Some open problems in mutual stationarity involving inner model theory: a commentary.

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November 1st 2004

We are interested here in particular problems in the theory of mutual stationarity as defined by Foreman and Magidor in [2]. There are many open questions, but we restrict ourselves here to one or two, that have used, or are likely to use, inner model theory in a deep way to establish the strengths of the various principles under question. The "deep" here should be read as meaning only that more is needed in the proofs concerned than simply quoting a Covering Lemma or the like. Indeed the purpose of the note is to sketch how one obtains just $O^{\#}$ from some of the stated principles. Once one has done this work, the framework is there for anyone with a sufficient knowledge of core model and inner model theory (see for example [5]) to work through the extra complexities from iteration theory, and the construction of global square sequences in core models, to obtain stronger results. For Question 1 a lower bound has been ascertained with P. Koepke in joint work in [3],[4].

Let $\langle \kappa_\alpha | \alpha < \delta \rangle$ for $\delta < \kappa_0$ be an ascending sequence of uncountable regular cardinals. Let $\kappa =_{df} \sup \kappa_\alpha$. For λ a regular cardinal let $\mathrm{cof}_\lambda =_{\mathrm{df}} \{\alpha \in \mathrm{On} \,|\, \mathrm{cf}(\alpha) = \lambda\}$.

Definition 1 *Let $\mathcal{S} = \langle S_\alpha | \alpha < \delta \rangle$ be a sequence of stationary sets, with $S_\alpha \subseteq \kappa_\alpha \cap \mathrm{cof}_\lambda$ ($\lambda < \kappa_0$). Then $\mathcal{S}$ is* mutually stationary *if the following set S is stationary:*

$$S =_{df} \{X \in [\kappa]^\lambda \mid \sup(X \cap \kappa_\alpha) \in S_\alpha\}.$$

Let $\mathrm{MS}(\langle \kappa_\alpha \rangle_{\alpha<\delta}, \lambda)$ abbreviate: *For all sequences $\mathcal{S} = \langle S_\alpha | \alpha < \delta \rangle$, with $S_\alpha \subseteq \kappa_\alpha \cap \mathrm{cof}_\lambda$, and S_α stationary$\rangle$, $\mathcal{S}$ is mutually stationary.*
We are thus asking that all sequences of priorily independently chosen stationary sets of the appropriate kind, form mutually stationary sequences.

Question 1 What is the consistency strength of $\mathrm{MS}(\langle \omega_n \rangle_{n<\omega}, \omega_1)(+CH)$?

Let $\langle\lambda_n\rangle_{n<\omega}$ be an ascending sequence of regular cardinals, with infinitely many not Mahlo. Let $\gamma < \lambda_0$ be regular.

Question 2 What is the consistency strength of $\mathrm{MS}(\langle\lambda_a\rangle_{n<\omega}, \gamma)$

Background: $\mathrm{MS}(\langle\kappa_n\rangle_{n<\omega}, \gamma)$ for any $\gamma < \kappa_0$(and more) can hold in a Prikry generic extension of a model with a measurable cardinal (Cummings-Foreman-Magidor, [1] Theorem 5.4). (In this model the κ_n come from a tail of the generic Prikry sequence.) This is an equiconsistency (Koepke-Welch, [3]). The questions above are about keeping the cardinals "small" in either sense. (In the Prikry generic model mentioned above the κ_n remain Ramsey, hence weakly compact, and hence Mahlo.) No upper bounds on consistency strength are known for such sequences of "small" κ_n. Lower bounds are in the order of inner models with measures of Mitchell order ν for every $\nu < \sup\{\omega_n\}$ (respectively $\nu < \sup\{\kappa_n\}$).

