Global Square and Mutual Stationarity at the $\aleph_n$

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Abstract

We give a proof of a theorem of Jensen and Zeman on the existence of a global $\square$ sequence in the Core Model below a measurable cardinal κ of Mitchell order ($o_M(\kappa)$) equal to κ^{++}, and use it to prove the following theorem on mutual stationarity at the $\aleph_n$.

Let ω_1 denote the first uncountable cardinal of V and set $\mathrm{Cof}(\omega_1)$ to be the class of ordinals of cofinality ω_1.

Theorem: *If every sequence $(S_n)_{n<\omega}$ of stationary sets $S_n \subseteq \mathrm{Cof}(\omega_1) \cap \aleph_{n+2}$, is mutually stationary, then there is an inner model with infinitely many inaccessibles $(\kappa_n)_{n<\omega}$ so that for every m the class of measurables λ with $o_M(\lambda) \geq \kappa_m$ is, in V, stationary in κ_n for all $n > m$. In particular, there is such a model in which for all sufficiently large $m < \omega$, the class of measurables λ with $o_M(\lambda) \geq \omega_m$ is, in V, stationary below $\aleph_{m+2}$.*

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1. Introduction

This paper extends previous investigations into the nature of *mutual stationarity*, a concept introduced by M. Foreman and M. Magidor [7] in order to transfer some combinatorial aspects of stationary subsets of regular cardinals to singular cardinals. They made particular use of this in investigating the non-saturation of the non-stationary ideals on $\mathcal{P}_\kappa(\lambda)$.

Our purpose here is to establish that the mutual stationarity property at $\aleph_\omega$ (or more precisely at the sequence of the first ω-many uncountable cardinals, $\langle \aleph_n \mid 0 < n < \omega \rangle$), is a *large cardinal* property, that is, it entails the consistency of *strong axioms of infinity* which concern measurable cardinals. The definition of mutual stationarity is more general than this however. We denote the domain of a first order structure $\mathfrak{B}$ by $|\mathfrak{B}|$:

Definition 1.1. *Let $(\kappa_n)_{n<\omega}$ be a strictly increasing sequence of regular cardinals $\geqslant \aleph_2$ with $\kappa_\omega = sup_{n<\omega}\kappa_n$. A sequence $(S_n)_{n<\omega}$ with each $S_n \subseteq \kappa_n$ is called* mutually stationary in

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$(\kappa_n)_{n<\omega}$ *if every first-order structure* $\mathfrak{A}$ *of countable type with* $\kappa_\omega \subseteq \mathfrak{A}$ *has an elementary substructure* $\mathfrak{B} \prec \mathfrak{A}$ *such that*

$$\forall n < \omega \sup |\mathfrak{B}| \cap \kappa_n \in S_n.$$

M. Foreman and M. Magidor, together with J. Cummings further investigated the status of such sequences in [3]. Note that if $(S_n)_{n<\omega}$ is mutually stationary in $(\kappa_n)_{n<\omega}$ then each S_n is stationary in κ_n. In the following we shall denote the class $\{\xi \in \mathrm{Ord} \mid \mathrm{cf}(\xi) = \lambda\}$ by $\mathrm{Cof}(\lambda)$.

Definition 1.2. *Let* $(\kappa_n)_{n<\omega}$ *be a strictly increasing sequence of regular cardinals and let* λ *be regular with* $\lambda < \kappa_0$. *The* mutual stationarity property $\mathrm{MS}((\kappa_n)_{n<\omega}, \lambda)$ *is the statement:*
"If $(S_n)_{n<\omega}$ *is a sequence of stationary sets* $S_n \subseteq \mathrm{Cof}(\lambda) \cap \kappa_n$, *then* $(S_n)_{n<\omega}$ *is mutually stationary in* $(\kappa_n)_{n<\omega}$*."*

M. Foreman and M. Magidor [7] proved the following two theorems:

Theorem 1.3. *For* $(\kappa_n)_{n<\omega}$ *any strictly increasing sequence of uncountable regular cardinals:*
(i) $\mathrm{MS}((\kappa_n)_{n<\omega}, \omega)$ *holds.*
(ii) $\mathrm{MS}((\kappa_n)_{n<\omega}, \omega_1)$ *implies* $V \neq L$.

This did not yet say that MS was a large cardinal property. That it was is the left to right direction of the following equivalence, proven in [13]:

Theorem 1.4. *The theories* $\mathrm{ZFC} + \exists(\kappa_n)_{n<\omega} \mathrm{MS}((\kappa_n)_{n<\omega}, \omega_1)$ *and* $\mathrm{ZFC} + \exists\kappa(\kappa$ *measurable*$)$ *are equiconsistent.*

The implication from right to left was first proven by Cummings, Foreman, and Magidor [2] *via* Prikry forcing. They proved more than this: they showed that a tail $(\kappa_n)_{n<\omega}$ of the Prikry generic sequence satisfies $\mathrm{MS}((\kappa_n)_{n<\omega}, \lambda)$ for any $\lambda < \kappa_0$ (or indeed the mutual stationarity of *any* sequence of stationary sets $S_n \subseteq \kappa_n$ irrespective of the cofinalities of the ordinals in the S_n). This is essentially obtained by utilising the fact that a tail of the Prikry generic sequence remains *coherently Ramsey* in the generic extension. The forward direction was proven in [13] using the core model K of A. J. Dodd and R. B. Jensen (see [6]). The deduction of the existence of $0^\sharp$ from $\mathrm{MS}((\kappa_n)_{n<\omega}, \omega_1)$ was done in detail, and the extension to proving the existence of the inner model with a measurable was sketched, using the hyperfine structure of S. Friedman and the first author ([8]). The proof involved the global square principle $\Box$ in L and techniques from the Jensen Covering theorem for L (see [5]). The purpose of this paper is to give a full account of the interaction of the proof of global $\Box$ with the MS property, (insofar as we are able) thus filling in the details of the above argument, but at the same time significantly strengthening the result to obtain models with many measures of high Mitchell order, in the case $(\kappa_n)_{n<\omega}$ consists of consecutive sequences of cardinals mentioned in the abstract:

Theorem 1.5. *If* $\mathrm{MS}((\aleph_n)_{1<n<\omega}, \omega_1)$ *holds then there is an inner model,* K, *and there is* $2 < k < \omega$ *so that below every* $\aleph_n$, *for* $k < n < \omega$, *there is a stationary set of ordinals* κ *which are, in* K, *measurable of Mitchell order* ω_{n-2}. *In fact, for such* $\aleph_n$ *the ordinals* $\alpha \in \mathrm{Cof}(\omega_{n-2})$ *which are singular in* K *are, in* V, *non-stationary below* $\aleph_n$.

One might wonder whether increasing the cofinality of the independently chosen stationary sets might yield increased Mitchell order. Well, perhaps, but seemingly not by our methods. The following is a corollary to the proof of the above theorem.

Corollary 1.6. *Let m be fixed, $1 \leq m < \omega$. Then if* $\mathrm{MS}((\aleph_{m+n})_{0<n<\omega}, \omega_m)$ *holds, exactly the same conclusion as that of Theorem 1.5 may be drawn.*

The methods here seem just short of allowing us to conclude that there is an inner model with a measurable κ with Mitchell order of κ equal to κ : ("$o_M(\kappa) = \kappa$").

It is important in the above statement that we use all the alephs below $\aleph_\omega$ (from some point on) since the first author has shown that omitting a cardinal above each one for which we wish to consider arbitrary stationary sets, has a much weaker consistency strength, (see [12]).

Theorem 1.7. (Koepke) *The theories* $\mathrm{ZFC} + \mathrm{MS}((\aleph_{2n+1})_{0<n<\omega}, \omega_1)$ *and* $\mathrm{ZFC} + \exists\kappa(\kappa$ *a measurable cardinal*) *are equiconsistent.*

It is unknown whether $\mathrm{MS}((\aleph_n)_{k<n<\omega}, \omega_k)$ (for any $k \geq 1$), when taking all the cardinals from some point on, is consistent relative to any large cardinals. One can speculate on this. That the mutual stationarity property can hold on sequences of cardinals with gaps, say on the even $\aleph_{2n}$, relative to a small large cardinal property such as measurability, but is much stronger (if consistent) when stipulated to hold on all the $\aleph_n$'s, puts it into a class of properties for which this phenomenon is well known. An outline reason is that typically members of a Prikry sequence $(\kappa_n)_n$ are collapsed to become, say, the $(\aleph_{2n+1})_n$ and will retain some weak vestige of large cardinal strength; in the process the $(\kappa_n^+)_n$ are preserved becoming the $(\aleph_{2n+2})_n$, without any such strong properties. Collapsing infinitely many supercompacts is seemingly required if one wishes to have strong properties at every $\aleph_n$. In contradistinction during this process, the successors of the supercompacts are collapsed too. Our results here are suggestive therefore that one might try to prove the consistency of $\mathrm{MS}((\aleph_n)_{1<n<\omega}, \omega_1)$ using again infinitely many supercompacts. Or perhaps using Radin forcing with a measurable of order $o(\kappa) \geq \kappa$.

The model K in Theorem 1.5 can be taken to be the core model built using measures (partial or full) only on its constructing extender sequence.

We shall need the following formulation of the Weak Covering Lemma due to W. Mitchell (*cf.* [14])

Theorem 1.8. (Weak Covering Lemma) *Assume there is no inner model with a measurable cardinal κ with $o_M(\kappa) = \kappa^{++}$. Let α be regular in K with $\omega_1 \leq \gamma = \mathrm{cf}(\alpha) < \mathrm{card}(\alpha)$. Then in K we have $o_M(\alpha) \geq \gamma$.*

We shall assume a development of the fine structure of such a core model K, as can be found in M. Zeman [18]. K is thus a model of the form $L[E]$ with E a sequence of partial or full extenders in the manner of Zeman's book. However no such extender requires any generator beyond its critical point. We shall need to consider the proof of the existence of Global $\square$ - a global square sequence - in such a model; this was shown by Jensen and Zeman to hold in [11]. The fine structural notation we shall adopt is that of the book (which is also that of the paper just cited). The indexing of extenders will be the Friedman-Jensen indexing whereby an extender is placed on the E sequence of a hierarchy at precisely the successor cardinal of the image of the critical point by that extender. Again this is following [11].

1.1. Outline of the proof

Foreman and Magidor in [7] 1.3 showed that in L the property $\mathrm{MS}((\kappa_n)_{n<\omega}, \omega_1)$ failed. Our method has its origins in this argument. There they took, assuming $V = L$ and to take a simplifying example, stationary sets $S_n \subseteq [\omega_{n-1}, \omega_n) \cap \mathrm{Cof}(\omega_1)$ for $n > 1$ with $\alpha \in S_n$ implying that

α was first collapsed in L at some least level $J_{\beta(\alpha)}$ by a $\Sigma_n(J_{\beta(\alpha)})$ function (possibly using some parameters in the definition) but by no $\Sigma_{n-1}(J_{\beta(\alpha)})$ function. (The precise levels of complexity of the defining functions are irrelevant, all that matters is that they differ infinitely often as n goes to infinity.) They took a structure H extending $\langle L_{\omega_\omega}, \in, \ldots \rangle$. If $X \prec H$ were a substructure of cardinality ω_1 satisfying the mutual stationarity property with regard to the above defined (S_n) sequence, we should have some $\beta_n^* =_{df} \sup X \cap \omega_n \in S_n$. H collapses transitively to some L_κ via some map $\pi : L_\kappa \longrightarrow H$. If $\pi(\beta_n) = \omega_n$, for $n \leq \omega$ then we look at the first point $\gamma \geq \kappa$ where one of the L_κ-cardinals β_n is definably collapsed over J_γ. Of course if β_n is so collapsed at some least level of complexity, Σ_{h_0} say, then so are all the β_m for $n \leq m \leq \omega$. Arguments similar to those appearing in Jensen's Covering Lemma involving a Pseudo-ultrapower (or what would now be called a long extender ultrapower) allow us for each $m \geq n$ to lift up the structure $\langle J_\gamma, \in \rangle$ to a structure $\langle J_{\gamma_m}, \in \rangle$ of cardinality ω_m, and with $\gamma_m \geq \beta_m^*$. This lift-up map $\pi_m : \langle J_\gamma, \in \rangle \longrightarrow \langle J_{\gamma_m}, \in \rangle$ has fine structural preserving properties. In particular β_m^* is $(\omega_m)^{J_{\gamma_m}}$, but crucially h_0 is still least so that it is Σ_{h_0} definably collapsed over $\langle J_{\gamma_m}, \in \rangle$. As m varies, h_0 does not, and so this is a contradiction.

The strategy of our proof (which first made its appearance in [13] applied to L and the Dodd-Jensen Core Model K^{DJ}) is to take a structure containing an initial segment of a canonical inner $L[E]$ model, and performing a collapse of a small substructure to some $\bar{H}$ say, look at some feature of the β_n (the preimages of the ω_n in that substructure) which can be extracted from the inner model $L[\bar{E}]$ part of $\bar{H}$. We could replicate the Foreman-Magidor argument, assuming $V = L[E]$, on the definability of the collapses of the β_n in some extension of the $L[\bar{E}]$ hierarchy of $\bar{H}$, but we want to do more. We want to step outside of $L[E]$ and obtain a result in V that the MS property implies that $L[E]$ has large cardinals. Since $L[E]$ models satisfy the existence of global $\square$ sequences, $\langle C_\beta \mid \beta \in Sing \rangle$, instead of looking at definable collapses in this hierarchy, we look and see what kind of C_{β_m} sets would be definable over $J_\gamma^{\bar{E}}$. The idea again is that on a tail of the $m < \omega$ the canonical method for defining C-sets in a construction of a global $\square$ yields C_{β_m} sets of a *bounded* order type, less than some β_{k_0} say. Although the $L[\bar{E}]$ hierarchy is not the $L[E]$ hierarchy in general, as there will be a failure of Condensation, nevertheless the lift-up maps produce target structures which are initial segments of the $L[E]$ hierarchy, and moreover a) these are precisely the structures we should wish to use to define the canonical $L[E]$ global $\square$ sequence, and b) an unvarying bound on the order types of the $C_{\beta_m^*}$ sets is maintained below $\pi(\beta_{k_0}) = \omega_{k_0}$[3].

In order for this to work as an argument in V, as opposed to an inner model such as L, we shall need an assumption that allows us to assume that there are stationarily many ordinals $\alpha \in [\omega_{n-1}, \omega_n) \cap \mathrm{Cof}(\omega_1)$ to form a stationary set S_n which actually do have a C_α set in $L[E]$ in the latter's canonical global $\square$ sequence. That is, they must be truly singular in $L[E]$. We thus formulate a negative large cardinal hypothesis for our $L[E]$ hierarchies *via* Mitchell's Weak Covering Lemma above that roughly says that the measurable cardinals of $L[E]$ below each ω_n are not too "thick". A simple argument (Theorem 4.1) allows us then to take the S_n sets to consist of α with C_α of increasing order type below ω_ω as n increases to infinity. This in turn allows us to obtain a final contradiction as our "lift up" process ensures that order types are kept bounded, reflecting as they do the order types obtained from the small structure $L[\bar{E}]$ for the $C_{\beta_n^*}$. We conclude then that there must be $L[E]$ models in which the large cardinal hypothesis holds; this

[3]Recently [4] used this argument (assuming $V = L$) on the bounding of canonical $\square$-sequence order types to show that the L-Coherent Squares sequence up to $\aleph_\omega$ does not exhibit the same MS property that [3] obtained by forcing over L.)

is our main theorem.

In order to see that we have sufficient canonically defined $C_{\beta_n^*}$ sets defined prior to any lift-up, we need to see that we have a sufficiently canonical structure extending $J^{\bar{E}}_{\beta_n^*}$. We obtain this by a standard dual iteration and comparison argument of $L[\bar{E}]$ with the core model K. As is familiar, under certain conditions the iterated ultrapowers taken all come from the K side of this co-iteration: the $L[\bar{E}]$ model does not move. On the K side some set sized mouse M iterates past the $L[\bar{E}]$ cardinals in turn. At the point where it has just iterated past β_n^* we may take a snapshot and define a $C_{\beta_n^*}$ set and calculate its order type. In order to do this we need to show that, at least on a tail of the β_n^*, the order types remain bounded. This results in two requirements a) to look at the proof of the canonical Global $\square$ sequence and more carefully compute the order types (this is effected in Lemma 3.38); and b) we need to know how iterating the structure over which the C-sets are to be obtained affects order types. We shall require that they are not (at least on a tail) prolongated by iterations of the mouse from which they are defined. Requirement b) is effected in the final section and is the statement labelled (6) there.

In section 3 we provide a proof of global $\square$ in small $L[E]$ models *ab initio*. One reason for doing this is that Jensen and Zeman's method of proof for Global $\square$ is to define a "smooth category" of structures and maps from which it is known that a Global $\square$ sequence can be derived. This latter *derivation* is purely combinatorial and so requires no inspection of the fine structure of the original model. The burden of their proof is the *construction* of the smooth category itself. However that construction does not yield an explicit computation for the order types of the various C_ν sequences. (It is the latter derivation that does that). We use [10] in order to compute those order types, and for this we need the notation of the of the proof to hand. We therefore give a proof of global $\square$ directly without going through the smooth category.

