A note on mutual indiscernibles.

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4.vii.2004

In [3] (Theorem 3.4) Paul Larson shows that it is consistent relative to the existence of a Woodin limit of Woodin cardinals, that for any function $f: \mathbb{R} \to \mathbb{R}$ definable in $L(\mathbb{R})$ there exist x, y with

$$(*) \qquad \min(I_x \cap I_y) = \min(I_{f(x)} \cap I_{f(y)}).$$

where:

Definition 1. *For $z \subset \mathrm{On}$, z a set, $I_z = \langle i^z_\tau \mid \tau \in \mathrm{On} \rangle$ is the class of Silver indiscernibles for $L[z]$.*

He points out:

Theorem 2. *If x, y instantiate (*) for the function $f(z) = z^\#$, then the set of reals in $L[x] \cap L[y]$ is closed under sharps in $L[x]$ and $L[y]$.*

It is the purpose of this note to indicate the following strengthening:

Theorem 3. *If x, y instantiate (*) for the function $f(z) = z^\#$, then the set of reals in $L[x] \cap L[y]$ is closed under daggers in $L[x]$ and in $L[y]$.*

There is a crucial use of Paris' *"Patterns of Indiscernibles"* result [2] in the proof of Theorem 2. Since Jensen extended Paris' result to the Dodd-Jensen Core Model (below a single measurable cardinal) we can use (a mild extension of) Jensen's argument to rerun Larson's proof of Theorem 2 to get the above. In the following we deal with $K = K_{\mathrm{DJ}}$ - the Dodd-Jensen core model mentioned, for $a \in \mathbb{R}$, we set $K^a = (K_{\mathrm{DJ}})^{L[a]}$.

Theorem 4. *(Jensen, [1] 21.22) Let $a \in \mathbb{R}$ be such that $K^a \neq K_{\mathrm{DJ}}$. Let N be the $<^*$ - least mouse with $N \notin L[a]$. Let $C = \langle \kappa_\iota | \iota \in \mathrm{On} \rangle$ with $\kappa_\iota = \kappa^N_\iota$ be the cub class of iteration points of the mouse N by its topmost measure. Then:*

$\exists \iota_0, \alpha, \beta$ so that for all $\theta \in \mathrm{On}$ $\qquad i^a_{\iota_0 + \theta} = \kappa_{\alpha + \beta\theta}$.

This "periodicity" phenomenon generalises the earlier theorem of Paris.

Proof: We show that $O^{\mathrm{dagger}} \in L[x] \cap L[y]$. The argument relativises straightforwardly for other $z \in L[x] \cap L[y]$. We set $\mu = \min(I_{x^\#} \cap I_{y^\#})$.

(1) $K^x = K^y$.

Proof: Suppose not and let, without loss of generality, $M \in L[x]$ be the $<^*$ -mouse $M \notin K^y$. Then $M \in L[y^\#]$ and is definable without parameters there. Hence $M \in L_\mu[y^\#] \cap L_\mu[x^\#]$. μ is inaccessible in both these models and if we apply Theorem 4 with $y = a$, and $N = M$, in $L[y^\#]$ we see that there are ι_0, α, β definable from M, y giving the required periodicity. Hence ι_0, α, β are all less than μ.

However then $D =_{\mathrm{df}} \{\kappa_{\alpha+\beta\theta} \mid \theta \in \mathrm{On}\} \cap \mu$ and $I_x \cap \mu$ are both in $L[x^\#]$ and are cub in μ. Hence if γ is in both, we have $\min(I_x \cap I_y) \leq \gamma < \mu$! Contradiction! QED(1).

This argument with some minor variations is simply repeated until we get what we want.

(2) $K^x = K_{\mathrm{DJ}} = K^y$.

Proof: Suppose $K^x \neq K_{\mathrm{DJ}}$. Then in $L[x^\#]$, or in $L[y^\#]$, let M be the $<^*$- least mouse not in K^x. Similarly to the above, if $\langle \kappa_\iota | \iota \in \mathrm{On}\rangle$ are the iteration points of M we have:

$$\exists \iota_0, \alpha, \beta \; \forall \, \theta \in \mathrm{On} \qquad i^x_{\iota_0+\theta} = \kappa_{\alpha+\beta\theta} \; ;$$

$$\exists \sigma_0, \gamma, \delta \; \forall \, \theta \in \mathrm{On} \qquad i^y_{\sigma_0+\theta} = \kappa_{\gamma+\delta\theta} \; .$$

The classes $D_0 =_{\mathrm{df}} \{\kappa_{\alpha+\beta\theta} \,|\, \theta \in \mathrm{On}\} \cap \mu$, and $D_1 =_{\mathrm{df}} \{\kappa_{\gamma+\delta\theta} \,|\, \theta \in \mathrm{On}\} \cap \mu$ are in $L[y^\#] \cap L[x^\#]$ and are cub below μ, as are I_x, I_y in the respective models $L[x^\#], L[y^\#]$. Hence again min $(I_x \cap I_y) < \mu$! QED(2).

