter $H(\kappa^+)$. By the inner model hypothesis with locally absolute parameters S_n has a CUB subset C_n in V for each n. But this is a contradiction, as the intersection of the C_n's is empty.

Now we refine the above argument. As not every real has a #, there exist reals R such that κ^+ equals κ^+ of $L[R]$, where κ is $\beth_\omega$. Let X be the set of such reals and for each R in X let $\langle C_\alpha^R \mid \alpha < \kappa^+,\ \alpha \text{ limit}\rangle$ be the $L[R]$-least $\square_\kappa$ sequence. Now for limit $\alpha < \kappa^+$, define C_α^* to be the intersection of the C_α^R, $R \in X$. Then $\langle C_\alpha^* \mid \alpha < \kappa^+,\ \alpha \text{ limit}\rangle$ is definable in $H(\kappa^+)$ without parameters and has the properties of a $\square_\kappa$ sequence with the sole exception that C_α^* is only guaranteed to be unbounded in α if α has cofinality greater than $2^{\aleph_0}$. Now repeat the above argument using $\langle C_\alpha^* \mid \alpha < \kappa^+,\ \alpha \text{ limit}\rangle$ in place of $\langle C_\alpha \mid \alpha < \kappa^+,\ \alpha \text{ limit}\rangle$, to obtain a contradiction. $\square$

To obtain the strong inner model hypothesis, we require more absoluteness. We say that the parameter p is *(globally) absolute* iff there is a parameter-free formula which has p as its unique solution in all outer models of V which have the same cardinals $\leq$ hcard (p) as V.

Strong inner model hypothesis (SIMH) Suppose that p is absolute, V^* is an outer model of V with the same cardinals $\leq$ hcard (p) as V and φ is a first-order sentence with parameter p which holds in an inner model of V^*. Then φ holds in an inner model of V.

Remark. If above we assume that the sentence φ holds not just in an inner model of V^* but in V^* itself, then in the conclusion we may demand that in an inner model of V witnessing φ, p is definable via the same formula ψ witnessing the absoluteness of p. (This inner model may, however, fail to have the same cardinals $\leq$ hcard (p) as V.) This is because we can replace the sentence φ by: "φ holds and p is defined by ψ".

Theorem 10 *([5]) Assume the SIMH. Then CH is false. In fact, $2^{\aleph_0}$ cannot be absolute and therefore cannot be $\aleph_\alpha$ for any ordinal α which is countable in L.*

Theorem 11 *The SIMH implies the existence of an inner model with a strong cardinal.*

Proof. Assume not, and let K be the core model below a strong cardinal (see [8]). As in the proof of Theorem 2, we let K' denote the iterate of K obtained by applying each order 0 measure exactly once. Then by Lemma 4, if λ is a cardinal then the K'-cofinality of λ is not measurable in K'. And by weak covering relative to K, if λ is a singular cardinal, then λ^+ is computed correctly in K (i.e., $(\lambda^+)^K = \lambda^+$).

Lemma 12 *For any singular cardinal λ, λ^+ is computed correctly in K'.*

Proof of Lemma 12. This is clear if the K-cofinality of λ is not measurable in K, for then λ is a fixed point of the iteration from K to K' and $(\lambda^+)^{K'} = (\lambda^+)^K = \lambda^+$. Otherwise let $\langle K_i \mid i \in \mathrm{Ord}\rangle$ result from the iteration of K to K' and choose i so that the ultrapower map $\sigma_i : K_i \to K_{i+1}$ applies the order 0 measure at $\kappa = \mathrm{cof}^{K_i}(\lambda)$. If $\langle \lambda_j \mid j < \kappa\rangle$ is a continuous and increasing sequence in K_i with supremeum λ, then λ^+ of K_{i+1} is represented in the ultrapower of K_i by $\langle \lambda_j^+ \mid j < \kappa\rangle$. In K_i, the product of the λ_j^+'s contains a subset of size $(\lambda^+)^{K_i}$, consisting of functions well-ordered by dominance on a final segment of κ. It follows that $(\lambda^+)^{K_{i+1}}$ has cardinality λ^+ and therefore K_{i+1} computes λ^+ correctly. As λ^+ is a fixed point of the remaining iteration from K_{i+1} to K', it follows that K' computes λ^+ correctly. This proves Lemma 12.

We say that λ is a *cut point* of K' iff no extender on the K' sequence with critical point less than λ has length at least λ (i.e., λ is *not overlapped* in K'). As we have assumed that there is no strong cardinal in K', there is a closed unbounded class of cut points of K'.

Let $\langle \lambda_n \mid n \in \omega \rangle$ be the first ω-many limit cardinals of V which are cut points of K', and let λ_ω be their supremum. Then each λ_n, and of course λ_ω, has cofinality ω.