In section 2 we state some fine structural lemmata that form the hard work of Jensen and Zeman's account in [11] which establish the right forms of parameter preservation and appropriate condensation results. We merely quote these as Condensation Lemmas (I) and (II). However in order to fulfill requirement b) we need to prove the preservation of certain fine structural d-parameters of [11]. This is done in Lemma 2.3. The analysis of the Condensation Lemmata excepted, we try to keep the rest of the proof as self-contained as possible. The proof of Lemma 3.9 is however a repetition of [11] 4.3 and is so omitted. Lemma 3.11 is related to [11] 4.5. These are key lemmata on the relationships between singularising structures and the maps between them, and are, in the Σ^* terminology, the successors to [1] Lemmas 6.15 and 6.18. From Definition 3.18 onwards this is an account very much following that of (the now out of print) [1], and which will also be in the forthcoming [16], but here uses a corresponding J_s notation incorporating mice. (The reader of [1] should be warned that the main exposition there in Chapter 6 is phrased in terms of constructing a more restricted "square-like" sequence, not on all singular ordinals but only on those ν where ν is a successor cardinal in the relevant structure. For much of the argument, but not all, the cases barely differ. The account in [16] constructs the full global square (not this restricted version) in L rather than $L[E]$. However the later sequence of combinatorial lemmas are really only notational variants.) The final calculation of order types follows the manuscript [10].

In section 4 we see how to use features of this proof to get the main Theorem 1.5; the reader who is completely familiar with the $\square$ proof in such $L[E]$ models and wants to discover the ideas in the application to mutual stationarity may wish to go straight there.

2. Fine structural prerequisites

For an acceptable J-structure M we assume familiarity with the notions of the uniformly defined Σ_1-Skolem function for M, h_M, the class of parameter sequences Γ^M, and the parameter sets $P^n_M, P_M, P^*_M, R^n_M, R_M$, and R^*_M. We shall write ρ_M as usual for the Σ_1-projectum of M. Similarly we shall write for the $n+1$*'st projectum* $\rho^{n+1}_M =_{\mathrm{df}} \min\{\rho_{M^{n,p}} \mid p \in \Gamma^n_M\}$. We may assume that parameters are finite sets of ordinals. This applies as well to the n*'th-standard parameter* and the *standard parameter* denoted here p^n_M, p_M respectively for a structure M as above. We wellorder $[\mathrm{On}]^{<\omega}$ by $u <^* v \leftrightarrow \max(u\Delta v) \in v$. For $X \subseteq \mathrm{Ord}$ a set, we write $\mathrm{ot}(X)$ for its order type, and by X^* we mean the set of limit points of X. Our discussion of fine structure is entirely in the language of $\Sigma^{(n)}_k$ relations due to Jensen (for which see [18] or [16]). Boldface relations such as $\mathbf{\Sigma}^{(n)}_1(M)$ denote those definable using parameters (in this case from M.) We remind the reader of the notation for various skolem functions here. We denote by $h^{n,p}_M = h_{M^{n,p}}$ the standard Σ_1 skolem function for the n'th projectum structure relative to the parameter $p \in \Gamma^n_M$. The Σ_1 hull of a set $X \subseteq M^{n,p}$ we shall denote by $h^{n,p}_M(X) =_{df} \{h^{n,p}_M(i,x)) \mid i \in \omega, x \in X\})$. We suppose a fixed Σ_0 formula $H(v_0, v_i, v_2, v_3)$ chosen so that $y = h_M(i,x) \leftrightarrow \exists z H(i,z,y,x)$.

The $\Sigma^{(n-1)}_1$ Skolem function for M is denoted by $\tilde{h}^n_M$ ([18],p.29). It is moreover uniformly $\Sigma^{(n-1)}_1$ definable over all such M. Note that $\tilde{h}^1_M(\langle j, y^0\rangle, p(0)) = h_M(j, \langle y, p(0)\rangle)$ for $p \in \Gamma_M$. If $p \in R^n_M$ then every $x \in M$ is of the form $\tilde{h}^n_M(z,p)$ for some $z \in H^n_M$. Hence if $p \in R^n_M$ then $\tilde{h}^n_M$ is a good, uniformly defined, $\Sigma^{(n-1)}_1(M)$ function mapping $\omega\rho^n_M$ onto M. We may similarly form hulls using $\tilde{h}^n_M$: if $X \subseteq M^{n,p}$ say, and $q \in M$ then the $\Sigma^{(n-1)}_1$ hull of $X \cup \{q\}$ is the set $\tilde{h}^n_M(X \cup \{q\}) =_{df} \{\tilde{h}^n_M(x,q)) \mid x \in X\}$.

Recall that a premouse M is *sound above* ν if $\omega\rho^{n+1}_M \leq \nu$ implies that $\tilde{h}^{n+1}_M(\nu \cup \{p_M\}) = |M|$. We also say that it is k*-sound* if it is sound above $\omega\rho^k_M$.

In order to have sufficient condensation Jensen and Zeman require certain parameters associated with canonical witness structures to be in the range of their maps. We only remind the reader of this definition here, and refer to the paper for a full discussion of its significance.

Definition 2.1. *Suppose* $\gamma \in p^n_M$ *and let* σ^M_γ *be the canonical witness map corresponding to* W^γ_M*; we define* $\delta(\gamma) = \sup(\mathrm{ran}(\sigma^M_\gamma) \cap \omega\rho^n_M)$ *if this is less than* $\omega\rho^n_M$*. Otherwise it is undefined. We then set:*
$\widetilde{p}^n_M =_{\mathrm{df}} \{\gamma \in p^n_M \mid \delta(\gamma) \text{ is defined}\}$; $\quad \widetilde{p}_M =_{\mathrm{df}} \bigcup_n \tilde{p}^n_M$;
$d^n_M =_{\mathrm{df}} \{\delta(\gamma) \mid \gamma \in \tilde{p}^k_M, k \leq n\}$; $d_M = \bigcup_n d^n_M$.

This finite (possibly empty) set d^n_M then collects together all the sups of those canonical witness maps σ_γ just for those γ for which the map is non-cofinal at the k'th levels for $k \leq n$. This allows for an appropriate form of the Condensation Lemma for hierarchies below mice M with any κ for which $(o_M(\kappa) = \kappa^{++})^M$.

As is usual $o_N(\kappa)$ denotes the extender order of κ in the hierarchies under consideration (and roughly corresponds to Mitchell order of measures).

Theorem 2.2. (Condensation Lemma II) (*cf*[11] 3.1) *Suppose there is no inner model for* $o_M(\kappa) = \kappa^{++}$*. Let* N, M *be mice and* $\sigma : \hat{N} \longrightarrow_{\Sigma^{(n)}_1} \hat{M}$*. Suppose further that* $\sigma(\bar{\alpha}) = \alpha, \sigma(\bar{p}) = p_M \backslash \alpha$*, and*

(i) $\omega\rho^{n+1}_M \leq \alpha < \omega\rho^n_M$ *and* M *is sound above* α*;*

(ii) $d^n_M \subseteq \mathrm{ran}(\sigma)$.

Then $\bar{p} = p_N \backslash \bar{\alpha}$ *; N is sound above* $\bar{\alpha}$*,* $\sigma(\tilde{p}_N \backslash \bar{\alpha}) = \tilde{p}_M \backslash \alpha$ *and* $\sigma(\delta^N(\gamma)) = \delta^M(\sigma(\gamma))$ *whenever* $\gamma \in \tilde{p}_N \backslash \bar{\alpha}$*.*

We shall ignore the hatted decorations to the mice M, N in the above: these are the *expansions* of the mice M, N to which the extenders are actually applied. This mechansim will play no role in what follows, and we again refer the concerned reader to [11] Sect. 2 or Ch. 8 of [18].

We shall need a lemma on preservation of these d-parameters under normal iterations. We prove this here.

Lemma 2.3. *Suppose* $\pi : M \longrightarrow N$ *is a normal iteration of the mouse* M*. Then* $\pi(d_M) = d_N$*.*

Proof: This would be by induction on the length of the iteration, but we simply do a one step ultrapower by an extender E with critical point κ and the reader can form the general and direct limit argument herself. This does not follow quite immediately from Condensation Lemma II as the latter assumes d_N is in the range of the map. We do know however that $\pi(p_M) = p_N$. We may express

$\tilde{p}_M = \{\nu \in p_M \mid$ *If* $\nu \in [\omega\rho^{k+1}_M, \omega\rho^k_M)$*then the canonical witness map is non-cofinal into* $\omega\rho^k_M\}$.

And so: $d_M = \{\delta^M(\nu) | \nu \in \tilde{p}_M\}$. Then if $\delta(\nu) \in d_M$ with $\nu \in [\omega\rho^{k+1}_M, \omega\rho^k_M)$ we have as in the proof of Thm 3.4 of [11]:

$$(*) \quad \forall\xi^k\forall\zeta^k(\xi^k < \nu \wedge \zeta^k = \widetilde{h}^{k+1}_M(\xi^k, p_M\backslash(\nu+1)) \longrightarrow \zeta^k \leq \delta(\nu)).$$

This is $\Pi^{(k)}_1$ in $\nu, \delta(\nu)$, and p_M. If $\mathrm{crit}(E) = \kappa \in [\omega\rho^{n+1}_M, \omega\rho^n_M)$ then π is $\Sigma^{(n)}_0$ preserving and cofinal into $\omega\rho^n_M$, hence $\Sigma^{(n)}_1$-preserving. This also holds of π for $k > n$ (since $\pi \restriction H^k_M = \mathrm{id} \restriction H^k_M$ and then $\mathrm{ran}(\pi)$ is trivially cofinal into $\omega\rho^k_N$). If $k < n$ then it is $\Sigma^{(k)}_2$ preserving. Consequently wherever ν lies we have from these preservation properties:

(1) $\nu \in \tilde{p}_M \longrightarrow \pi(\nu) \in \tilde{p}_N \wedge \pi(\delta^M(\nu)) \geq \delta^N(\pi(\nu))$.

We want equality here. For $k < n$, $\Sigma^{(k)}_2$preservation suffices to guarantee this: if

$$\exists\delta^k < \pi(\delta^M(\nu))[\forall\xi^k\forall\zeta^k(\xi^k < \pi(\nu) \wedge \zeta^k = \widetilde{h}^{k+1}_N(\xi^k, \pi(p_M)\backslash(\pi(\nu)+1)) \longrightarrow \zeta^k \leq \delta^k]$$

held in N then this would go down to M and give a contradiction. For $k = n$ we can reason as follows. We have:

$$W^\nu_M \cap H^n_M = \tilde{h}^{n+1}_M(\nu \cup p_M\backslash\nu+1) \cap H^n_M = h^{n,p_M}_M(\nu \cup (p_M\backslash\nu+1 \cap \omega\rho^n_M)).$$

As $\delta^M(\nu) < \omega\rho^n_M$ we may write, setting $\delta = \delta^M(\nu)$, $p_0 = p_M\backslash\nu+1 \cap \omega\rho^n_M$, using our fixed Σ_0 formula $H(v_0, v_i, v_2, v_3)$ from above:

$$\forall\tau_0 < \delta\exists\tau < \delta(\tau_0 < \tau \wedge \exists z^n\exists i < \omega\exists\vec{\xi} < \nu H(z^n, i, \tau, \langle\vec{\xi}, p_0\rangle)),$$

and by the definition of δ, this is satisfied in M^{n,p_M}. However note that the quantifier $\exists z^n$ can be considered bounded by S^M_δ too: we are essentially taking a Σ_1-Skolem hull in M^{n,p_M} and consequently if δ bounds that Skolem hull below $\mathrm{On} \cap H^n_M$ then S_δ also bounds locations for a search for Σ_1 witnesses to any Σ_1 definition of an object in that hull. Hence M satisfies:

(2) $\exists y(y = S^M_\delta \wedge (\forall\tau_0 < \delta\exists\tau < \delta(\tau_0 < \tau \wedge \exists z^n \in y\exists i < \omega\exists\vec{\xi} < \nu H(z^n, i, \vec{\xi}, p_0, \tau))))$.

Since $\pi(p_0) = (p_N\backslash\pi(\nu) + 1 \cap \omega\rho^n_N)$, the latter statement about $\pi(\delta)$ and $\pi(y) = S^N_{\pi(\delta)}$ shows that $\delta^N(\pi(\nu)) \geq \pi(\delta)$. Hence

(3) $\nu \in \tilde{p}_M \longrightarrow \pi(\delta^M(\nu)) = \delta^N(\pi(\nu))$.

Now note:

(4) $\pi(\nu) \in \tilde{p}_N \longrightarrow \nu \in \tilde{p}_M$ and hence again $\pi(\delta^M(\nu)) = \delta^N(\pi(\nu))$.

For $k < n$ this follows from $\Sigma^{(k)}_2$ preservation. For $k = n$ this follows from the cofinality of π into $\omega\rho^n_N$: if $\delta^N(\pi(\nu))$ is defined, then it is less than some $\pi(\delta)$ and the formula $(*)$ written out for N and $\pi(\delta)$, $\pi(\nu)$, replacing $\delta(\nu)$, ν then goes down to M, so this suffices.

If $k > n$ then $\pi \restriction H^k_M = \mathrm{id} \restriction H^k_M$. Thence $\nu \in p_M \cap \omega\rho^k_M \longleftrightarrow \nu \in p_N \cap \omega\rho^k_N$. The previous argument shows that $\delta^M(\nu) \leq \delta^N(\nu)$. Again, as above, since π is $\Sigma^{(k)}_1$ preserving, and using $(*)$, we have $\delta^M(\nu) \geq \delta^N(\nu)$ also. Q.E.D.

We shall also be assuming familiarity with the construction of fine-structural pseudo-ultrapowers (*cf* [18]). We shall be using these in the form of the Pseudo-Ultrapower Theorem below, and an Interpolation Lemma.

Definition 2.4. *Let M be an acceptable J-structure, and $\nu \in M$ a regular cardinal of M. Then $k(M, \nu)$ is defined to be the least k (if it exists) so that there is a good $\Sigma^{(k)}_1$-definable (possibly with parameters) function whose domain is a bounded subset of ν and whose range is unbounded in ν. (Such a function is said to* singularise ν *and we say that ν is* $\mathbf{\Sigma}^{(k)}_1(M)$-singularised over *$M$.)*

In the next theorem, and on other occasions, "$f \in \mathbf{\Sigma}^{(n)}_1(\overline{M})$" for $n = -1$ should be read as "$f \in \overline{M}$."

Theorem 2.5. (Pseudo-Ultrapower Theorem) *Let $\bar{M}$ be an acceptable J-structure, $\bar{\nu}$ a regular cardinal of $\bar{M}$ but with $k = k(\bar{M}, \bar{\nu})$ defined. Let $\overline{Q} =_{df} J^{\bar{M}}_{\bar{\nu}}$. We suppose that we have a structure Q' with $\nu = On \cap Q'$ and a cofinal map $\sigma : Q \rightarrow_{\Sigma_0} Q'$. Define $\Gamma = \Gamma^k_{\bar{M},\bar{\nu}} =_{\mathrm{df}}$*

$$\{f \mid \mathrm{dom}(f) \in \overline{Q} \wedge (\exists n < k)(f \in \mathbf{\Sigma}^{(n)}_1(\overline{M}) \wedge \omega\rho^{n+1}_{\overline{M}} \geq \bar{\nu})\}.$$

Then there is a map $\tilde{\sigma} : \bar{M} \longrightarrow_{\Sigma_0} M$ (the "canonical k-extension" of σ) satisfying:
(i) $\tilde{\sigma}$ is Q-preserving, M is an acceptable end extension of Q;
(ii) a) $\tilde{\sigma}$ is $\Sigma^{(n)}_2$ preserving for $n < k$;
b) $\tilde{\sigma}$ is $\Sigma^{(k)}_0$ preserving and cofinal (thus $\Sigma^{(k)}_1$-preserving); and

$$M = \{\tilde{\sigma}(f)(u) \mid u \in \sigma(\mathrm{dom}(f)), f \in \Gamma\}.$$

(iii) $\tilde{\sigma}(\bar{\nu}) = \nu$ and the latter is regular in M ;
(iiv) $k = k(M, \nu)$: k is least so that there is a $\mathbf{\Sigma}^{(k)}_1(M)$ map singularising ν.