We roughly sketch a proof of the following

Theorem 1 [3] $\mathrm{ZFC} \vdash \mathrm{MS}(\langle\omega_n\rangle_{n<\omega}, \omega_1) + CH \longrightarrow O^{\#}$ *exists.*

This indicates that $\mathrm{MS}(\langle\omega_n\rangle_{n<\omega}, \omega_1)(+CH)$ must be a large cardinal property. Before sketching this we remark that Foreman and Magidor showed that in L $\mathrm{MS}(\langle\omega_n\rangle_{n<\omega}, \omega_1)$ fails:

Theorem 2 [2] *(V=L) Let $T^h_n \subseteq \omega_n$ be defined by*
$T^h_n = \{\alpha < \omega_n \cap \mathrm{cof}_{\omega_1} \mid \mathrm{defcol}(\alpha) = h\}$
where defcol *of an ordinal α is the least h so that α is cofinalised by a $\Sigma_h(L_{\beta(\alpha)})$ function, where in turn $\beta(\alpha)$ is the least β so that $L_{\beta+1} \models$"α is singular". Let $f : \omega \longrightarrow \omega$. Then: (i) Each T^h_n is stationary. (ii) $\mathcal{S} = \langle S^{f(n)}_n\rangle$ is mutually stationary iff f is eventually constant. Hence: (iii) $\neg\,\mathrm{MS}(\langle\omega_n\rangle_{n<\omega}, \omega_1)$.*

Their argument can be seen to work in any of the current extender models of the form $L[E]$ model for E a coherent sequence of extenders - such as those described in [5]:

Theorem 3 *(V=L[E]]) If the definitions of $\beta(\alpha)$ and* defcol *are amended to those appropriate for the $L[E]$ hierarchy, then (i)-(iii) still hold.*

Proof: The essential ingredient of the above argument for Theorem 3 is the Condensation Lemma for L. In general Condensation will fail for $L[E]$ hierarchies. But note we are only looking at the very bottom part of this hierarchy inside of $L[E]$. Below $\aleph_\omega$ all cardinals are small (!) and we do in fact have the requisite condensation, for the simple reason that any failure of condensation, when some hull is collapsed, requires that transitivised collapse to contain a local inaccessible cardinal. But none such can appear in any hull of $H_{\aleph_\omega}$

Q.E.D.

Sketch Proof of Theorem 1: We wish to step outside of the inner model L. We suppose that $\neg O^{\#}$. We may thus use both the Jensen Covering Lemma for L and the fact that L has a Global Square sequence $\square$, which is then, by Covering, a Global Square sequence in V:

Definition 2 *Let* $\mathrm{Sing} = \{\beta \in \mathrm{Ord} \mid \lim(\beta) \wedge \mathrm{cof}(\beta) < \beta\}$ *be the class of singular limit ordinals. Global square* ($\square$) *is the assertion: there is a system* $(C_\beta)_{\beta \in \mathrm{Sing}}$ *satisfying:*
(a) C_β *is a closed cofinal subset of* β*;*
(b) $\mathrm{otp}(C_\beta) < \beta$*;*
(c) if $\overline{\beta}$ *is a limit point of* C_β *then* $\overline{\beta} \in \mathrm{Sing}$ *and* $C_{\overline{\beta}} = C_\beta \cap \overline{\beta}$*.*

Under our assumption of $\neg O^{\#}$ the Covering Lemma implies that, with $\omega_2 = \omega_2^V$, $\mathrm{Sing} \setminus \omega_2 = (\mathrm{Sing} \setminus \omega_2)^L$. Since the other clauses in the definition of $\square$ are absolute, we have that the $\square$ sequence defined in L is truly an $\square$ sequence for V.

Lemma 1 *Let* κ *be a regular cardinal* $\geq \aleph_2$ *and* λ *a regular cardinal* $< \kappa$*. Then for every ordinal* θ *such that* $\theta^+ < \kappa$ *the set*

$$\{\beta \in \mathrm{Cof}_\lambda \cap \kappa \mid \mathrm{otp}(C_\beta) \geq \theta\}$$

is stationary in κ*.*

Proof: Let $C \subseteq \kappa$ be closed unbounded in κ. Let $\mu = \max(\lambda, \theta^+)$ which is an uncountable regular cardinal $< \kappa$. Take a singular limit point γ of C of cofinality μ. Then $C \cap C_\gamma$ is closed unbounded in γ of ordertype $\geq \mu$. Take β to be a singular limit point of $C \cap C_\gamma$ such that $\mathrm{cof}(\beta) = \lambda$ and $\mathrm{otp}(C \cap C_\gamma \cap \beta) \geq \theta$. By the coherency property Def. 2(c), $C_\beta = C_\gamma \cap \beta$. Thus $\beta \in C \cap \{\beta \in \mathrm{Cof}_\lambda \cap \kappa \mid \mathrm{otp}(C_\beta) \geq \theta\} \neq \emptyset$. Q.E.D.