(3) Both $L[x], L[y] \models$ "There exists an inner model with a measurable cardinal".

Proof: Suppose this failed, for example, for $L[y]$. By (2) $L[y^\#]$ sees there is an embedding $K_{\mathrm{DJ}} \to K_{\mathrm{DJ}}$ (coming from the shift of indiscernibles and $L[y] \to L[y]$). Hence $L[y^\#]$ sees there is an inner model with a measurable. The same is true then for $L[x^\#]$. Let L^V and L^U be the minimal ρ-models on τ_0, ρ_0 respectively in $L[y^\#]$ and $L[x^\#]$. By Kunen one of L^V and L^U is an iterate of the other. Without loss of generality we shall assume $\exists \upsilon(\rho_\upsilon = \tau_0)$. Hence the iteration points of L^V are $\langle \rho_{\upsilon+\iota} | \iota \in \mathrm{On}\rangle$, where $C = \langle \rho_\iota | \iota \in \mathrm{On}\rangle$ are those of L^U. An examination of Jensen's proof of Theorem 4 shows that

$$\exists \iota_0, \alpha, \beta \; \forall \, \theta \in \mathrm{On} \qquad i^x_{\iota_0+\theta} = \rho_{\alpha+\beta\theta} \; ;$$

$$\exists \sigma_0, \gamma, \delta \; \forall \, \theta \in \mathrm{On} \qquad i^y_{\sigma_0+\theta} = \rho_{\upsilon+\gamma+\delta\theta} \; .$$

The point here is that contained in [1] Lemma 21.18: the class $C = \bigcap_{X \in U} \widetilde{X}$, where $\widetilde{X} = \bigcup_{\iota \in \mathrm{On}} \pi_{0\iota}(X)$ where $\pi_{\iota\theta} : L^{U_\iota} \to L^{U_\theta}$ are the iteration maps. This proof works for us here just as well as for Jensen's argument where C was the class of iteration points of a mouse (and will work again for us when C is replaced by Λ below in (4)). (It is the failure of this Lemma for classes of iteration points generated by mice with extenders, that contributes to the failure of generalising such periodicity phenomena beyond the realm of mice with sequences of measures.)

Now the sets $E_0 =_{\mathrm{df}} \{\rho_{\alpha+\beta\theta} \,|\, \theta \in \mathrm{On}\} \cap \mu$, and $E_1 =_{\mathrm{df}} \{\rho_{\upsilon+\gamma+\delta\theta} \,|\, \theta \in \mathrm{On}\} \cap \mu$ are in $L[y^\#] \cap L[x^\#]$ and are cub below μ, and we finish as before. QED(3).

(4) Both $L[x], L[y] \models$ "O^{dagger} exists."

Proof: Again suppose without loss of generality $L[y] \models$ "O^{dagger} does not exist", for a contradiction. Let:

$$O^{\mathrm{dagger}} = \langle J^{E,F}_\alpha, \in, E, F\rangle \models \text{"}E \text{ is a normal measure on } \kappa, \, F \text{ is a normal measure on } \lambda > \kappa.\text{"}$$

Let M be the iterate of O^{dagger} using the bottom measure E only, so that if $M = \langle J^{V,F'}_\alpha, \in, V, F'\rangle$, then V is the measure in $L[y]$'s minimal ρ-model. Then the iteration $O^{\mathrm{dagger}} \longrightarrow M$ is definable in $L[x^\#]$, and thus $M \in L[x^\#]$. Let $\Lambda = \langle \lambda_\iota | \iota \in \mathrm{On}\rangle$ be the class of iteration points of M by the top filter F'. Then Λ is a definable class of both $L[x^\#]$ and $L[y^\#]$ and is cub below μ. Again Jensen's argument can be reworked to show that:

$\exists \iota_0, \gamma, \delta \; \forall \theta \in \mathrm{On} \;\; i^x_{\iota_0+\theta} = \lambda_{\gamma+\delta\theta}$ with all of $\iota_0, \gamma, \delta, \lambda_\gamma < \mu$.

However then $E =_{\mathrm{df}} \{\lambda_{\gamma+\delta\theta} \,|\, \theta \in \mathrm{On}\} \cap \mu$ is a definable set in $L[x^\#]$, and is cub below μ.

QED(4) and Theorem 2.

One suspects that the existence of such x, y implies much more about the reals they have in common. There are extensions of the "Patterns" to models with sequences of measures (see Mitchell [4], but it seems more worthwhile at the very least to work towards:

Conjecture: Let x, y be as above. Then the reals common to both $L[x]$ and $L[y]$ are closed under the pistol operation: $z \to z^{\text{pistol}}$ (the latter being the sharp for an inner model containing z and a strong cardinal).

As remarked parenthetically in (3) above, it seems that some other argument than periodicity is going to have to be deployed.

Question: Suppose $u, v \in \mathbb{R}$ are such that $\min(I_u \cap I_v) = \min(I_{u^{\#\#}} \cap I_{v^{\#\#}})$. Can one say more more about the reals common to both $L[u], L[v]$?

Bibliography

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