Lemma 2.6. (Interpolation Lemma) *Suppose $\overline{M} = \langle J^{\overline{A}}_{\overline{\beta}}, \overline{B}\rangle$ is a structure such that $\overline{\nu}$ is regular in $\overline{M}$, but with $k = k(\bar{M}, \bar{\nu})$ defined. Suppose further that $f : \overline{M} \longrightarrow_{\Sigma^{(k)}_1} M = \langle J^A_\beta, B\rangle$. Let $\widetilde{\nu} = \sup f``\overline{\nu}$. Then there is a structure $\widetilde{M} = \langle J^{\widetilde{A}}_{\widetilde{\beta}}, \widetilde{B}\rangle$, a map $\widetilde{f} : \overline{M} \longrightarrow \widetilde{M}$ with $\widetilde{f} \supseteq f \upharpoonright J^{\overline{A}}_{\overline{\nu}}$ and $\widetilde{f}, \Sigma^{(k)}_0$-cofinal (and hence $\Sigma^{(k)}_1$-preserving), and a unique $f' : \widetilde{M} \longrightarrow_{\Sigma^{(k)}_0} M$, with $f = f' \circ \widetilde{f}$ and $f' \upharpoonright \tilde{\nu} = \mathrm{id} \upharpoonright \tilde{\nu}$.*

3. Global $\square$ in K.

Definition 3.1. *Let* $\mathrm{Sing} = \{\beta \in \mathrm{Ord} \mid \lim(\beta) \wedge \mathrm{cf}(\beta) < \beta\}$ *be the class of singular limit ordinals.* Global $\square$ *is the assertion: there is a system $(C_\beta)_{\beta \in \mathrm{Sing}}$ satisfying:*
(a) C_β is a closed cofinal subset of β;
(b) $\mathrm{ot}(C_\beta) < \beta$;
(c) if $\overline{\beta}$ is a limit point of C_β then $\overline{\beta} \in \mathrm{Sing}$ and $C_{\overline{\beta}} = C_\beta \cap \overline{\beta}$.

Jensen [9] introduced the principle and proved it held in L. The format of the proof we shall follow will be that of [1], which was a proof in the setting of generalised $L[A]$ hierarchies suitable for the use of Jensen's Coding Theorem. The second author [15] proved that global $\square$ held in the Dodd-Jensen core model K. The first proof of $\square$ which used the Baldwin-Mitchell arrangement of the $L[E]$ hierarchies, was for Jensen's model for K with measures of order zero, and was by Wylie [17]. From the order types of the square sequence elements C_ξ we shall define stationary sets S_n to which we shall apply the MS-principle.

We consider how a global $\square$ sequence can be derived in K. For clarity we shall assume there is no inner model with a measure of Mitchell order $o_M(\kappa) = \kappa^{++}$ (see [11]) and that K is built under this assumption. We assume for the rest of this section $V = K$. Jensen and Zeman prove (more than) the following.

Theorem 3.2. *Let S be the class of all singular limit ordinals that are limits of admissibles. There is a class $\langle C_\nu | \nu \in S\rangle$ so that:*

(i) C_ν is a set of ordinals closed below ν and, if $\mathrm{cf}(\nu) > \omega$, then it is also unbounded;
(ii) $\mathrm{ot}(C_\nu) < \nu$;
(iii) $\overline{\nu} \in C_\nu \longrightarrow \overline{\nu} \in S \wedge C_{\bar{\nu}} = \overline{\nu} \cap C_\nu$;

Unstated in the last theorem is the very uniform method of defining the C_ν sequence from the pertinent singularising structure. It is well known that once one has a global $\square$ sequence defined on the singular ordinals of some cub class that contains all singular cardinals and is cub beneath each successor cardinal, then this can be filled out to a global sequence on *all* singular ordinals to satisfy Definition 3.1. Hence proving the above theorem suffices. As $V = K = L[E]$ for $E = E^K$ a fixed sequence of extenders, if ν is a singular ordinal, then there will be a least level $J^E_{\beta(\nu)}$ of the J^E-hierarchy over which ν is definably singularised, *i.e.* there will be a partial $\boldsymbol{\Sigma_\omega}(J^E_{\beta(\nu)})$ definable good function mapping a subset of some $\gamma < \nu$ cofinally into ν. This level of the hierarchy $J^E_{\beta(\nu)}$ will also be our main singularising structure M_ν. Note that by the soundness of the K hierarchy, using [18] Lemma 1.18.4, any such function is also $\boldsymbol{\Sigma}^{(n)}_1(J^E_{\beta(\nu)})$ for some n. That is, $k(\nu, J^E_{\beta(\nu)})$ in the sense of Definition 2.4 is defined.

However there will be many other mice over which ordinals are singularised and we must consider these in addition.

Definition 3.3. *S^+ is the class of $s = \langle \nu_s, M_s \rangle$ where*
(a) $\nu_s \in \mathrm{Sing}$;
(b) M_s is a mouse satisfying the following:
(i) ν_s is regular in M_s or possibly is $On \cap M_s$, and $J_s =_{\mathrm{df}} J_{\nu_s}^{E^{M_s}}$ is a union of admissible structures $\langle J_\tau^{E^{M_s}}, \in, E^{M_s} \upharpoonright \omega\tau \rangle$;
(ii) for some m, ν_s is $\mathbf{\Sigma}_1^{(m)}(M_s)$ singularised, that is $k(\nu_s, M_s)$ is defined;
(iii) M_s is sound above ν_s, and if $\nu_s = \kappa^{+M_s}$ where $\kappa \in \mathrm{Card}^{M_s}$, then M_s is sound above κ.

Recall that if $M = \langle J_\alpha^E, \in \rangle$ and $\nu \leq \alpha$ then $M||\nu =_{df} \langle J_\nu^E, \in, E_\nu \rangle$. We then note the following facts:

Lemma 3.4. *(i) If $\langle \nu, M \rangle, \langle \nu, N \rangle$ satisfy (b)(i),(ii) above but are both sound above ν, with $M||\nu = N||\nu$, then $M = N$.*
(ii) If $\langle \nu, M \rangle, \langle \nu, N \rangle \in S^+$and $J_\nu^{E^M} = J_\nu^{E^N}$ then $M = N$.

Proof: This is proved in [11] and is a straightforward iteration and comparison argument.
Q.E.D.

The following definition encapsulates the essential concepts associated with singularising structures.

Definition 3.5. *We associate the following to $s \in S^+$:*

a) $n_s =_{\mathrm{df}} k(\nu_s, M_s)$, the least $n \in \omega$ so that ν_s is $\mathbf{\Sigma}_1^{(n)}(M_s)$ singularised over M_s.

b) $M_s^l =_{\mathrm{df}} M_s^{l, p_{M_s} \upharpoonright l}$ for $l \leq n_s$.

c) $h_s^l =_{\mathrm{df}} h_{M_s}^{l, p_{M_s} \upharpoonright l}$; $h_s =_{\mathrm{df}} h_s^{n_s}$; $\widetilde{h}_s =_{\mathrm{df}} \widetilde{h}_{M_s}^{n_s+1}$.

d) $\kappa_s \simeq$ the largest cardinal of J_s, if such exists; $\omega\rho_s =_{\mathrm{df}} \mathrm{On} \cap M_s^{n_s}$;
$\beta(s) =_{\mathrm{df}} \mathrm{On} \cap M_s$.

e) $p_s =_{\mathrm{df}} p_{M_s} \backslash \nu_s$ if ν_s is a limit cardinal of J_s; $p_s =_{\mathrm{df}} p_{M_s} \backslash \kappa_s$ otherwise;
$q_s =_{\mathrm{df}} p_s \cap \omega\rho_{M_s}^{n_s}$;
$d_s =_{\mathrm{df}} d_{M_s}$

f) $\alpha_s =_{\mathrm{df}} \max\{\alpha < \nu_s \mid \nu_s \cap \tilde{h}_s(\alpha \cup \{p_s\}) = \alpha\}$, setting $\max \varnothing = 0$.

g) $\gamma_s \simeq \min\{\gamma < \nu_s \mid \exists f (f$ a good $\Sigma_1^{(n_s) M_s}(\{p_s\})$ function singularising ν_s, $dom(f) \subseteq \gamma)\}$.

Notice that κ_s, if defined, may be in p_s. Note that the closure of the set in *f*) ensures that α_s is always defined; additionally α_s must be strictly less than γ_s. Note also that if we set $\gamma' = \max\{\gamma_s, \max\{p_{M_s} \cap \nu_s\} + 1\}$ ($\max\{\gamma_s, \max\{p_{M_s} \cap \kappa_s\} + 1\}$ if κ_s is defined), then $\widetilde{h}_s(\gamma' \cup p_s)$ must be cofinal in ν_s since we shall have enough parameters in the domain of this hull to define our singularising map).

Lemma 3.6. $\omega\rho_{M_s}^{n_s} \geq \nu_s \geq \omega\rho_{M_s}^{n_s+1}$

Proof The first inequality is proven in [11] Lemma 4.2. If the second inequality failed, then the partial function $\mathbf{\Sigma}_1^{(n_s)}(M_s)$ singularising ν_s would be a subset of $\nu_s \times \nu_s$ and thus coded as a bounded subset of $\omega\rho_{M_s}^{n_s+1}$ belonging to M_s. Q.E.D.

Definition 3.7. *For* $s, \bar{s} \in S^+$ *: (i) We set* $f : \bar{s} \Longrightarrow s$ *if there is* $|f|$ *with* $|f| : J_{\bar{s}} \longrightarrow_{\Sigma_1} J_s$, *and* $|f|$ *is the restriction of* $f : M_{\bar{s}} \longrightarrow_{\Sigma_1^{(n)}} M_s$ *where* $n = n_s, \nu_s = f(\nu_{\bar{s}})$ *(if* $\nu_s \in M_s$*);* $\kappa_s \in \mathrm{ran}(|f|)$ *(if* κ_s *is defined); and* α_s, p_s, d_s *are all in* $\mathrm{ran}(f)$.

(ii) $\mathbb{F} = \{\langle \bar{s}, |f|, s\rangle | f : \bar{s} \Longrightarrow s\}$*; we write here* $\bar{s} = d(f), s = r(f)$*;*

(iii) If $\nu_s \in M_s$*, we set:*

$p(s) =_{\mathrm{df}} p_s \cup \{d_s, \alpha_s, \nu_s, \kappa_s\}$ *(including* κ_s *if it is defined); otherwise*

$p(s) =_{\mathrm{df}} p_s \cup \{d_s, \alpha_s, \kappa_s\}$ *(again including* κ_s *only if it is defined).*

(iv) $f_{(\delta,q,s)}$ *is the inverse of the transitive collapse of the hull* $\widetilde{h}_s(\delta, \{p(s), q\})$ *in* M_s.

(v) We shall write "$f \Longrightarrow s$*" if there is* $\bar{s}$ *with* $f : \bar{s} \Longrightarrow s$.

Lemma 3.9 will show for the $f_{(\delta,q,s)}$ in clause *(iv)*, that there is some $\bar{s}$ with $\langle \bar{s}, |f_{(\delta,q,s)}|, s\rangle \in \mathbb{F}$.

Lemma 3.8. *If* $\exists \bar{s}(f : \bar{s} \Longrightarrow s)$ *then* $|f|$ *and* f *are uniquely determined by* $\mathrm{ran}(|f|) \cap \nu_s$.

Proof: As M_s is sound above ν_s, we have by our definitions, that $\widetilde{h}_s(\omega\nu_s \cup \{p_s\}) = M_s$. We have a $\Delta_1(J_s)$ onto map $g : \omega\nu_s \twoheadrightarrow J_s$. Thus, if $Y = \widetilde{h}_s(\omega\nu_s \cap \mathrm{ran}(|f|) \cup \{p_s\})$, then also

$$Y = \widetilde{h}_s(\mathrm{ran}(|f|) \cup \{p_s\}) = \mathrm{ran}(f).$$

Q.E.D.

Lemma 3.8 justifies us in calling f the *canonical extension* of $|f|$, and sometimes we abuse notation and write $f : J_{\bar{s}} \longrightarrow_{\Sigma_1} J_s$ where more correctly we should write $f \upharpoonright J_{\bar{s}} : J_{\bar{s}} \longrightarrow_{\Sigma_1} J_s$. By virtue of the last lemma, this does not cause any ambiguity. Note that we also have identity triples for $s \in S^+$: $id_s =_{df} \langle s, |id \upharpoonright J_s|, s\rangle \in \mathbb{F}$.

The next two lemmata are fundamental and concern relationships between singularising structures, and associated maps between them. The proof of the first is *verbatim* from [11] and so is omitted.

Lemma 3.9. (cf [11] 4.3) *Let* $f : \overline{M} \longrightarrow_{\Sigma_1^{(n)}} M_s$*; suppose* $f(\bar{d}, \bar{\alpha}, \bar{p}) = d_s, \alpha_s, p_s$*, and (where appropriate)* $f(\bar{\kappa}, \bar{\nu}) = \kappa_s, \nu_s$*. (The latter if* $\nu_s \in M_s$ *; if* $\nu_s = \mathrm{On} \cap M_s$ *then we take* $\bar{\nu} = \mathrm{On} \cap M_{\bar{s}}$*.) Then* $\bar{s} = (\bar{\nu}, \overline{M}) \in S^+$ *and thus* $f : \bar{s} \Longrightarrow s$*; moreover* $n, \bar{d}, \bar{\alpha}, \bar{p}, \bar{\kappa}$ *(the latter defined if* κ_s *is) are* $n_{\bar{s}}, d_{\bar{s}}, \alpha_{\bar{s}}, p_{\bar{s}}, \kappa_{\bar{s}}$.

Definition 3.10. *Suppose* $f : \bar{s} \Longrightarrow s$*. Then let* $\lambda(f) =_{\mathrm{df}} \sup f``\nu_{\bar{s}}$*;* $\rho(f) =_{\mathrm{df}} \sup f``\rho_{\bar{s}}$.

The following is essentially [11] Lemma 4.5, but we include a proof in order to obtain as a corollary to it the lemma following.

Lemma 3.11. *Suppose* $f : \bar{s} \Longrightarrow s$*, and let* $\lambda = \lambda(f)$*. Then* $\lambda \in \mathrm{Sing}$ *and there exist unique* s'*,* $f_0 : \bar{s} \Longrightarrow s'$ *with* $f \upharpoonright \nu_{\bar{s}} = f_0 \upharpoonright \nu_{\bar{s}}$ *and* $\nu_{s'} = \lambda$*. We set* $s|\lambda = s'$ *and call* f_0 *the* reduct *of* f *(to* $\lambda(f)$*) :* $f_0 = red(f)$.

Proof: Let $n = n_s$. We directly apply the Interpolation Lemma with λ as $\widetilde{\nu}$, and $M_{\bar{s}}$, M_s as $\overline{M}, M$ respectively, and using $f^* : M_{\bar{s}} \longrightarrow_{\Sigma_1^{(n)}} M_s$ (where f^* is the canonical extension of f) we have the structure $\widetilde{M} = M_{s'}$ and maps $\widetilde{f}, f'$ as specified.

(1) $s' = \langle \lambda, \widetilde{M} \rangle \in S^+, n = n_{s'}$.

We have that $\gamma_{\bar{s}}$ is defined and $n = n_{\bar{s}}$. As $\widetilde{h}_{\bar{s}}(\gamma_{\bar{s}} \cup \{p_{M_{\bar{s}}}, r\})$ is cofinal in $\overline{\nu}$ for some parameter r, then $\lambda \cap \widetilde{h}^{n+1}_{\widetilde{M}}(\widetilde{f}(\gamma_{\bar{s}}) \cup \{p', \widetilde{f}(r)\})$ is cofinal in λ for such an r (setting $p' = \widetilde{f}(p_{\bar{s}}) = f'^{-1}(p_s)$). Thus λ is $\Sigma_1^{(n)}$-singularised over $\tilde{M}$. Hence $n \geq n_{s'}$. We need to show that λ is not $\Sigma_1^{(n-1)}$-singularised over $\tilde{M}$. Suppose this fails and thus that $\{\alpha | \sup(\lambda \cap \widetilde{h}^n_{\widetilde{M}}(\alpha \cup \{r\})) = \alpha\}$ is bounded in λ, by α' say, for some choice of a parameter $r \in \widetilde{M} = M_{s'}$. By the construction of the pseudo-ultrapower we may assume that r is of the form $\widetilde{f}(\bar{g}_0)(\eta)$ for some good $\mathbf{\Sigma}_1^{(n-1)}(M_{\bar{s}})$ function $\bar{g}_0$ and some $\eta < \lambda$. Define

$$\widetilde{H}(\xi^n, \zeta^n, d) \longleftrightarrow \widetilde{h}^n_{\tilde{M}}(\omega\xi^n \cup \{d\}) \cap \lambda \subseteq \zeta^n;$$
$$\bar{H}(\xi^n, \zeta^n, d) \longleftrightarrow \widetilde{h}^n_{\bar{s}}(\omega\xi^n \cup \{d\}) \cap \bar{\nu} \subseteq \zeta^n.$$

These are (uniformly defined) $\Pi_1^{(n)}$ relations over their respective structures - in the parameters $\lambda, \bar{\nu}$. By the leastness in the definition of $n_{\bar{s}}$ we have that there are arbitrarily large $\bar{\tau}^n < \bar{\nu}$ with $\widetilde{h}^n_{\bar{s}}(\omega\bar{\tau}^n \cup \{p_{\bar{s}}\}) \cap \bar{\nu} \subseteq \bar{\tau}^n$; using the soundness of $M_{\bar{s}}$ above $\bar{\nu}$, this implies that for arbitrary $\bar{\xi}^n, \zeta^n < \bar{\tau}$: $\widetilde{h}^n_{\bar{s}}(i, \bar{\xi}^n, \bar{g}_0(\zeta^n)) \cap \bar{\nu} \subseteq \bar{\tau}^n$. In other words:

$$\forall \zeta^n < \bar{\tau}^n \overline{H}(\bar{\tau}^n, \bar{\tau}^n, \bar{g}_0(\zeta^n)).$$

As the substituted $\bar{g}_0$ is a good $\Sigma_1^{(n-1)}$ function we have that this is a $\Pi_1^{(n)}$ statement, and so is preserved upwards to $M_{s'}$:

$$\forall \zeta^n < \widetilde{f}(\bar{\tau}^n) \tilde{H}(\widetilde{f}(\bar{\tau}^n), \widetilde{f}(\bar{\tau}^n), \widetilde{f}(\bar{g}_0)(\zeta^n)).$$

However as $\widetilde{f} \restriction \bar{\nu}$ is cofinal into λ, we may choose $\bar{\tau}^n$ so that $\widetilde{f}(\bar{\tau}^n) > \max\{\alpha', \eta\}$. This contradicts our definition of α'. Q.E.D.(1)

(2) $p' = p_{s'}$.

