Games and Abstract Inductive definitions

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Introduction

1) Ordinals and operators.
   (i) Ordinals (ii) Operators, monotone and non-monotone.
2) Circular Definitions (Gupta-Belnap).
   Example over $\mathbb{N}$, quasi-inductive operators.
3) Games
4) Closure points of operators and connections with strategies
   Solovay, Svenonius, Martin
5) Consequences in weak systems of analysis

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• This enables schemes of definition by *transfinite recursion*.

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For any monotone operator there is a least *countable* ordinal $\mu = \mu(\Gamma)$ with $\Gamma_\mu(\varnothing) = \Gamma_{\mu+1}(\varnothing)$, the *least fixed point*. We write Γ_∞ for $\Gamma_\mu(\varnothing)$.

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A set $Y \subseteq \mathbb{N}$ is accordingly Π^0_n/Σ^0_n if it can be defined over $\mathbb{N}$ by a Π^0_n/Σ^0_n formula:

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- Thus an operator Γ may be classified as Σ^0_n if "$n \in \Gamma(X)$" can be written out with a Σ^0_n definition.

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• A Σ^1_1-*operator* is defined similarly if "$n \in \Gamma(X)$" can be written using a negated universal (so, equivalently an *existential* second order formula.

Let[1] $\mathcal{L}$ be a first order language, suitable for a structure $\mathcal{M}$ and let $\mathcal{L}^+$ be its extension by a possibly infinite set of new predicate symbols $\dot{G}_n(x_1, \ldots x_{k(n)})$. For each $\dot{G}$ there is a definition from the set of definitions $\mathcal{L}$ of the form

$$(1)_n \quad \dot{G}_n(x_1, \ldots x_{k(n)}) =_{\mathrm{df}} A_{G_n}(x_1, \ldots x_{k(n)}).$$

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If we specialise $\mathcal{M}$ to $\mathbb{N}$, and have a single definition arising from a fixed first order formula $\varphi(v_0, \dot{X})$, we set:

$$X_{\alpha+1} = \Gamma_\varphi(X_\alpha) =_{\text{df}} \{n | \langle \mathbb{N}, +, \times, 0,', \ldots, X_\alpha\rangle \models \varphi(n, \dot{X})\}$$

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The problem here is to decide what to do at limit stages, as there is no question here that such an operator is monotone or even *progressive* (*i.e.*, $X \subseteq \Gamma_\varphi(X)$).

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A Liminf rule and quasi-inductive operators

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$X_\lambda = \liminf \langle X_\alpha | \alpha < \lambda \rangle =_{\mathrm{df}} \bigcup_{\alpha<\lambda} \bigcap_{\alpha<\beta<\lambda} X_\beta$.

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Remark There will be a *countable* ordinal $\zeta = \zeta(\Gamma)$ so that $\Gamma_\zeta(\varnothing) = \Gamma_\infty$.

Examples from Computer Science, philosophical logic

Arithmetical Quasi-inductive Sets

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(ii) (Burgess) The a.q.i. Y also can be characterized in terms of the Gödel hierarchy of sets.

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Infinite 2 person games

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• Thus a strategy for player I is a rule or function that takes the even length sequence $k_0, \dots, k_{2n-1}$ played so far, and tells him/her what to play for k_{2n}. Analogously for II.

Strategies (More formally)

Definition

(1) A *strategy for* I is a function

$$\sigma_I : \{p \in \mathbb{N}^{<\mathbb{N}} \mid \ell h(p) \text{ is even}\} \longrightarrow \mathbb{N}$$

(2) A *strategy for II* is a function

$$\sigma_{II} : \{p \in \mathbb{N}^{<\mathbb{N}} \mid \ell h(p) \text{ is odd}\} \longrightarrow \mathbb{N}.$$

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Definition

1. A strategy σ for I is a *winning strategy* for $G(A)$ if I wins each play by using σ. A winning strategy for II is similarly defined.
2. $G(A)$ is *determined* if either I or II has a winning strategy for $G(A)$
3. 3. If Σ is a class of sets in $\mathcal{P}(\mathbb{N})$, then we say $\text{Det}(\Sigma) \iff \forall A \in \Sigma \quad G(A)$ is determined. (We say that $G(A)$ is a "Σ-game.")

Classification of sets via games and their strategies

Theorem

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We can view the last two theorems (which are theorems in analysis) as provable in *weak subsystems of analysis* ($\mathrm{RCA}_0 + \mathrm{Determinacy}(\boldsymbol{\Sigma^0_1})$ and $\mathrm{RCA}_0 + \mathrm{Determinacy}(\boldsymbol{\Sigma^0_2})$ respectively).

Theorem

(Svenonius, Moschovakis) For any Π^1_1 *set of integers* Y, *there is a* Σ^0_1 *set* $U \subseteq \mathbb{N} \times \mathbb{N}^{\mathbb{N}}$ *so that* Y *can be represented as:*
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Theorem

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Question: Do a.q.i. sets Y have game representations as above?

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On $\mathcal{P}(\mathbb{N})$,
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with the first two corresponding to Determinacy(Σ^0_1) and Determinacy(Σ^0_2).

Question: Do Σ^0_3 games have a.q.i. strategies?

Question: Do a.q.i. sets Y have game representations as above?

Det(Σ^0_1)	Det(Σ^0_2)	??	Det(Σ^0_3)
mon. Π^1_1-ind.	mon. Σ^1_1-ind.	A.Q.I.	???

Remark

By a result of H. Friedman (modified by Martin) $\mathrm{Det}(\Sigma^0_4)$ *can* not *be proven in analysis: it requires* ZFC^- *: the full axioms of Set Theory + AC - Power Set.*

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It is thus of interest to see if we can use a.q.i. definitions to find strategies of Σ^0_3-games.

Main Theorem

(i) Games with payoff sets which are Boolean combinations of Σ^0_2 *sets, have a.q.i. winning strategies, but*
(ii) not so for Σ^0_3 *games.*

Moreover:

Theorem

For any arith. quasi-inductive set Y*, there is a* Σ^0_3 *set* $U \subseteq \mathbb{N} \times \mathbb{N}^{\mathbb{N}}$ *so that* Y *can be represented as:*
$Y = \{n | I \text{ has a w.s. in } G_{(n)}\}$ *where* $(n) = \{x \in \mathbb{N}^{\mathbb{N}} | (n, x) \in U\}$.

If we consider Comprehension Axiom schemes formulated in second order number theory we may relate these to the above notions.

Theorem

(i) Π^1_3*-Comprehension Axiom proves* $\mathrm{Det}(\Sigma^0_3)$.
(ii) Δ^1_3*-Comprehension Axiom does not prove* $\mathrm{Det}(\Sigma^0_3)$

Theorem

(Tanaka-MedSalem) Δ^1_2*-monotone induction* $+$ Π^1_3*-Transfinite Induction proves* $\mathrm{Det}(\Delta^0_3)$.

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Conclusion

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- However their analysis is not without point: one can show that they are intimately involved with levels of the Gödel hierarchy of constructible sets, which are the first building block of a proof-theoretic ordinal analysis of Π^1_3-CA namely links in the level of the L-hierarchy involving chains of Σ_2-elementary end extensions, just as analysing Π^1_2-CA invloves such chains for Σ_1(work of Rathjen, Arai).

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• Hence: if anyone is ever to perform such a proof theoretic analysis, they will first have to analyse the proof theoretic ordinal of a.q.i. definitions.