Lemma 13 *Each λ_n, and λ_ω as well, is an absolute parameter.*

Proof of Lemma 13. We first show that λ_0 is absolute. Let V^* be an outer model of V with the same cardinals as V up to λ_0. Note that for some real R in V, no $L_\alpha[R]$ satisfies ZFC and therefore $R^\#$ does not exist in V^*. It follows that for any singular cardinal λ of V^*, λ is singular in V and λ^+ is computed correctly in V. In particular, V^* and V have the same cardinals, and the same singular cardinals, up to λ_0^+.

It follows by Lemma 12 that for any singular cardinal λ of V^*, λ^+ is computed correctly in both K' and $(K^*)'$, where $(K^*)'$ denotes the K' of V^*, obtained from K^*, the K of V^*, by applying each order 0 measure exactly once. Note that K is universal in V^*, and therefore is a simple iterate of K^*. It follows that K' is a simple iterate of $(K^*)'$, obtained by lifting the iteration map from K^* to K along the iteration from K^* to $(K^*)'$.

Claim. The iteration from $(K^*)'$ to K' fixes singular cardinals of V^* which are cut points either of K' or of $(K^*)'$.

Proof of Claim. Let λ be a singular cardinal of V^*. If λ is a cut point of $(K^*)'$ then as λ has non-measurable cofinality in $(K^*)'$, λ is fixed by the iteration. If λ is a cut point of K' then as λ has non-measurable cofinality in $(K^*)'$, λ can only move if an extender overlapping λ is applied. As by assumption λ is not overlapped in K' it must be that the least extender overlapping λ was applied. But then λ^+ is not computed correctly in the resulting ultrapower and therefore not computed correctly in K', the result of the iteration, contradicting Lemma 12. This proves the Claim.

It follows from the Claim that λ_0 is the least limit cardinal of V^* which is a cut point of $(K^*)'$. As V^* is an arbitrary outer model of V with the same cardinals as V up to λ_0, we have shown that λ_0 is an absolute parameter. The same argument shows that each λ_n is absolute, and therefore so is λ_ω, the supremum of the first ω limit cardinals which are cut points of K'. This proves Lemma 13.

Now let $\langle C_\alpha \mid \alpha < \lambda_\omega^+, \alpha \text{ limit}\rangle$ be the least $\Box_{\lambda_\omega}$ sequence of K'; this is also a $\Box_{\lambda_\omega}$ sequence in V, as $(\lambda_\omega^+)^{K'} = \lambda_\omega^+$. As in the proof of Theorem 9, there are generic extensions of V preserving $H(\lambda_\omega^+)$ which add CUB subsets to each $S_n = \{\alpha < \lambda_\omega^+ \mid \text{ordertype } C_\alpha > \lambda_n\}$. It follows from the strong inner model hypothesis (and the Remark immediately following its statement) that for each n there is an inner model M_n, with the correct λ_ω^+ and λ_n, in which $S_n^{M_n}$ contains a CUB subset C_n, where $S_n^{M_n}$ is defined using the least $\Box_{\lambda_\omega}$ sequence of $(K')^{M_n}$. The latter may of course differ from the least $\Box_{\lambda_\omega}$ sequence of K'. However as λ_ω^+ is computed correctly in each $(K')^{M_n}$ and λ_ω is a cut point of non-measurable cofinality in each $(K')^{M_n}$, it follows that the $(K' \mid \lambda_\omega^+)^{M_n}$'s compare to a common K'' of height λ_ω^+ with all ordinals in some CUB subset C of λ_ω^+ as closure points. But if α is such a closure point in the intersection of the C_n's and α_n is the image of α under the comparison embedding of $(K' \mid \lambda_\omega^+)^{M_n}$ into K'', then C_{α_n} as defined in K'' contains elements cofinal in α and therefore C_α as defined in K'', an initial segment of C_{α_n}, has ordertype at least that of C_α as defined in $(K' \mid \lambda_\omega^+)^{M_n}$. It follows that C_α as defined in K'' has ordertype greater than λ_n for each n, which is a contradiction. $\Box$

Remarks. (a) It is likely that Theorem 11 can be improved to obtain an inner model with a Woodin cardinal. But it is not possible to obtain an iterable inner model with a Woodin cardinal and an inaccessible above it (unless the SIMH is inconsistent): Otherwise every real would be generic for Woodin's extender algebra defined in an iterate of such an inner model, implying that for every real R there is an inaccessible in $L[R]$; this contradicts Theorem 1. (b) David Asperó and the first author observed that the consistency of the SIMH for the parameter ω_1 follows as in the proof of Theorem 7 from that of a Woodin cardinal with an inaccessible above. In particular this yields the consistency of the natural extension of Lévy absoluteness asserting Σ_1 absoluteness with parameter ω_1 for arbitrary ω_1-preserving extensions. (c) For any finite set of absolute parameters, the version of the SIMH where V^* is required to be a *set-generic* extension of V is consistent for those parameters.

Question. Is the strong inner model hypothesis consistent relative to large cardinals?

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