By the pseudo-ultrapower construction, we have $\widetilde{M} = \widetilde{h}^{n+1}_{\widetilde{M}}(\lambda \cup p') = \widetilde{h}^{n+1}_{\widetilde{M}}(\tilde{\kappa} \cup p')$ (where $\tilde{\kappa} = \widetilde{f}(\kappa_{\bar{s}})$ if $\kappa_{\bar{s}}$ is defined) and is sound above λ (or $\tilde{\kappa}$). The solidity of $p_{\bar{s}}$ above $\bar{\nu}$ transfers via the $\Sigma_1^{(n)}$-preserving map f' to show that p' is solid above λ (see [18] 3.6.8). Then the minimality of the standard parameter and the definition of $p_{s'}$ shows that $p_{s'} \leq^* p'$. However if $p_{s'} <^* p'$ held, we should have for some $i \in \omega$, and some $\vec{\xi}$, that $p' = \widetilde{h}^{n+1}_{\widetilde{M}}(i, \langle \vec{\xi}, p_{s'} \rangle)$, and thus $p_s = \widetilde{h}^{n+1}_s(i, \langle f'(\vec{\xi}), f'(p_{s'}) \rangle)$ whence $M_s = \widetilde{h}^{n+1}_s(\nu \cup f'(p_{s'}))$. This is a contradiction as $f'(p_{s'}) <^* p_s$. Q.E.D.(2)

(3) If $\widetilde{d} =_{\mathrm{df}} \widetilde{f}(d_{\bar{s}})$ then $\widetilde{d} = d_{s'}$.

Proof: This is very similar to Lemma 2.3, using the $\Sigma_1^{(n)}$-preservation properties of $\widetilde{f}$, and is left to the reader. Q.E.D.(3)

(4) If $\widetilde{\alpha} =_{\text{df}} \widetilde{f}(\alpha_{\bar{s}})$ then $\widetilde{\alpha} = \alpha_{s'}$.
Proof: Define

$$\widetilde{H}(\xi^n, \zeta^n) \longleftrightarrow h_{s'}(\omega\xi^n \cup \{p_\lambda\}) \cap \lambda \subseteq \zeta^n \; ; \; \bar{H}(\xi^n, \zeta^n) \longleftrightarrow h_{\bar{s}}(\omega\xi^n \cup \{p_{\bar{s}}\}) \cap \bar{\nu} \subseteq \zeta^n.$$

That $\widetilde{\alpha}$ is sufficiently closed, and hence $\widetilde{\alpha} \leq \alpha_{s'}$, is proven using $\bar{H}(\alpha_{\bar{s}}, \alpha_{\bar{s}})$, and that $\bar{H}$ is $\Pi_1^{(n)\bar{M}}(\{p_{\bar{s}}\})$. As $\widetilde{f}$ is $\Sigma_1^{(n)}$-preserving and $\widetilde{H}(\xi^n, \zeta^n)$ has the same $\Pi_1^{(n)\widetilde{M}}(\{p_{s'}\})$ definition, we conclude $\widetilde{H}(\alpha_{s'}, \alpha_{s'})$. Conversely for $\widetilde{\alpha} < \eta^n < \lambda$ we set $\bar{\eta} = f^{-1}``\eta^n$. Then we have $\neg\bar{H}(\bar{\eta}, \bar{\eta})$ (as $\bar{\eta} > \alpha_{\bar{s}}$). Hence for some $i \in \omega$, and for some $\bar{\xi} < \bar{\eta}$ we have $\bar{\eta} \leq h_{\bar{s}}(i, \langle \bar{\xi}, p_{\bar{s}} \rangle) < \bar{\nu}$. As $f(\bar{\eta}) \geq \eta^n$ and as $\widetilde{f}$ is $\Sigma_0^{(n)}$-preserving we have $\eta^n \leq h_{s'}(i, \langle \widetilde{f}(\bar{\xi}), p_{s'} \rangle) < \lambda$. Hence $\widetilde{\alpha} \geq \alpha_{s'}$.

Q.E.D.(4)

We have shown enough now to see that we may set $f_0^* = \widetilde{f}$. Q.E.D.(Lemma)

The next lemma corresponds to Lemma 6.19 of [1] and the $(\rightarrow)$ direction is Lemma 4.6 of [11].

Lemma 3.12. *Suppose $f : \bar{s} \Longrightarrow s$. Then $\lambda(f) < \nu_s \longleftrightarrow \rho(f) < \rho_s$.*

Proof: Let $\lambda = \lambda(f)$, $\nu = \nu_s$, $n = n_s$. We use the notation of the previous proof.
$(\rightarrow)$ Suppose $\rho(f) = \rho_s$. Then the map f' is not only $\Sigma_0^{(n)}$ but is cofinal at the n'th level, and thus $\Sigma_1^{(n)}$-preserving. We also have that $f'(p_{s'}) = p_s$ and $f' \restriction \lambda = id \restriction \lambda$. This implies that

$$\nu \cap f' `` h_{s'}(\lambda \cup p_{s'}) \supseteq \nu \cap h_s(\lambda \cup p_s) = \lambda.$$

If this last equality held, and were $\lambda < \nu$, we should have by definition that $\lambda \leq \alpha_s$; this would contradict the fact that α_s is in $\text{ran}(f)$.
$(\leftarrow)$ Suppose $\rho(f) < \rho_s$ but $\lambda = \nu$. Choose a good $\Sigma_1^{(n)}(M_{\bar{s}})$ function, $\overline{F}$ say, singularising $\bar{\nu}$ definably in some parameter $\overline{q}$. Suppose $\varphi(x^n, y^n, w) \equiv \exists z^n \psi(x^n, y^n, z^n, w)$ is a functionally absolute definition for $\overline{F}$ with ψ a $\Sigma_0^{(n)}$ formula satisfying $\overline{F}(a) = b \leftrightarrow M_{\bar{s}} \models \exists z^n \psi(a, b, z^n, \overline{q})$.

The formula $\tilde{\varphi} \equiv \exists z^n \in M|\rho(f)\ \psi(x^n, y^n, z^n, f^*(\overline{q}))$ now defines a function F since $\tilde{\varphi}(a, b, f^*(\overline{q})) \Rightarrow \varphi(a, b, f^*(\overline{q}))$ holds and φ is functionally absolute. As f^* is $\Sigma_0^{(n)}$-preserving we conclude that $\overline{F}(a) = b \Rightarrow F(f^*(a)) = f^*(b)$. F is then a function singularising $\nu = \lambda$ but by virtue of $\tilde{\varphi}$ its graph is a $\boldsymbol{\Sigma}_0^{(n)}(M_s)$ subset of $\rho(f) < \rho$, and hence is a member of M_s. Contradiction! Q.E.D.

The construction of the C_s-sets attached to $s = (\nu_s, M_s)$ will follow in essence the construction in [16]. The main point is that we can give an estimate to the order types of the C_s sets.

We may state immediately what the C_s-sets for $s = (\nu_s, M_s) \in S^+$ will be:

Definition 3.13. *Let $s \in S^+$; $C_s^+ =_{\text{df}} \{\lambda(f) \mid \exists \bar{s}\, f : \bar{s} \Longrightarrow s\}$; $C_s =_{\text{df}} C_s^+ \backslash \{\nu_s\}$.*

Definition 3.14. *Let $f : \bar{s} \Longrightarrow s$. Then $\beta(f) =_{\text{df}} \max\{\beta \leq \nu_{\bar{s}} \mid f \restriction \beta = \text{id} \restriction \beta\}$.*

Elementary closure considerations show that $\beta(f)$ is defined, and is essentially the critical point of the embedding f or else is $\nu_{\bar{s}}$ The next lemma corresponds to Lemma 6.22(b) of [1].

Lemma 3.15. *Let $f : \bar{s} \Longrightarrow s$. Set $\beta = \beta(f)$. (i) $\beta = \nu_s$ iff $f = \mathrm{id}_{\bar{s}}$ iff $f(\beta) \not> \beta$.*
(ii) β is regular in $M_{\bar{s}}$.

Proof: (i) Suppose $f : \bar{s} \Longrightarrow s$. Then $\nu_{\bar{s}} \leq \nu_s$. If $\beta = \nu_s$ then $\nu_s = f(\nu_{\bar{s}}) = \nu_{\bar{s}}$ and so $f \restriction \nu_s = id \restriction \nu_s$ and $\operatorname{ran} f^* = \tilde{h}_s(\omega\nu_s \cup \{p(s)\}) = M_s$. So $f^* = id_{\bar{s}} = id_s$ and $M_{\bar{s}} = M_s$. Suppose $f^* = id_s$; then $f(\beta) = id(\beta) \not> \beta$. Lastly if $f(\beta) = \beta$ and $\beta \leq \nu_{\bar{s}}$ trivially by definition of β implies $f = id_{\bar{s}}$.

(ii) Note that if $\beta < \nu_{\bar{s}}$ then it is the critical point of the embedding f and is regular in $M_{\bar{s}}$ by a standard argument. If $\beta = \nu_{\bar{s}}$ then it is regular in $M_{\bar{s}}$ by the latter's definition. QED

The next lemma lists some properties of $f_{(\gamma,q,s)}$ which were given at Definition 3.7. It crucially depends on Lemma 3.9, and corresponds to Lemma 6.23 of [1]. Firstly a *minimality property* of $f_{(\gamma,q,s)}$.

Lemma 3.16. *(i) If $\gamma \leq \nu_s$ then $f_{(\gamma,q,s)}$ is the least f such that $f \restriction \gamma = \mathrm{id} \restriction \gamma$ with $q, p(s) \in \operatorname{ran}(f^*)$, in that if g is any other such with these two properties, (meaning that if $g \Longrightarrow s$ with extension g^* is so that $\gamma \cup \{q, p(s)\} \subseteq \operatorname{ran}(g^*)$) then $g^{-1} f_{(\gamma,q,\nu_s)} \in \mathbb{F}$.*

(ii) $f_{(\gamma,q,s)} = f_{(\beta,q,s)}$ where $\beta = \beta(f_{(\gamma,q,s)})$.

(iii) $f_{(\nu,0,s)} = \mathrm{id}_s$;

(iv) Let $f : \bar{s} \Longrightarrow s$ with $\bar{\gamma} \leq \nu_{\bar{s}}, f``\bar{\gamma} \subseteq \gamma \leq \nu_s, \bar{q} \in J_{\bar{s}}, f^*(\bar{q}) = q$, then $\operatorname{ran}(f^* f^*_{(\bar{\gamma},\bar{q},\bar{s})}) \subseteq \operatorname{ran}(f^*_{(\gamma,q,s)})$.

With (i) this implies: if $\beta(f) \geq \bar{\gamma}$ then $f f_{(\bar{\gamma},\bar{q},\bar{s})} = f_{(\bar{\gamma},q,s)}$.

(v) Set $g = f_{(\gamma,q,s)}$; $\lambda = \lambda(g)$ and $g_0 = \mathrm{red}(g)$. Then $q \in J_{s|\lambda}$ and $g_0 = f_{(\gamma,q,s|\lambda)}$.

Proof: *(i) - (iv)* are easy consequences of the definitions. (For *(i)* note this makes sense since we have specified in effect that $\operatorname{ran}(g^*) \supseteq \operatorname{ran}(f_{(\gamma,q,s)})$.) We establish *(v)*. We know that $g_0 \Longrightarrow s|\lambda$. Set $g_0' = f_{(\gamma,q,s|\lambda)}$ and we shall argue that $g_0 = g_0'$. Let $k = g_0^{-1} g_0'$. The argument of Lemma 3.11 shows that $d(g_0) = d(g)$; as $g_0 \restriction \gamma = \mathrm{id} \restriction \gamma$, and $q \in \operatorname{ran}(g_0)$ then by *(i)* the minimality of $g_0' \Longrightarrow s|\lambda$ implies we have such a k defined. Thus $k \in \mathbb{F}$. But $k \Longrightarrow d(g_0)$ so we conclude, as $d(g_0) = d(g)$, that $gk \in \mathbb{F}$. But $\operatorname{ran}((gk)^*) \cap \lambda = \operatorname{ran}(g^*) \cap \lambda$. So, using that $gk \restriction \gamma = \mathrm{id} \restriction \gamma$, and $q, p(s) \in \operatorname{ran}(gk)$, and then *(i)* again, we have $(gk)^{-1} g = k^{-1} \in \mathbb{F}$. Hence $k = \mathrm{id}_{d(g_0')}$ and thus $g_0 = g_0'$. Q.E.D.

Our definitions are preserved through $\Longrightarrow$ when a map f is *cofinal*, meaning that $|f|$ is cofinal into $r(f)$ (as is assumed in the next lemma which corresponds to 6.24 of [1]):

Lemma 3.17. *Let $f : \bar{s} \Longrightarrow s$ with $\lambda(f) = \nu_s$. Set $\bar{\nu} = \nu_{\bar{s}}$, $\nu = \nu_s$, and let $\bar{\gamma} < \bar{\nu}, \gamma = f(\bar{\gamma}), \bar{q} \in J_{\bar{s}}, f(\bar{q}) = q$. Set*
$\bar{g} = f_{(\bar{\gamma},\bar{q},\bar{s})}$; $g = f_{(\gamma,q,s)}$. Then

(i) $\lambda(\bar{g}) < \bar{\nu} \longleftrightarrow \lambda(g) < \nu$;

(ii) If $\lambda(\bar{g}) < \bar{\nu}$ then $f(\lambda(\bar{g})) = \lambda(g)$ and $f(\beta(\bar{g})) = \beta(g)$.

Proof: Assume $\lambda(\bar{g}) < \bar{\nu}$. Set $\overline{h} = \tilde{h}_{\bar{s}}$, $\lambda' = f(\lambda(\bar{g}))$. The following is $\Pi_1^{(n)M_{\bar{s}}}(\{\lambda(\bar{g}), \bar{\gamma}, p(\bar{s})\})$:

$$\forall x^n \forall \xi^n < \bar{\gamma} \forall i < \omega (x^n = \overline{h}(i, \langle \xi^n, \bar{q}, p(\bar{s}) \rangle) \wedge x^n < \bar{\nu} \longrightarrow x^n < \lambda(\bar{g}));$$

if $\bar\nu = \mathrm{On} \cap M_{\bar s}$ then we drop the conjunct $x^n < \bar\nu$. Then

$$\forall x^n \forall \xi^n < \gamma \forall i < \omega (x^n = \tilde h_s(i, \langle \xi^n, q, p(s)\rangle) \wedge x^n < \nu \longrightarrow x^n < \lambda')$$

as f is $\Pi_1^{(n)}$-preserving. Hence $\lambda' \geq \lambda(g)$.

Claim 1: $\lambda' \leq \lambda(g)$.

As $\lambda(\bar g) < \bar\nu$ we have $\omega\rho(\bar g) < \omega\rho_{\bar s}$ by Lemma 3.12. Hence if we set $A = A^{n, p_{\bar s} \restriction n}_{M_{\bar s}}$, and $\bar N = \langle J^A_{\rho(\bar g)}, A \cap J_{\rho(\bar g)}\rangle$ we have that $\overline{N} \in M_{\bar s}$ and is an amenable structure, with

$$\lambda(\bar g) = \sup(\bar\nu \cap h_{\overline{N}}(\bar\gamma \cup \{\bar q, p(\bar s) \cap \omega\rho_{\bar s}\}).$$

Applying f^*, and with $N = f(\overline{N})$, we have

$$\lambda' = \sup(\nu \cap h_N(\gamma \cup \{q, p(s) \cap \omega\rho_\nu\}).$$

For amenable structures (such as N) we have a uniform definition of the canonical $\Sigma_1(N)$ Skolem function h_N. From $\langle N, A_N\rangle \subseteq \langle M^n_s, A^n_s\rangle$, we have that $h_N \subseteq h_s$, and thus

$$\lambda' = \sup(\nu \cap h_s(\gamma \cup \{q, p(s) \cap \omega\rho_s\})) = \sup(\nu \cap \tilde h_s(\gamma \cup \{q, p(s)\})).$$

Thus $\lambda' \leq \lambda(g)$ and *Claim 1* is finished.

Claim 2 $f(\beta(\bar g)) = \beta(g)$.