Note that $(S_n)_{n<\omega}$ with

$$S_n = \{\beta \in \mathrm{Cof}_{\omega_1} \cap \aleph_{n+3} \mid \mathrm{otp}(C_\beta) \geq \aleph_{n+1}\}$$

is a sequence of stationary sets to which we could apply the MS-principle. However if we let $\mathfrak{A}$ be a first order structure of countable type with $\omega_\omega + 1 \cup H_{\aleph_\omega} \subseteq \mathfrak{A}$, and if there is an elementary substructure $\mathfrak{B} \prec \mathfrak{A}$ such that $\forall n < \omega \; \chi_n =_{\mathrm{df}} \sup(|\mathfrak{B}| \cap \omega_{n+3}) \in S_n$, this would require that the element C_{χ_n} of L's global $\square$ sequence be defined in a radically different manner from that of C_{χ_m} for $m \neq n$. Indeed as m goes to infinity, the order types of the C_{χ_m} must also go to ω_ω.

We may also assume (by adding ordinals $\zeta \leq \omega_2$ into the domain $|\mathfrak{B}|$) that $\mathrm{card}(|\mathfrak{B}|\} = \aleph_2$ (we may assume that we can do this without increasing the values χ_n, if $\mathfrak{A}$ is sufficiently rich). Now let $\pi : \mathfrak{H} \longleftrightarrow \mathfrak{B}$ be the transitivisation of $\mathfrak{B}$, and let $L^{\mathfrak{H}} = L_\gamma$, by the Condensation Lemma for L. Let $\pi(\beta_n) = \omega_n$ for $n \leq \omega$. Each β_n is regular in L_γ for $n < \omega$, but is in cof_{ω_1}. Such β_n cannot be regular in L (as otherwise $\mathrm{cf}(\beta_n) \geq \omega_2$ by the Covering Lemma again). Hence there is a least $\delta \geq \gamma$ where, for some $m, h < \omega \; n > m \longrightarrow \beta_n$is *singularized* by some function $\Sigma_h(L_\delta)$ - where we assume h chosen least, so that there is a function at this level of definability mapping some ordinal $\overline{\beta} < \beta_n$ cofinally into β_n, in terms of the notion defined above $h = \text{defcol}(\beta_n)$.

Thus precisely at $\delta+1$ L sees a tail of the β_n are made singular, by functions of the same order of complexity. L_δ is thus a form of a "singularising" structure for a tail of the β_n. However it is at the same time the requisite structure for defining the C_{β_n}- sequences that go into L's $\square$ sequence. If one goes into the proof of $\square$ at this point one sees that there is $m < \omega$, and a single fixed ordinal $\alpha_0 < \beta_m$ so that for $n > m$ $\text{ot}(C_{\beta_n}) \leq \alpha_0$. Now arguments of the type used in the Jensen Covering Lemma show that, again for some $m \leq m_1 < \omega$, that $n > m_1 \longrightarrow \text{ot}(C_{\chi_n}) < \pi(\alpha_0)$ by demonstrating that the lift-up of the singularising structure for β_n is that of χ_n. Note that $\pi(\alpha_0)$ is a *fixed* ordinal between these various lifted-up structures. This contradicts our assumption on increasing order types for the S_n sets. Q.E.D.

This argument can be repeated using the Weak Covering Lemmas due to Mitchell over models with measurables of varying Mitchell orders. (Obtaining an inner model with a measure is also proven in [3] - for larger models see [4].)