Let $\beta = f(\beta(\bar g))$; as $\bar g = f_{(\beta(\bar g), \bar q, \bar s)}$ we have $\beta(\bar g) \notin \mathrm{ran}(\bar g)$.

$$\beta = f(\beta(\bar g)) = f(\sup\{\bar\delta < \bar\nu \mid \bar\delta \subseteq \mathrm{ran}(\bar g)\}) = f(\sup\{\bar\delta < \bar\nu \mid \bar\delta \subseteq h_{\overline{N}}(\bar\delta \cup \{\bar q, p(\bar s) \cap \omega\rho\})\}) =$$

$$= \sup\{\delta < \nu \mid \delta \subseteq h_{\bar N}(\delta \cup \{q, p(s) \cap \omega\rho_s\})\}.$$

By the above $\beta \leq \sup\{\delta < \nu \mid \delta \subseteq h_\nu(\delta \cup \{q, p(s) \cap \omega\rho_s\}) = \beta(g)$. Suppose however $\beta < \beta(g)$. Then in M_s we have:

$$\forall \beta^n \leq \beta \exists \xi^n < \gamma \exists i < \omega (\beta^n = \tilde h_s(i, \langle \xi, q, p(s)\rangle)).$$

However f is $\Sigma_1^{(n)}$-preserving, so this goes down to $M_{\bar s}$ as:

$$\forall \bar\beta^n \leq \beta(\bar g) \exists \bar\xi^n < \bar\gamma \exists i < \omega (\bar\beta^n = \tilde h_{\bar s}(i, \langle \bar\xi^n, \bar q, p(\bar s)\rangle)).$$

But this, with $\bar\beta^n \leq \beta(\bar g)$ implies $\beta(\bar g) \in \mathrm{ran}(\bar g)$ which is a contradiction! This finishes *Claim 2* and *(ii)*. Finally, just note for $(\longleftarrow)$ of *(i)* that as $\rho(f) = \rho_s$, if $\lambda(g) < \nu$ then by Lemma 3.12 there is $\eta = f(\bar\eta) < \rho(f)$ with $\tilde h_s(\gamma \cup \{q, p(s)\}) \cap \omega\rho_s \subseteq \eta$. This $\Pi_1^{(n)}$ statement goes down to $M_{\bar s}$ as $\tilde h_{\bar s}(\bar\gamma \cup \{\bar q, p(\bar s)\}) \cap \omega\rho_{\bar s} \subseteq \bar\eta$. Hence $\lambda(\bar g) < \lambda$. Q.E.D.

From this point onwards in the proof up until Lemma 3.35 we are very much following, almost verbatim, the development of [1] Lemmata 6.25 - 6.34 or correspondingly in [16] Lemmata 2.20-2.37: the fine structural arguments specific to our level of mice have all been dealt with, and the rest is very much combinatorial reasoning that is common to whatever model we are trying to define a $\square$ sequence for.

Definition 3.18. *Let $s = \langle \nu_s, M_s \rangle \in S^+, q \in J_s$. $B(q,s) =_{\rm df} B^+(q,s) \backslash \{\nu_s\}$ where*

$$B^+(q,s) =_{\rm df} \{\beta(f_{(\gamma,q,s)}) \mid \gamma \leq \nu_s\}.$$

$B(q,s)$ is thus the set of those $\beta < \nu_s$ so that $\beta = \beta(f)$ where $f = f_{(\beta,q,s)}$. (Recall that B^* is always the class of limit points of B for any set $B \subseteq \mathrm{On}$.)

Lemma 3.19. *Let f abbreviate $f_{(\gamma,q,s)}$. Assume $q \in J_s$.*
(i) Suppose $\gamma \in B(q,s)^*$. Then $\mathrm{ran}(f) = \bigcup_{\beta \in B(q,s) \cap \gamma} \mathrm{ran}(f_{(\beta,q,s)})$.
(ii) Let $\bar{s}$ be such that $f : \bar{s} \Longrightarrow s$ with $f(\bar{q}) = q$, and with $\gamma \leq \nu_{\bar{s}}$. Then $\gamma \cap B(q,s) = B(\bar{q},\bar{s})$.
(iii) Let $\lambda = \lambda(f)$; let $\gamma \leq \nu_{\bar{s}}$. Then $\gamma \cap B(q,s|\lambda) = \gamma \cap B(q,s)$.

Proof: *(i)* is clear; *(ii)* follows from Lemma 3.16*(iv)*, and *(iii)* from *(ii)* and Lemma 3.16*(v)*.
Q.E.D.

Definition 3.20. *Let $s \in S^+, q \in J_s$. Set:*

$$\Lambda^+(q,s) =_{\rm df} \{\lambda(f_{(\gamma,q,s)}) \mid \gamma \leq \nu_s\}; \qquad \Lambda(q,s) =_{\rm df} \Lambda^+(q,s) \backslash \{\nu_s\}.$$

The sets $\Lambda(q,s) \subseteq C_s$ are first approximations to C_s if q is allowed to vary. We first analyse these sets.

Lemma 3.21. *Let $s \in S^+, q \in J_s$. (i) $\Lambda(q,s)$ is closed below ν_s; (ii) $\mathrm{ot}(\Lambda(q,s)) \leq \nu_s$; (iii) if $\lambda \in \Lambda(q,s)$ then $q \in J_{s|\lambda}$ and $\Lambda(q,s|\lambda) = \lambda \cap \Lambda(q,s)$.*

Proof: Set $\Lambda = \Lambda(q,s)$. *(i):* Let $\eta \in \Lambda^*$. We claim that $\eta \in \Lambda^+(q,s)$. For each $\lambda \in \Lambda(q,s) \cap \eta$ pick β_λ with $\lambda(f_{(\beta_\lambda,q,s)}) = \lambda$. Now just let γ be the supremum of these β_λ. Then clearly $\lambda(f_{(\gamma,q,s)}) = \sup_\lambda \lambda(f_{(\beta_\lambda,q,s)}) = \eta$.

(ii) is obvious; *(iii)*: Let $\lambda \in \Lambda$, and let $\lambda = \lambda(g)$ where $g = f_{(\beta,q,s)}$, where we take $\beta = \beta(g)$. Suppose $g : \bar{s} \Longrightarrow s$. Let $g(\overline{q}) = q$ and set $g_0 = \mathrm{red}(g)$. Then by Lemma 3.16*(v)* $g_0 = f_{(\beta,q,s|\lambda)}))$. If $\gamma \geq \beta$ then $\lambda = \lambda(f_{(\gamma,q,s|\lambda)})$. If $\gamma \leq \beta$ then

$$|f_{(\gamma,q,s|\lambda)})| = |g_0||f_{(\gamma,\bar{q},\bar{s})}| = |g||f_{(\gamma,\bar{q},\bar{s})}| = |f_{(\gamma,q,s)}|.$$

where the first equality is justified by Lemma 3.16*(iv)*. Whence $\lambda(f_{(\gamma,q,s|\lambda)}) = \lambda(f_{(\gamma,q,s)})$ and (iii) holds.
Q.E.D.

Lemma 3.22. *If $f : \bar{s} \Longrightarrow s$, $\mu = \lambda(f), \overline{q} \in J_{\bar{s}}, f(\overline{q}) = q$, then:*
(i) $\Lambda(\overline{q},\bar{s}) = \varnothing \longrightarrow \mu \cap \Lambda(q,s) = \varnothing$,
(ii) f“$\Lambda(\overline{q},\bar{s}) \subseteq \Lambda(q,s|\mu)$,
(iii) If $\overline{\lambda} = \max \Lambda(\overline{q},\bar{s})$ and $\lambda = f(\overline{\lambda})$ then $\lambda = \max(\mu \cap \Lambda(q,s))$.

Proof: *(i)* By its definition, if $\Lambda(\overline{q},\bar{s}) = \varnothing$ then $f_{(0,\overline{q},\bar{s})}$ is cofinal into $\bar{\nu}$. Hence $\mathrm{ran}(f f_{(0,\overline{q},\bar{s})})$ is both cofinal in μ, and contained in $\mathrm{ran}(f_{(0,q,s)})$ by Lemma 3.16*(iv)*, thus $\mu \cap \Lambda(q,s) = \varnothing$. This finishes *(i)*. Note that By 3.21*(iii)* $\Lambda(q,s|\mu) = \mu \cap \Lambda(q,s)$. Let $f_0 = \mathrm{red}(f)$.

(ii) Let $\overline{\lambda} = \lambda(f_{(\bar{\beta},\overline{q},\bar{s})}) \in \Lambda(\overline{q},\bar{s})$, and let $f(\overline{\beta},\overline{\lambda}) = \beta, \lambda = f_0(\overline{\beta},\overline{\lambda})$. Then by Lemma 3.17(ii)

$$f_0(\lambda(f_{(\bar{\beta},\bar{q},\bar{s})})) = \lambda(f_{(\beta,q,s|\mu)}) \in \Lambda(q,s|\mu).$$

(iii) Let $\overline{\beta} = \sup\{\gamma | \lambda(f_{(\gamma,\bar{q},\bar{s})}) \leq \overline{\lambda}\}$. Then $\lambda(f_{(\bar{\beta},\bar{q},\bar{s})}) = \overline{\lambda}$, and by the assumed maximality of $\overline{\beta}$ we have $\lambda(f_{\bar{\beta}+1,\bar{q},\bar{s}}) = \bar{\nu}$. Set $\beta = f(\bar{\beta}) = f_0(\bar{\beta})$, then by Lemma 3.17(ii):

$$\lambda = f_0(\bar{\lambda}) = \lambda(f_{\beta,q,s|\mu}).$$

However $\lambda(f_{\beta+1,q,s|\mu}) \geq \mu$, since, again by Lemma 3.16*(iv)*,

$$\mathrm{ran}(f_0 f_{(\bar{\beta}+1,\bar{q},\bar{s})}) \subseteq \mathrm{ran}(f_{(\beta+1,q,s|\mu)}).$$

Thus $\lambda = \max(\Lambda(q, s|\mu)) = \max(\mu \cap \Lambda(q, s))$, the latter equality being by the last lemma.

Q.E.D.

The p.r. definitions of $\lambda(f)$, $B(q, s)$, $\Lambda(q, s)$, are uniform in the appropriate parameters. If $s = \langle \mu, M_\mu \rangle \in S^+$, then we may define:

$$F_s = \{f_{(\gamma,q,s|\nu)} \mid \nu \in S \cap \mu, q \in J_{s|\nu}, \gamma \leq \nu\};$$

$$E_s = \{\langle \nu, M_{s|\nu}, p(s|\nu), \tilde{h}_{s|\nu} \rangle \mid \nu \in S \cap \mu\};$$

$$G_s = \{\langle\langle s|\nu, q\rangle, \Lambda(q, s|\nu\rangle \mid q \in J_{s|\nu}, \nu \in S \cap \mu\}.$$

We then have:

Lemma 3.23. *(i) E_s, F_s, G_s are uniformly $\Delta_1(J_s)$ for $s \in S^+$;*
(ii) $\mu' < \mu \Longrightarrow E_{\mu'}, F_{\mu'}, G_{\mu'} \in J_s$.

Lemma 3.24. *Let $f : \bar{s} \Longrightarrow s$ with $\overline{q} \in J_{\bar{s}}, f(\overline{q}) = q$. Then*
(i) If f is cofinal into ν_s then $|f| : \langle J_{\bar{s}}, \Lambda(\bar{q}, \bar{s})\rangle \longrightarrow_{\Sigma_1} \langle J_s, \Lambda(q, s)\rangle$;
(ii) Otherwise: $|f| : \langle J_{\bar{s}}, \Lambda(\bar{q}, \bar{s})\rangle \longrightarrow_{\Sigma_0} \langle J_s, \Lambda(q, s)\rangle$

Proof: *(i)* It suffices to show that $|f|(\Lambda(\bar{q}, \bar{s}) \cap \overline{\tau}) = \Lambda(q, s) \cap f(\tau)$ for arbitrarily large $\overline{\tau} < \nu_{\bar{s}}$. This will follow from the last lemma and 3.22.

However, if $\bar{\lambda} \in \Lambda(\bar{q}, \bar{s})$, then $\Lambda(\bar{q}, \bar{s}) \cap \bar{\lambda} = \Lambda(\bar{q}, \bar{s}|\bar{\lambda})$ by Lemma 3.21, and by the last lemma, if $f(\bar{\lambda}) = \lambda$, we have

$$f(\Lambda(\bar{q}, \bar{s}|\bar{\lambda})) = \Lambda(q, s|\lambda) = \lambda \cap \Lambda(q, s)$$

(with the latter equality by Lemma 3.21 again). If $\Lambda(\bar{q}, \bar{s})$ is unbounded in $\nu_{\bar{s}}$, this suffices; if it is empty or bounded, then the Lemma 3.22 takes care of these cases.

For non-cofinal maps *(ii)* we still have, if $\lambda(f) = \mu$, that

$$|f_0| : \langle J_{\bar{s}}, \Lambda(\bar{q}, \bar{s})\rangle \longrightarrow_{\Sigma_1} \langle J_{s|\mu}, \Lambda(q, s|\mu)\rangle$$

where $f_0 = \mathrm{red}(f)$. But $\Lambda(q, s|\mu) = \mu \cap \Lambda(q, s)$, and $|f_0| = |f|$. Q.E.D.

The C_s sets may be decomposed into a finite sequence of sets of the form $\Lambda(l^i_s, s)$.

Definition 3.25. *Let $s \in S^+, \eta \leq \nu_s$. $l^i_{\eta s} < \nu_s$ is defined for $i < m_{\eta s} \leq \omega$ by induction on i :*

$$l^0_{\eta s} = 0; \quad l^{i+1}_{\eta s} \simeq \max(\eta \cap \Lambda(l^i_{\eta s}, s)$$

We also write l^i for $l^i_{\eta s}$ if the context is clear; also we set $l^i_s \simeq l^i_{\nu_s s}$; $m_s = m_{\nu_s s}$.

Some facts about this definition may be easily checked:
Fact
• $l^i_{\eta s} \leq l^{i+1}_{\eta s}$ $(i+1 < m_{\eta s})$ is monotone
• $i > 0 \longrightarrow l^i_{\eta s} \in \eta \cap C_s$.
• Let $l^i_{\eta s}$ be defined, and suppose $l^i_{\eta s} < \mu \leq \eta$. Then $l^i_{\eta s} = l^i_{\mu s}$.
(The last here is by induction on i.)

Lemma 3.26. *Let $f : \bar{s} \Longrightarrow s$.*
(i) If $\lambda = \lambda(f)$ then $l^i_{\lambda s} \simeq f(l^i_{\bar{s}})$;
(ii) let $\overline{\eta} < \nu_{\bar{s}}, f(\overline{\eta}) = \eta$; then $l^i_{\eta s} \simeq f(l^i_{\overline{\eta}\bar{s}})$.

Proof *(i)* By induction on i. If $i = 0$ this is trivial. Suppose $i = j + 1$. Then, as inductive hypothesis $l^j_{\lambda s} = f(l^j_{\bar{s}})$, and thus

$$|f| : \langle J_{\bar{s}}, \Lambda(l^j_{\bar{s}}, \bar{s})\rangle \longrightarrow_{\Sigma_1} \langle J_{s|\lambda}, \Lambda(l^j_{\lambda s}, s|\lambda)\rangle,$$

by the last lemma, as $|\operatorname{red}(f)| = |f|$. However

$$\Lambda(l^j_{\lambda s}, s|\lambda) = \lambda \cap \Lambda(l^j_{\lambda s}, s),$$

by 3.21. Hence:

$$f(l^i_{\bar{s}}) \simeq f(\max \Lambda(l^j_{\bar{s}}, \bar{s})) \simeq \max(\lambda \cap \Lambda(l^j_{\lambda s}, s)) \simeq l^i_{\lambda s}$$

with the middle equality holding by Lemma 3.22(iii). *(ii)* is proved similarly.

Q.E.D.

Corollary 3.27. *(i) Let $f : \bar{s} \Longrightarrow s$ cofinally. Then $l^i_s \simeq f(l^i_{\bar{s}})$.*
(ii) Let $\lambda \in C_s$. Then $l^i_{\lambda s} \simeq l^i_{s|\lambda}$.

Proof *(i)* is immediate. For *(ii)* choose $f : \bar{s} \Longrightarrow s$ with $\lambda = \lambda(f)$, and set $f_0 = \operatorname{red}(f)$. Then $l^i_{\lambda s} \simeq f(l^i_{\bar{s}}) \simeq f_0(l^i_{\bar{s}}) \simeq l^i_{s|\lambda}$ with the last equality holding from *(i)*. Q.E.D.

Lemma 3.28. *Let $\eta \leq \nu, \lambda = \min(C^+_s \backslash \eta)$. Then $l^i_s \simeq l^i_{\lambda s} \simeq l^i_{\eta s}$ (for any $i < \omega$ for which either side is defined).*

Proof Induction on i, again $i = 0$ is trivial. Suppose $l^j_s = l^j_{\eta s} = l^j_{\lambda s}$ and $i = j + 1$. Set $l = l^j_{\eta s}$, then we have: $\Lambda(l, s) \cap \eta = \Lambda(l, s) \cap \lambda$, since $\Lambda(l, s) \subseteq C_s$ and $C_s \cap [\eta, \lambda) = \varnothing$. Suppose, without loss of generality that $l^i_{\eta s}$ is defined. Then

$$l^i_{\eta s} = \max(\eta \cap \Lambda(l, s)) = \max(\lambda \cap \Lambda(l, s)) = l^i_{\lambda s} = l^i_{s|\lambda}.$$

Q.E.D.

Lemma 3.29. *Let $j \leq i < m_s$. Set $l = l^i_s$. Then $l^j_s \in \operatorname{ran}(f_{0,l,s})$.*

Proof Set $f = f_{(0,l,s)}$. Suppose $f : \bar{s} \Longrightarrow s$, and $\lambda = \lambda(f)$. Then $l^j_{\lambda s} \simeq f(l^j_{\bar{s}})$ by Lemma 3.26*(i)*. But l^j_s exists, and $l^j_s < \lambda \leq \nu_s$. Hence $l^j_s = l^j_{\lambda s} = f(l^j_{\bar{s}})$. Q.E.D.

Importantly the $\langle l^j_{\lambda s}\rangle$ sequences are finite.

Lemma 3.30. *Let $s \in S^+, \eta \leq \nu_s$. Then $m_{\eta s} < \omega$.*

Proof Suppose this fails. Then for some $\eta \leq \nu_s$ we have that $l^i_{\eta s}$ is defined for all $i < \omega$. Let $\lambda = \min(C^+_s \backslash \eta)$. Then $l^i_{\lambda s} = l^i_{\eta s}$ by Lemma 3.28. Choose $f : \bar{s} \Longrightarrow s$ with $\lambda = \lambda(f)$. Then $l^i_{\lambda s} = l^i_{s|\lambda} = f(l^i_{\bar{s}})$ for $i < \omega$ by Cor. 3.27*(ii)* & Lemma 3.26(i). Taking λ for ν_s, we assume, without loss of generality, that l^i_s is defined for $i < \omega$ for some $s \in S$. We obtain an infinite descending chain of ordinals by showing that as i increases, and with it l^i_s, the maximal β^i that must be contained in the range of any $f \Longrightarrow s$ together with l^i_s in order for $\mathrm{ran}(f)$ to be unbounded in s strictly *decreases.* This is absurd.

Set $l = l^i_s$. Define:

$$\beta^i = \beta^i_s =_{\mathrm{df}} \max\{\beta | \lambda(f_{(\beta,l,s)}) < \nu_s\}.$$

By the definition of l^{i+1}_s we have that

$$\lambda(f_{(\beta,l,s)}) < \nu_s \longleftrightarrow \lambda(f_{(\beta,l,s)}) \leq l^{i+1}_s.$$

Furthermore, by the definition of β^i :

(1) $\lambda(f_{(\beta^i,l,s)}) \leq l^{i+1}_s$;
(2) $\lambda(f_{(\beta^i+1,l,s)}) = \nu_s$.

Claim: $\beta^{i+1} < \beta^i$ for $i < \omega$.

Proof Set $f = f_{(\beta^{i+1},l^{i+1},s)}$. Then $\lambda(f) = l^{i+2}$, dropping the subscript ν. Let $f : \bar{s} \Longrightarrow s$. Then $l^j_{\bar{s}}$ exists and

$$f(l^j_{\bar{s}}) = l^j_{l^{i+1},s} = l^j_s \quad \text{for } j < i+1$$

since $l^j < l^{i+1} < \nu_s$ (with the first equality from Lemma 3.26*(i)* and (1), the second from Lemma 3.28).

(3) $\beta^i \geq \beta^{i+1}$.

Proof of (3): Suppose not, then $(\beta^i + 1) \cup \{l^i\} \subseteq \mathrm{ran}(f)$. Hence $\mathrm{ran}(f_{(\beta^i+1,l^i,s)}) \subseteq \mathrm{ran}(f)$. hence by (2), $\lambda(f) = \nu_s > l^{i+2}$. Contradiction!

(4) $\beta^i \neq \beta^{i+1}$.

Proof of (4): Suppose not. As β^{i+1} is the first ordinal moved by f we conclude that $f(\beta^i) > \beta^i$. Set $g = f_{(\beta^i,l,s)}, \bar{g} = f_{(\beta^i,\bar{l},\bar{s})}$ where $\bar{l} = l^i_{\bar{s}}$. Then $g = f\bar{g}$, since $f \restriction \beta^i = \mathrm{id}, f(\bar{l}) = l (= l^i_s)$. Hence

$$l^{i+1} = \lambda(g) = \lambda(f\bar{g}) < l^{i+2} = \lambda(f).$$

Hence $\lambda(\bar{g}) < \nu_{\bar{s}}$. Now we set: $g' = f_{(f(\beta^i),l,s)}$ and $g_0 = f_{(\beta^i,l,s|l^{i+2})}$. If further $f_0 = \mathrm{red}(f)$, then we have also $g_0 = f_0\bar{g}$ by 3.16*(iv)*. As $l^{i+1} = \lambda(g) < l^{i+2}$, Lemma 3.17*(ii)* applies and:

$$f(\beta(\bar{g})) = f_0(\beta(\bar{g})) = \beta(g_0) = \beta(g) = \beta^i.$$

Hence $\beta^i \in \mathrm{ran}(f)$ which is a contradiction. This proves the *Claim* and hence the Lemma.Q.E.D.

We now set $l_{\eta s} = l^{m-1}_{\eta s}$, where $m = m_{\eta s}$. Again we write l_s for $l_{\nu_s s}$. Notice that then $\Lambda(l_{\eta s}, \nu_s) \cap \eta$ is either unbounded in η or is empty. We first analyze the latter case.

Lemma 3.31. *Suppose $\Lambda(l_{\eta s}, s) \cap \eta = \varnothing$. Set $l = l_{\eta s}$. Then:*
(i) $l = 0 \longrightarrow C_s \cap \eta = \varnothing$,
(ii) $l > 0 \longrightarrow l = \max(C_s \cap \eta)$,
(iii) $\eta \in C^+_s \longrightarrow \eta = \lambda(f_{(0,l,s)})$.

Proof Set $\rho = \min(C^+_s \backslash (l + 1)$.

(1) $l = l_{\rho s}$.

Proof: Set $n = m_{\eta s} - 1$. Then $l = l^n_{\eta s} < l + 1 < \eta$. Hence (by the third bullet point of the Fact after 3.26) $l = l^n_{l+1,s}$. But $\Lambda(l, s) \cap (l+1) = \varnothing$. Hence $l^{n+1}_{l+1,s}$ is undefined and $l = l_{l+1,s}$. Hence $l = l_{\rho,s}$ by Lemma 3.28. Q.E.D.(1)

(2) $\lambda(f_{(0,l,s)}) = \rho$.

Proof: Choose $f : \bar{s} \Longrightarrow s$, with $\lambda(f) = \rho$ witnessing that $\rho \in C_s$. Then, by Lemma 3.26(i), $f(l_{\bar{s}}) = l_{\rho s} = l$. Set $\bar{l} = l_{\bar{s}}$. Now note that we must have that $\lambda(f_{(0,\bar{l},\bar{s})}) = \bar{s}$. For, if this failed then

$$f(\lambda(f_{(0,\bar{l},\bar{s})})) = \lambda(f_{(0,l,s)}) < \rho$$

by Lemma 3.17 and so the latter is in $C^+_s \cap (l, \rho)$, which is absurd! Then

$$\lambda(f_{(0,l,s)}) = \lambda(ff_{(0,\bar{l},\bar{s})}) = \lambda(f) = \rho.$$

Q.E.D.(2)

From (2) and the definition of l as $l_{\eta s}$ it follows that $\rho \geq \eta$. There are thus three alternatives:
If $l = 0$ then *(i)* holds:

$$\rho = \min(C^+_s \backslash 1) = \min(C^+_s) \geq \eta.$$

If $l > 0$ then $l = \max(C_s \cap \eta)$ since

$$(C_s \cap \eta) \backslash (l + 1) \subseteq (C_s \cap \rho) \backslash (l + 1) = \varnothing$$

and thus we have *(ii)*; finally for *(iii)*:

$$\eta \in C^+_s \longrightarrow \eta = \max(C^+_s \backslash (l + 1) = \rho = \lambda(f_{(0,l,s)}).$$

Q.E.D.

We now get a characterisation of the closed sets C^+_s.

Lemma 3.32. *Let λ be an element or a limit point of C^+_s. Let $l = l_{\lambda s}$. Then there is β such that $\lambda = \lambda(f_{(\beta,l,s)})$. Hence C_s is closed in ν_s, and*

$$C^+_s = \{\lambda(f_{(\beta,\mu,s)}) \mid \beta \leq \nu_s, \mu < \nu_s\}.$$

Proof *Case 1* $\lambda \cap \Lambda(l, s) = \varnothing$

Then $C_s \cap \lambda = \varnothing$ or $l = \max(C_s \cap \lambda)$ by the last lemma. Hence λ is not a limit point of C_s^+. Hence $\lambda \in C_s^+$, and thus $\lambda = \lambda(f_{(0,l,s)})$ by *(iii)* of that lemma.

Case 2 $\lambda \cap \Lambda(l, s)$ is unbounded in λ.

Given $\mu \in \Lambda(l, s) \cap \lambda$, let β_μ be such that $\lambda(f_{(\beta_\mu,l,s)}) = \mu$. Then $\lambda(f_{(\beta,l,s)}) = \lambda$ where $\beta = \sup_\mu \beta_\mu$. The last sentence is immediate from the previous one. Q.E.D.

We remark that we have just shown that the first conjunct of (i) of Theorem 3.2 holds. We move towards proving the other clauses. The following is (iii).

Lemma 3.33. $\lambda \in C_s \longrightarrow \lambda \cap C_s = C_{s|\lambda}$.

Proof Assume inductively the result proven for all ν' with $\nu' < \nu_s$ and $s|\nu' \in S$, (that is, the lemma is proven with $s|\nu'$ replacing s) and we shall prove the lemma for ν_s by induction on λ. Let $l = l_{\lambda s}$. Hence by Cor.3.27 $l = l_{s|\lambda}$. By Lemma 3.32 $\lambda \in \Lambda(l, s)$. Set $\Lambda = \lambda \cap \Lambda(l, s)$. Then by Lemma 3.21*(ii)* $\Lambda = \Lambda(l, s|\lambda)$.

Case 1 $\Lambda = \varnothing$.

If $l = 0$, then $C_{s|\lambda} \subseteq \lambda \cap C_s = \varnothing$ (the latter by Lemma 3.31). If $l > 0$, then

$$l = l_{s|\lambda} = \max(C_{s|\lambda} \cap \lambda) = \max(C_{s|\lambda}) = l_{\lambda s} = \max(\lambda \cap C_s)$$

by the same lemma. As $l < \lambda$, we use the inductive hypothesis on λ:

$$l \cap C_s = C_{s|l} = l \cap C_{s|\lambda}$$

where the second equality is the inductive hypothesis taking $\lambda = \nu' < \nu_s$. Hence

$$C_{s|\lambda} = \lambda \cap C_s = C_{s|l} \cup \{l\}.$$

Case 2 Λ is unbounded in λ.

Then $\mu \in \Lambda \longrightarrow \mu \in C_s \cap C_{s|\lambda}$. Hence by the overall inductive hypothesis $C_{s|\mu} = \mu \cap C_{s|\lambda}$ and (as $\mu < \lambda$) $C_{s|\mu} = \mu \cap C_s$. Hence

$$C_{s|\lambda} = \lambda \cap C_s = \bigcup_{\mu \in \Lambda} C_{s|\mu}.$$

Q.E.D.

Now (i) of the Theorem follows easily:

Lemma 3.34. $\sup(C_s) < \nu_s \longrightarrow \mathrm{cf}(\nu_s) = \omega$.

Proof Let $l = \sup(C_s) = l_s$. Then $\mathrm{ran}(f_{(0,l,\nu_s)})$ is countable, and cofinal in ν_s. Q.E.D.

Lemma 3.35. *Let $f : \bar{s} \Longrightarrow s$. Then $|f| : \langle J_{\bar{s}}, C_{\bar{s}} \rangle \longrightarrow_{\Sigma_0} \langle J_s, C_s \rangle$.*

Proof: It suffices to show that for arbitrarily large $\tau < \nu_{\bar{s}}$, $|f|(C_{\bar{s}} \cap \tau) = C_s \cap |f|(\tau)$. As usual we continue to write "f" for "$|f|$". Set $l_{\bar{s}} = \bar{l}$.

Case 1 $\Lambda(\bar{l}, \nu_{\bar{s}})$ is unbounded in $C_{\bar{s}}$.

If $\bar{\lambda} \in C_{\bar{s}}$ and $\lambda = f(\bar{\lambda})$ then by 3.22 (and 3.21) $\lambda \in \Lambda(f(\bar{l}), s) \subseteq C_s$. By Lemma 3.23 we have $E_{\bar{s}|\bar{\lambda}} \in J_{\bar{s}}$ and $f(E_{\bar{s}|\overline{\lambda}}) = E_{s|\lambda}$. By Lemma 3.31

$$C_{\bar{s}|\bar{\lambda}} = \{\lambda(f_{(0,l,\bar{s})}) < \bar{\lambda} | l < \bar{\lambda}\} \in J_{\bar{s}}$$

and is uniformly Σ_0 from $E_{\bar{s}|\bar{\lambda}}$ over $J_{\bar{s}}$. Consequently $|f|(C_{\bar{s}|\overline{\lambda}}) = C_{s|\lambda}$, by Σ_1-elementarity of $|f|$. But $C_{\bar{s}|\overline{\lambda}} = \overline{\lambda} \cap C_{\bar{s}|\bar{\nu}}$, $C_{s|\lambda} = \lambda \cap C_s$.

Case 2 $\Lambda(\bar{l}, \bar{\nu}) = \varnothing$.

Let $f(\bar{l}) = l$. Then $l = l_{\lambda\nu}$ where $\lambda = \lambda(f)$. However $\lambda(f_{0,\bar{l},\bar{s}}) = \nu_{\bar{s}}$ by our case hypothesis. Thus $\lambda(f_{(0,l,s)}) = \lambda(ff_{(0,\bar{l},\bar{s})}) = \lambda$. Hence $\Lambda(l, \nu) \cap \lambda = \varnothing$. By Lemma 3.31 we are reduced to the following two subcases:

Case 2.1 $\bar{l} = l = 0$. Then, $C_{\bar{s}} = C_s \cap \lambda = \varnothing$, and so the result is trivial.

Case 2.2 $\bar{l} = \max C_{\bar{s}}$. Then $l > 0$ and thus $l = \max(C_s \cap \lambda)$. Hence for sufficiently large

$$\bar{\tau} > \bar{l}\, f(\bar{\tau} \cap C_{\bar{s}}) = f(C_{\bar{s}}) = f(C_{\bar{s}} \cap \bar{l} \cup \{\bar{l}\}) = (C_s \cap l) \cup \{l\} = C_s \cap \lambda = f(\bar{\tau}) \cap C_s.$$

Q.E.D.

We now proceed towards calculating the order types of the C_s-sequences. This is done (in a somewhat speedy manner) in [1], but the following comes from [10]. We first generalise the definition of β^i.

Definition 3.36. *For $\eta \leq \nu_s$ set :* $\beta^i_{\eta s} \simeq \max\{\beta \mid \lambda(f_{(\beta, l^i_{\eta s}, s)}) < \eta\}$.

In very close analogy to the $\beta^i = \beta^i_s$ we have parallel properties for the $\beta^i_{\eta s}$:

1. $\lambda(f_{(\beta, l^i_{\eta s}, s)}) < \nu_s \longleftrightarrow \lambda(f_{(\beta, l^i_{\eta s}, s)}) \leq l^{i+1}_{\eta s}$.

2. $\beta^i_{\eta s}$ is defined if and only if $l^{i+1}_{\eta s}$ is defined - *i.e.* $i + 1 < m_{\eta s}$.

3. $\beta^i_{\eta s} \simeq \beta^i_{\lambda s}$ if $\lambda = \min(C^+_s \backslash \eta)$. $\lambda(f_{(\beta, l^i_{\eta s}, s)}) < \eta \longleftrightarrow \lambda(f_{(\beta, l^i_{\eta s}, s)}) < \lambda$.

4. $\beta^{i+1}_{\eta s} < \beta^i_{\eta s}$ when defined. (By the same argument as for $\beta^{i+1} < \beta^i$.)

Now we set $b_\eta = b_{\eta s} =_{\text{df}} \{\beta^i_{\eta s} \mid i + 1 < m_{\eta s}\}$. For $\eta \in C_s$ we then set $d_\eta = d_{\eta s} =_{\text{df}} b_{\eta^+ s}$ where $\eta^+ = \min(C^+_s \backslash (\eta + 1))$. The subscript s on ordinals remains unaltered throughout the rest of the argument so we shall drop it. Then we have:

5. Let $\eta \in C_s$, with $l^i_{\eta^+} < \eta$. Then by induction on i: $l^i_{\eta^+} = l^i_\eta$.

6. Let $\eta \in C_s$, with $l^i_{\eta^+} < \eta$ then:

$$l^{i+1}_{\eta^+} = \eta \text{ if } \eta \in \Lambda(l^i_\eta, s), \text{ and equals } l^{i+1}_\eta \text{ otherwise .}$$

Proof of 6: $l^i_{\eta^+} = l^i_\eta$ by 5. If $\eta \in \Lambda(l^i_\eta, s)$ then η is maximal in this set below η^+. So the first alternative holds. If $\eta \notin \Lambda(l^i_\eta, s)$ note that $i \neq m_{\eta s} - 1$ (otherwise by Lemma 3.32 for some β, $\eta = \lambda(f_{(\beta, l^i_{\eta s}, s)}) \in \Lambda(l^i_\eta, s)$). Thus l^{i+1}_η is defined and $l^{i+1}_{\eta^+}$ must equal this.