If $A \subseteq \omega$, then let us define: $S_A =_{df}$

$$\left\{ X \in [\aleph_\omega]^{\omega_1} \mid \ \forall m > 1 \ \ \text{cf}(\sup(X \cap \omega_m)) = \begin{array}{ll} \omega & \text{if } \ m \in A \\ \omega_1 & \text{if } \ m \notin A \end{array} \right\}.$$

Question 3 What is the consistency strength of *"For some $A \subseteq \omega$ which infinitely often both contains and omits successor pairs $n, n+1$ of integers, S_A is stationary."*?

Background: One could ask many variants of this question. If the set A simply alternates, for example if A = Evens, then it has been shown (by Magidor) equiconsistent that this S_A is stationary, with the existence of infinitely many measurable cardinals. If one raises the cofinalities to be ω_1 and ω_2, and take A = Evens, one gets, perhaps unsurprisingly, more.

Theorem 4 ZFC $+2^{\aleph_0} < \aleph_\omega \vdash$ *"If $A \subseteq \omega$ is infinite and coinfinite, and $S_A =_{\text{df}}$*

$$\left\{ X \in [\aleph_\omega]^{\omega_1} \mid \ \forall m > 1 \ \ \text{cf}(\sup(X \cap \omega_m)) = \begin{array}{ll} \omega_1 & \textit{if } \ m \in A \\ \omega_2 & \textit{if } \ m \notin A \end{array} \right\}$$

is stationary, then there is an inner model with infinitely many measurables of Mitchell order ω_1.

However it is much harder to arrange other patterns of alternation of cofinality. As upper bound for Question 3, we have [1] (Theorem 6.7) the consistency of ZFC together with the existence of infinitely many supercompact cardinals (in fact this cited theorem produces a model from this hypothesis where all such qualifying S_A are stationary, for arbitrary $A \subseteq \omega$.)

As a lower bound, in Question 3, current methods lead us to believe the following conjecture (we have an as yet unchecked proof based on the sketch below):

Conjecture: *If for a single A as in Question 3, S_A is stationary, then there is an inner model with a strong cardinal.*

Why should this be a large cardinal property? Fix such an A with S_A stationary. Let $\mathfrak{B} \prec \mathfrak{A}$ as before, with $\omega_\omega \subseteq |\mathfrak{A}|$ and $\omega_2 \subseteq B$. Let $\mathfrak{H}$ be the transitivisation of $\mathfrak{B}$ as above. Again let us suppose $\neg O^{\#}$. Let L_γ, be as above, and let again δ be least so that definably over L_δ a tail of the β_n are definably made singular. Reasonably elementary arguments show that all the successor cardinals of L_γ have, again on a tail, the same cofinality (that inherited from δ). So, consider the set of cardinals β_n^{+L}. By our assumption on A, S_A we must have that infinitely many of the β_n^{+L} must satisfy $\beta_n^{+L} < \beta_n^{+\mathfrak{K}}$ (otherwise we'd have alternating cofinalities depending on the cofinalities of the $\beta_n^{+\mathfrak{K}}$); but for such n, as $\mathfrak{H} \models$"*The Covering Lemma holds over L*", we have that $\mathfrak{H} \models$"$\mathrm{cf}(\beta_n^{+L}) = \beta_n$". Take a particular case: suppose $\mathrm{cf}(\delta) = \omega_1$; and both $m, m+1 \in A$. Then $\beta_m^{+L} < \beta_m^{+\mathfrak{K}} = \beta_{m+1}$ as the former has cofinality ω_1. But then it has cofinality that of β_n by Covering applied inside $\mathfrak{H}$! Contradiction! One argues similarly in the other case. Why is the conjecture phrased in terms of a strong cardinal rather than something stronger, such as Woodins? Because to perform the argument in the case of core models, and preserve cofinalities in the objects being iterated whilst using the comparison lemma, we need to work in the world of linear iterations (*cf.* [5] Ch.8); further we need to have that any universal weasel is an iterate of the true K (Jensen, *cf* [5] Theorem 7.4.9). This may also appear in [4].

References

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