Lemma 3.37. *Let $\eta, \mu \in C_s$, with $\eta < \mu$. Then $d_\eta <^* d_\mu$.*

Proof Let $\eta^+ = \min(C^+_s \backslash (\eta + 1))$, $\mu^+ = \min(C^+_s \backslash (\mu + 1))$. Let i be maximal so that $l^i_{\mu^+} = l^i_{\eta^+}$.Then $\beta^j_{\mu^+} = \beta^j_{\eta^+}$ for $j < i$. As $l^i_{\mu^+} \leq \eta < \mu$, we have by 6. above that $l^{i+1}_{\mu^+}$ is defined and $l^{i+1}_{\mu^+} = \mu$ or l^{i+1}_μ. Moreover then $\beta^i_{\mu^+}$ is defined, and by maximality of i, $l^{i+1}_{\eta^+} \neq l^{i+1}_{\mu^+}$.

Claim $l^{i+1}_{\eta^+} < l^{i+1}_{\mu^+}$.

That $l^{i+1}_{\mu^+} < \eta^+$ is ruled out: otherwise $l^{i+1}_{\eta^+} = l^{i+1}_{\mu^+}$ again. So $l^{i+1}_{\eta^+} < \eta^+ \leq l^{i+1}_{\mu^+}$.

Q.E.D. *Claim.*

As $\beta^i_{\mu^+}$ is defined, if $\beta^i_{\eta^+}$ is undefined, then we'd be finished. Otherwise set $l = l^i_{\mu^+} = l^i_{\eta^+}$. Then

$$\lambda(f_{(\beta^i_{\eta^+}, l, s)}) = l^{i+1}_{\eta^+} \text{ and } \lambda(f_{(\beta^i_{\mu^+}, l, s)}) = l^{i+1}_{\mu^+}.$$

Hence $\beta^i_{\eta^+} < \beta^i_{\mu^+}$ and thus $d_\eta <^* d_\mu$ as required. Q.E.D.

Lemma 3.38. *Let α be p.r. closed so that for some $\alpha_0 < \alpha$ $\lambda(f_{(\alpha_0, 0, s)}) = \nu_s$. Then* $\mathrm{ot}(C_s) < \alpha$.

Proof: First note that p.r. closure implies $\mathrm{ot}(\langle [\alpha]^{<\omega}, <^* \rangle) = \alpha$. Let $\alpha_0 < \alpha$ be least with the property that $\lambda(f_{(\alpha_0, 0, s)}) = \nu_s$. Then $\{\beta^i_{\eta s} \mid \eta \leq \nu_s, i + 1 < m_{\eta s}\} \subseteq \alpha_0$. Thus

$$\mathrm{ot}\langle \{d_\eta \mid \eta \in C_s\}, <^* \rangle \leq \mathrm{ot}(\langle [\alpha_0]^{<\omega}, <^* \rangle) < \alpha.$$

Thus $\mathrm{ot}(C_s) < \alpha$. Q.E.D.

To obtain the requisite $\langle C_\nu \mid \nu \in S \rangle$ for a global sequence in K, we assign the appropriate level $K_{\beta(\nu)}$ as M_s over which ν is definably singularised in K. Then $s = \langle \nu, K_{\beta(\nu)} \rangle \in S^+$.

This completes the proof of Theorem 3.2 on the existence of a global $\Box$.

4. Obtaining Inner Models with measurable cardinals

We assume that we have a global $\Box$ sequence $\langle C_\nu \mid \nu \in S \rangle$ in K constructed as in the last section. We have:

Theorem 4.1. *Assume $n > 3$ and $\{\alpha < \omega_n \mid \alpha \in \mathrm{Cof}(\omega_{n-2}) \cap K\text{-Sing}\}$ is, in V, stationary below ω_n. Then*

$$T_n =_{\mathrm{df}} \{\beta \in \mathrm{Cof}(\omega_1) \cap \omega_n \mid \mathrm{ot}(C_\beta) \geq \omega_{n-3}\}$$

is stationary in ω_n.

Proof Let $C \subseteq \omega_n$ be an arbitrary closed and unbounded set in ω_n. Take $\gamma \in C^* \cap \mathrm{Cof}(\omega_{n-2})$ with γ a K-singular; in other words with C_γ defined. As $\mathrm{cf}(\gamma) > \omega$, C_γ is cub in γ. Then $C \cap C_\gamma$ is closed unbounded in γ of ordertype $\geq \omega_{n-2}$. Take $\beta \in (C \cap C_\gamma)^*$ such that $cf(\beta) = \omega_1$ and $\mathrm{ot}(C \cap C_\gamma \cap \beta) \geq \omega_{n-3}$. By the coherency property 3.1(c), $C_\beta = C_\gamma \cap \beta$. Thus $\beta \in C \cap T_n \neq \emptyset$. $\square$

Note that $(T_n)_{3<n<\omega}$ as above would be a sequence of sets to which we could apply the MS-principle, if we knew that they were (in V) stationary beneath the relevant $\aleph_n$. This is what the assumption in the above theorem achieves. The following is essentially our main Theorem 1.5.

Theorem 4.2. *If* $\mathrm{MS}((\aleph_n)_{1<n<\omega}, \omega_1)$ *holds then there exists* $k < \omega$ *so that for all* $n > k$, *there is* D_n*, closed and unbounded in* ω_n*, so that*

$$D_n \cap \mathrm{Cof}(\omega_{n-2}) \subseteq \{\alpha < \omega_n \mid o^K(\alpha) \geq \omega_{n-2}\}.$$

Proof: We suppose not. Then for arbitrarily large $n < \omega$:

$$S^0_n =_{\mathrm{df}} \{\alpha < \omega_n \mid \alpha \in \mathrm{Cof}(\omega_{n-2}) \wedge \mathrm{Sing}^K(\alpha)\}$$

is stationary in ω_n by appealing to Mitchell's Weak Covering Lemma for K, 1.8.

We shall define a sequence $(S_n)_{1<n<\omega}$ of stationary sets. By Theorem 4.1, for arbitrarily large $n < \omega$, T_n is stationary in ω_n; for such n (which we shall call *relevant*) let $S_n = T_n$; for all other $n > 1$ take $S_n = \mathrm{Cof}(\omega_1) \cap \omega_n$.

Define the first-order structure $\mathfrak{A} = (H_{\omega_{\omega+1}}, K_{\omega_{\omega+1}}, \in, \lhd, \langle f_n \rangle_{n<\omega}, \cdots)$ with a wellordering $\lhd$ of the domain of $\mathfrak{A}$, and the sequence of finitary functions f_n including a complete set of Skolem functions for $\mathfrak{A}$. The mutual stationarity property yields some $X \prec H_{\omega_{\omega+1}}$ such that

$$\{\omega_n \mid n \leq \omega\} \subseteq X, \qquad \forall n > 2\, (\sup X \cap \omega_n) \in S_n, \text{ and } \omega_2 \subseteq X.$$

We may assume without loss of generality the latter clause, since a direct argument shows that all ordinals less than, say, ω_k may be added to the hull X without increasing the $\sup X \cap \omega_n$ for any $n > k$. (This goes as follows: let X_0 be a hull that satisfies the MS property and the first two requirements above: $\{\omega_n \mid n \leq \omega\} \subseteq X_0$, $\forall n > 2\ (\sup X_0 \cap \omega_n) \in S_n$. We now consider the enlarged hull of $X =_{\mathrm{df}} X_0 \cup \omega_k$ in $\mathfrak{A}$. Let $n > k$. Consider for each m, and each $\vec{x} \in [X_0]^p$, $\sup\{f_m(\vec{\xi}, \vec{x}) \cap \omega_n \mid \vec{\xi} \in [\omega_k]^l\}$ where we have assumed that f_m is $l + p$-ary. But this is a supremum definable in X_0 from f_m, $\vec{x}$, ω_n, and ω_k. Hence it is less than $\sup(X_0 \cap \omega_n)$. By choice of $\langle f_n \rangle$, every $y \in X$ is of the form $f_m(\vec{\xi}, \vec{x})$ so this suffices.)

Let $\pi : (\bar{H}, \overline{K}, \in, \ldots) \cong (X, K \cap X, \in, \ldots)$, be the inverse of the transitive collapse, and let $\beta_n =_{\mathrm{df}} \pi^{-1}(\omega_n)$ for $n \leq \omega$. For each $2 < n < \omega : \beta_n > \aleph_2$ and $\mathrm{cof}(\beta_n) = \omega_1$. Let $\beta^*_n =_{\mathrm{df}} \sup(\pi``\beta_n)$. We now consider the coiteration of K with $\overline{K}$. Let the resulting coiteration of $(K, \overline{K})$ be $((M_i, \pi_{i,j}, \nu_i)_{i \leq j \leq \theta}\ ,\ (N_i, \sigma_{i,j}, \nu_i)_{i \leq j \leq \theta})$. Just as in the proof of the Covering Lemma, we show two things, firstly, that the coiteration requires a truncation on the K side (indeed in the very first ultrapower) and secondly that on the $\overline{K}$ side the iteration is trivial: no ultrapowers are taken at all. The arguments here are close to the corresponding points in the proof of the Weak Covering Lemma for K^c. However here we are not dealing with the ω complete measures of the K^c-sequence, but rather the measures in $\overline{K}$. The components of this argument are all in [18], but we assume the reader would prefer us to assemble them together here, which we now do.

(1) *The first ultrapower on the K side is taken after a truncation. In fact $\pi_{0,1} : M_0^* \longrightarrow M_1$, where $\pi_{0,1} \neq \mathrm{id}$ and M_0^* is a proper initial segment of K.*

Proof: Note that β_3 is a cardinal of $\overline{H}$, whilst $K_{\beta_3} = \bar{K}_{\beta_3}$ as $X \cap \omega_3$ is transitive. However $\mathrm{cf}(\beta_3) = \omega_1$ and is thus not a true cardinal of K (by the Covering Lemma for K). Hence the first action of the comparison will be a truncation on the K side to a structure M_0^* in which β_3 is a cardinal, and thence the ultrapower map $\pi_{0,1}$ as stated. Q.E.D.(1)

(2) *On the $\overline{K}$ side of the coiteration all the maps $\sigma_{i,j}$ are the identity: $\forall i \leq \theta$ $N_i = \overline{K}$.*

Proof: Suppose this is false for a contradiction and let ι be the least index where an ultrapower of $N_\iota = \overline{K}$ is taken by some $E^N_{\nu_\iota}$ with critical point κ_ι. On the K side let ζ be least so that $\mathcal{P}(\kappa_\iota) \cap M_\iota \| \zeta = \mathcal{P}(\kappa_\iota) \cap N_\iota$. Let us set M^* to be this $M_\iota \| \zeta$. (Note that no truncation is ever taken in the comparison on the $\overline{K}$ side.) Further note that since M_0^* was a truncate of K, we have that thereafter each M_i is sound above κ_i and that always $\omega\rho^{n+1}_{M_i} \leq \kappa_i < \omega\rho^n_{M_i}$ for some $n = n(i)$ (*cf.* the argument in [18] p.207 for 6.6.3). We set now $n = n(\iota)$. As $E^N_{\nu_\iota}$ is a total measure on $N_\iota = \overline{K}$ we have that $\widetilde{E} =_{\mathrm{df}} E^K_{\pi(\nu_\iota)} = \pi(E^N_{\nu_\iota})$ is a full measure in K with critical point $\tilde{\kappa} =_{\mathrm{df}} \pi(\kappa_\iota)$.

We apply the measure $E^N_{\nu_\iota}$ to M^* itself and form the fine structural ultrapower

$$\widetilde{M} = \mathrm{Ult}^*(M^*, E^N_{\nu_\iota}) \text{ with map } t : M^* \longrightarrow \widetilde{M}.$$

Note that by the weak amenability of $E^N_{\nu_\iota}$, $\widetilde{M} \cap \mathcal{P}(\kappa_\iota) = M^* \cap \mathcal{P}(\kappa_\iota)$, and that t is $\Sigma^{(n)}_0$ and cofinal.

We should like to compare M^* with $\widetilde{M}$ but for this we need the following Claim.

Claim 1 $\widetilde{M}$ is normally iterable above κ_ι.

Proof: The tactic is to show that $\widetilde{M}$ is $\Sigma^{(n)}_0$- embeddable into an initial segment of an iterate of K. Since the latter is normally iterable, so will be the former. First note:

(i) M^* and $\overline{K}$ agree up to ν_ι, hence if E_ι is the extender sequence on M_ι we have that $\pi \restriction J^{E_\iota}_{\nu_\iota} : J^{E_\iota}_{\nu_\iota} \longrightarrow J^{E^K}_{\widetilde{\nu}}$ cofinally for $\widetilde{\nu} =_{\mathrm{df}} \sup \pi``\nu_\iota$.

The following substitutes for the ω-complete measures of the K^c argument of [18] p.208.

(ii) $\mathrm{cf}(\nu_\iota) > \omega$ and hence we have a canonical extension $\pi^* \supseteq \pi \restriction J^{E_\iota}_{\nu_\iota}$; with $\pi^* : M^* \longrightarrow M'$ with $\omega\rho^{n+1}_{M^*} \leq \kappa_\iota < \omega\rho^n_{M^*}$ implying that $\omega\rho^{n+1}_{M'} \leq \tilde{\kappa} < \omega\rho^n_{M'}$, M' sound above $\tilde{\kappa} =_{df} \pi(\kappa_\iota)$, and π^* $\Sigma^{(n)}_0$ preserving.

Proof of (ii): Note that $\mathrm{cf}(\nu_\iota) = \mathrm{cf}(\kappa_\iota^{+M_\iota})$ and that $\kappa_\iota^{+M_\iota}$ is a $\overline{K}$ cardinal. Either it is equal to some β_i or else H will think, by the Weak Covering Lemma applied inside H, that it has cofinality equal to some β_i. In either case it has uncountable cofinality, as $\mathrm{cf}(\beta_i) > \omega$. By the definition of $n = n(\iota)$ we have that $\omega\rho^{n+1}_{M^*} \leq \kappa_\iota < \omega\rho^n_{M^*}$ and that M^* is sound above κ_ι. Consequently ν_ι is definably singularised over M^* and we have the right conditions to apply Lemma 2.5. The other properties mentioned in (ii) follow from this lemma. Q.E.D.(ii)

Note that M' is coiterable with K: it agrees with the latter up to $\tilde{\kappa}$, and as $cf(\kappa_\iota^{+M_\iota}) = cf(\tilde{\kappa}^{+M'}) > \omega$ a standard argument shows that any countable witness to an illfounded iteration of M' with critical points above $\tilde{\kappa}$ can be defined in a hull of such an iterate, and collapsed to an element of $(H_{\tilde{\kappa}^+})^{M'}$ also witnessing an illfounded iteration; this yields a contradiction. A simple comparison argument of M' with K then shows:

(iii) If $\tilde{\kappa}$ a K-cardinal, $\omega\rho_{M'}^{n+1} \leq \tilde{\kappa}$, and M' sound above $\tilde{\kappa}$ then M' is an initial segment of K.

Applying the full measure $\widetilde{E}$ yields $\sigma : K \longrightarrow_{\tilde{E}} \widetilde{K}$. Let $\widetilde{M}' = \sigma(M')$, and this is also an initial segment of $\widetilde{K}$. As $\pi^* \supseteq \pi \restriction J_{\nu_\iota}^{E_\iota}$ we have:

(iv) $X \in E_{\nu_\iota}^N \longleftrightarrow \pi^*(X) = \pi(X) \in \widetilde{E}$.

Defining $\mathbb{D}(M^*, E_{\nu_\iota}^N)$ the term model for the ultrapower and $\eta : \mathbb{D}(M^*, E_{\nu_\iota}^N) \cong M'$ its transitivisation, we have:

(v) (a) The map $d([f]) = \sigma\circ\pi^*(f)(\tilde{\kappa})$ is a structure preserving map $d : \mathbb{D}(M^*, E_{\nu_\iota}^N) \longrightarrow \widetilde{M}'$.
(b) The map $k = d \circ \eta^{-1} : \widetilde{M} \longrightarrow \widetilde{M}'$ is $\Sigma_0^{(n)}$-preserving with $k(\kappa_\iota) = \tilde{\kappa}$.
Proof: This is a standard computation for (a), and for (b) note by the elementarity of σ and (ii) that $\omega\rho_{\widetilde{M}'}^{n+1} \leq \sigma(\tilde{\kappa}) < \omega\rho_{\widetilde{M}'}^{n}$. Q.E.D.(v)

By (v)(b) since $\widetilde{M}'$ is normally iterable above $\tilde{\kappa}$, $\widetilde{M}$ will be normally iterable above κ_ι, as required. Q.E.D. *Claim 1.*

Claim 2 $E_{\nu_\iota}^N = E_{\nu_\iota}^{M^*}$.

(The proof of this Claim follows that of Lemma 6.6.4 of [18] with K here replacing K^c there.)

Proof: Since M^* and $\widetilde{M}$ agree up to ν_ι the coiteration of these two is above κ_ι. By *Claim 1* this coiteration is successful (meaning that all ultrapowers occurring are wellfounded) with iteration embeddings $i : \widetilde{M} \longrightarrow \widetilde{M}_\theta$ and $j : M^* \longrightarrow M_\theta^*$ say.

(vi) The iteration i of $\widetilde{M}$ is above $(\kappa_\iota^+)^{\widetilde{M}} = (\kappa_\iota^+)^{M^*}$.

Proof: We have seen above that $\widetilde{M} \cap \mathcal{P}(\kappa_\iota) = M^* \cap \mathcal{P}(\kappa_\iota)$, and thus $(\kappa_\iota^+)^{\widetilde{M}} = (\kappa_\iota^+)^{M^*}$. Also $\overline{K}, M^*, \widetilde{M}$ all agree up to ν_ι and forming $\widetilde{W} = \mathrm{Ult}(J_{\nu_\iota}^{E^{M^*}}, E_{\nu_\iota}^N)$ we see therefore that it is an initial segment of $\widetilde{M}$. From coherence of our extender sequences we know that

$$E^{\widetilde{M}} \restriction \nu_\iota = E^{\overline{K}} \restriction \nu_\iota = E^{M^*} \restriction \nu_\iota \text{ and } E_{\nu_\iota}^{\widetilde{M}} = \emptyset = E_{\nu_\iota}^{\widetilde{W}}.$$

By the initial segment condition of extender sequences we have that there are no further extenders on the $E^{\widetilde{M}}$ sequence with critical point κ_ι. Hence all critical points used in forming the iteration map i are above $(\kappa_\iota^+)^{\widetilde{M}}$. Q.E.D. (vi)

The rest of the argument is fairly standard.

(vii) $\widetilde{M}_\theta = M^*_\theta$.

Proof: Let $A \in \Sigma_1^{(n)}(M^*)$ in p_{M^*} be such that $A \cap \kappa_\iota \notin M^*$, and then note that $A \cap \kappa_\iota \notin \widetilde{M}$ as they agree about subsets of κ_ι. Hence if the iteration j is simple, then M^*_θ is not a proper initial segment of $\widetilde{M}_\theta$. But if j is non-simple then we reach the same conclusion as no proper initial segment of $\widetilde{M}_\theta$ can be unsound. Hence $\widetilde{M}_\theta$ is an initial segment of M^*_θ. But again we cannot have that it is a proper initial segment, since otherwise using the $\Sigma_0^{(n)}$ preservation property of t we'd have $A \cap \kappa_i$ in M^*_θ and so in M^*. Q.E.D. (vii)

(viii)

1. $\omega\rho^{n+1}_{\widetilde{M}} = \omega\rho^{n+1}_{M^*} = \omega\rho^{n+1}_{M^*_\theta}$.
2. If $p = p_{M^*} \backslash \omega\rho^{n+1}_{M^*}$ then $i \circ t(p) = p_{M^*_\theta, n+1}$.
3. t is Σ^*-preserving.

Proof: These are standard arguments from the proof of solidity for mice - *cf.* [18] p153-4. In 2. one first sees that $i \circ t(p) \in P^{n+1}_{M^*_\theta}$; a solidity argument on witnesses $W^{\alpha,p}_{M^*}$ shows that in fact $i \circ t(p) = p_{M^*_\theta, n+1}$.

(ix) $j \restriction \kappa_\iota = \mathrm{id} = i \circ t \restriction \kappa_\iota$; however $\mathrm{crit}(j) = \kappa_\iota$.

Proof: As the first clause is immediate, we argue that $j(\kappa_\iota) > \kappa_\iota$. As j is an iteration map $j(p) \in P^{n+1}_{M^*_\theta}$. By the Dodd-Jensen Lemma (*cf.* [18] Theorem 4.3.9) $j(p) \leq^* i \circ t(p)$, and hence by (viii)(ii) we have $j(p) = i \circ t(p)$. By the soundness of M^* above κ_ι we have that $\kappa = \widetilde{h}^{n+1}_{M^*}(i, \xi, p)$ for some $i < \omega$ and some $\xi < \kappa_\iota$. Hence $j(\kappa_\iota) = \widetilde{h}^{n+1}_{M^*_\theta}(i, \xi, j(p))$. As $j(p) = i \circ t(p)$ we have

$$j(\kappa_\iota) = i \circ t(\widetilde{h}^{n+1}_{M^*}(i, \xi, p)) = i \circ t(\kappa_\iota) > \kappa_\iota.$$

Q.E.D. (ix)

Hence κ_ι is the first point moved by j and thus some measure $E^{M^*}_\gamma$ is applied as the first ultrapower on the M^* side of the coiteration with $\mathrm{crit}(E^{M^*}_\gamma) = \kappa_\iota$ and γ being least with $E^{M^*}_\gamma \neq E^{\widetilde{M}}_\gamma$. As $E^{M^*} \restriction \nu_\iota = E^{\widetilde{M}} \restriction \nu_\iota$ and (see the proof of (vi)) $E^{\widetilde{M}}_{\nu_\iota} = \emptyset$ we must have $\gamma = \nu_\iota$ here. But then

$$X \in E^{M^*}_{\nu_\iota} \longleftrightarrow \kappa_\iota \in j(X) \longleftrightarrow \kappa_\iota \in i \circ t(X) \longleftrightarrow \kappa_\iota \in t(X) \longleftrightarrow X \in E^N_{\nu_\iota}.$$

Hence $E^N_{\nu_\iota} = E^{M^*}_{\nu_\iota}$ which is our *Claim 2*. Q.E.D. *Claim 2*

At the θ'th stage therefore, M_θ is an end extension of $\overline{K}$. For $n < \omega$, let i_n be the least stage i where $\kappa_i \geq \beta_n$ if such an i exists, otherwise set $i_n = \theta$. Let $k_0 < \omega$ be the least k such that any truncations performed on the K iteration side have been performed before stage i_k. We may also assume from this point i_{k_0} on then, that the least $m > 0$ with $\omega\rho^m_{M_\iota} < \kappa_\iota$ is fixed for all $\iota \geq i_{k_0}$; for this m then, we set for the rest of the proof $\rho = \omega\rho^m_{M_\iota}$ for any $\iota \geq i_{k_0}$, and we shall have that any M_ι is sound above κ_ι for $\iota \geq i_{k_0}$, and thus that $M_\iota = \widetilde{h}^m_{M_\iota}(\kappa_\iota \cup \{p_{M_\iota}\})$. Further by choice of m note that for $n > k_0$, $\rho^{m-1}_{M_{i_n}} > \kappa_{i_n} \geq \beta_n$. As we have in the iteration that $\pi_{i,j}(\langle d_{M_i}, p_{M_i}\rangle) = \langle d_{M_j}, p_{M_j}\rangle$, and parameters are finite sequences, we may further assume that k_0 has also been chosen sufficiently large so that for any $n \geq k_0$: (i) $d_{M_{i_n}}, p_{M_{i_n}} \cap [\beta_{n-1}, \beta_n) = \emptyset$, and lastly that (ii) k_0 is itself relevant.

(3) *Suppose $\langle \kappa_i \mid i < i_n \rangle$ is unbounded in β_n, where n is relevant. Then for no $\eta < i_n$ do we have $\pi_{\eta,i}(\kappa_\eta) = \kappa_i$ for unboundedly many $\kappa_i < \kappa_{i_n}$.*

Proof: If the conclusion failed then we should have that $\beta_n = \kappa_{i_n}$ and thus i_n is a limit ordinal of cofinality ω_1. By the normality of the iteration we then should have $\pi_{i,j}(\kappa_i) = \kappa_j$ for a closed ω_1-sub-sequence of the sequence of critical points $\langle \kappa_i \mid i < i_n \rangle$; let us choose such an ω_1-sub-sequence, and call the set of its elements $\overline{D}$ with $\overline{D}$ closed below β_n. These are all inaccessible in $\overline{K}$. Applying π, if we set $D = \pi$"$\overline{D}$, then we have that D is a cub set of order type ω_1 below β^*_n of K-inaccessibles. This will follow once we check that π is continuous on $\overline{D}$. Since $\overline{H}$ is correct about whether any ordinal α has cofinality ω or not, and since all the $\beta_n (n < \omega)$ have uncountable cofinality, easily we see that if κ_λ is a limit point of $\overline{D}$, then it has cofinality ω in $\overline{H}$. If $f : \omega \longrightarrow \kappa_\lambda$ is the least function in $\overline{H}$ witnessing this, then

$$\pi(\kappa_\lambda) = \pi(\sup\{f \text{ `` } \omega\}) = \sup\{\pi(f(n)) \mid n \in \omega\}.$$

(We are using here that the MS property is formulated using *all* the $\aleph_n$'s and not just a subsequence.) But n is relevant so $\beta^*_n \in T_n$ and thus is singular in K, but of uncountable cofinality. Thus the closed $C_{\beta^*_n}$ set of K of K-singular ordinals from the global $\square$ sequence, has non-empty intersection with D, which is absurd, as the latter consists of K-inaccessibles.

(4) *If $n \geq k_0$ is relevant then*
(i) M_{i_n} *is sound above* β_n;
(ii) β_n *is* $\Sigma^{(m-1)}_1$ *singularised over* M_{i_n};
(iii)

$$(*) \quad \rho > \alpha_{\beta_n} =_{\text{df}} \max\{\alpha \mid \sup(\widetilde{h}^m_{M_{i_n}}(\alpha \cup \{p_{M_{i_n}}\}) \cap \beta_n) = \alpha\}.$$

Proof: (i) follows from the definition of $i_n : M_{i_n}$ is sound above $\widetilde{\kappa} =_{\text{df}} \sup\{\kappa_i \mid i < i_n\}$. Now take an arbitrary $\delta < \beta_n$, $\delta \geq \rho$. (Recall that by the choice of k_0, $\rho = \rho_{M_{i_{k_0}}} = \rho_{M_{i_m}}$ for any $m \geq k_0$.) Divide into the two cases of $\tilde{\kappa} = \beta_n$ or $\tilde{\kappa} < \beta_n$. In the first case take i minimal such that $\kappa_i \in [\delta, \beta_n)$. Then $M_i = \widetilde{h}^m_{M_i}(\delta \cup \{p_{M_i}\})$ and in particular

$$\kappa_i \in \widetilde{h}^m_{M_i}(\delta \cup \{p_{M_i}\}).$$

By (3) take $\gamma < \beta_n$ such that whenever $\kappa_j \in (\gamma, \beta_n)$ then $\kappa_j \neq \pi_{ij}(\kappa_i)$; now then fix an index j with $\kappa_j \in (\gamma, \beta_n)$. By elementarity:

$$\pi_{ij}(\kappa_i) \in \widetilde{h}^m_{M_j}(\delta \cup \{p_{M_j}\}).$$

Since $\kappa_j > \pi_{ij}(\kappa_i)$, the point $\pi_{ij}(\kappa_i)$ is not moved in the further iteration past stage j, and so:

$$\pi_{ij}(\kappa_i) \in \widetilde{h}^m_{M_{i_n}}(\delta \cup \{p_{M_{i_n}}\}).$$

As δ was arbitrary above ρ this establishes (iii). In the second case the reasoning is similar but simpler. Note that if $\delta \geq \tilde{\kappa}$ then $\widetilde{h}^m_{M_{i_n}}(\delta \cup \{p_{M_{i_n}}\}) = M_{i_n}$. If $\delta < \tilde{\kappa}$ and κ_j is least with $\delta < \kappa_j < \beta_n$ then again:

$$\widetilde{h}^m_{M_j}(\delta \cup \{p_{M_j}\}) \cap \kappa_j = \kappa_j;$$

applying π_{j,i_n} we see that

$$\widetilde{h}^m_{M_{i_n}}(\delta \cup \{p_{M_{i_n}}\}) \cap \beta_n \neq \delta.$$

Hence (iii) holds in this second case as well.

However now there must be some $\gamma < \beta_n$ with $\sup(\widetilde{h}^m_{M_{i_n}}(\gamma \cup \{p_{M_{i_n}}\}) \cap \beta_n) = \beta_n$. Because if this failed we could choose a sequence

$$\gamma_0 = \rho, \gamma_{i+1} = \sup(\widetilde{h}^m_{M_{i_n}}(\gamma_i \cup \{p_{M_{i_n}}\}) \cap \beta_n) < \beta_n,$$

and then take $\gamma = \sup_i \gamma_i$. As $\mathrm{cf}(\beta_n) > \omega$, $\gamma < \beta_n$. However we have then that

$$\gamma = \sup(\widetilde{h}^m_{M_{i_n}}(\gamma \cup \{p_{M_{i_n}}\}) \cap \beta_n) < \beta_n$$

and simultaneously $\gamma > \alpha_{\beta_n}$. This contradicts $(*)$. Hence (ii) holds and (4) is thus proven.

Q.E.D.(4)

(5) *If n is relevant, then in the notation of* (4), *if $m > 1$ then for no smaller $m' < m$ is β_n $\Sigma_1^{(m'-1)}$ singularised over M_{i_n}.*

Proof: Just note that as $\rho^{m'-1}_{M_{i_n}} \geq \rho^{m-1}_{M_{i_n}} > \beta_n$, any purported $\Sigma_1^{(m'-1)}$-singularisation over M_{i_n} yields a singularising function *in* M_{i_n}. This is absurd as β_n is regular in M_{i_n}. Q.E.D.(5)

We thus have, by (4) and (5), that for relevant n, $s_n =_{\mathrm{df}} \langle \beta_n, M_{i_n} \rangle \in S^+$. The definitions of the α_{β_n} from $(*)$ of (4)(iii) thus conform to the definition of the α_{s_n} of Def. 3.5(f). We therefore have C_{s_n} sequences associated to such s_n as in the Global $\Box$ proof of the previous section.

(6) *For relevant $n \geq k_0$, we have* $\mathrm{ot}(C_{s_n}) < \beta_{k_0}$.

Proof: Set $i = i_{k_0}$; $j = i_n$. Then by the usual property of such ultrapowers $\pi_{i,j}``\omega\rho^{m-1}_{M_i}$ is cofinal in $\omega\rho^{m-1}_{M_j}$. Set $s = s_{k_0}$ and let δ be least such that:

(a) $\delta > \gamma$ if β_{k_0} is a K-successor cardinal, and $(\gamma^+)^K = \beta_{k_0}$;

(b) $\lambda(f_{(\delta,0,s)}) = \beta_{k_0}(= \nu_s)$ where $f_{(\delta,0,s)} \Longrightarrow s$.

Then $\delta < \beta_{k_0}$. Let $Y =_{\mathrm{df}} \pi_{i,j}``\ \mathrm{ran}(f^*_{(\delta,0,s)})$. We note that $\mathrm{ran}(f^*_{(\delta,0,s)})$ is a $\Sigma_1^{(m-1)}$ hull in $M_s(= M_i)$ and that $\pi_{i,j}$ is $\Sigma_1^{(m-1)}$ preserving. We have that Y is a $\Sigma_1^{(m-1)}$ hull in $M_j(= M_{s_n})$. As remarked just before the start of (6), we note that α_s, α_{s_n} (in the sense of Definition 3.5 f)) are below ρ by $(*)$ of (4)(iii). Consequently if we define $\widetilde{Y} =_{\mathrm{df}} \mathrm{ran}(f^*_{(\delta,0,s_n)})$ then $\widetilde{Y}$ is a $\Sigma_1^{(m-1)}$ hull of M_j. However $\widetilde{Y} \supseteq Y$, as $\pi_{i,j}(p_s, d_s) = p_{s_n}, d_{s_n}, \pi_{i,j}$ is $\Sigma_1^{(m-1)}$-preserving, and

$\pi_{i,j} \restriction \delta = \mathrm{id}$. (We need Lemma 2.3 here on the preservation of the d_s parameters under iteration. Note we are not claiming $\pi_{i,j}(p(s)) = p(s_n)$ as β_n might also be a K-successor, $(\bar{\gamma}^+)^K$ say, and $\bar{\gamma}$ may not be in $\pi_{i,j}$"Y.)

By choice of δ and Lemma 3.12 $\rho(f_{(\delta,0,s)}) = \omega\rho_s$. Hence Y is cofinal in $\omega\rho_{s_n}$. However then $\widetilde{Y}$ is also so cofinal. That is $\rho(f_{(\delta,0,s_n)}) = \omega\rho_{s_n}$ which again by Lemma 3.12 implies $\lambda(f_{(\delta,0,s_n)}) = \nu_{s_n} = \beta_n$. By Lemma 3.38 this implies $\mathrm{ot}(C_{s_n}) < \beta_{k_0}$. Q.E.D.(6)

For relevant n we form the "lift-up" map $\pi^*_n : M_{i_n} \longrightarrow M^*_n$ which extends $\pi \restriction (\overline{K}|\beta_n)$. We obtain the structure M^*_n and the map π^*_n as a pseudo-ultrapower using the Pseudo-ultrapower Lemma 2.5.

(7)(a) *For relevant n, π^*_n is $\Sigma_1^{(m-1)}$-preserving, and β^*_n is $\Sigma_1^{(m-1)}$-singularised over M^*_n; further, if $m > 1$, then for no smaller $m' < m$, is β^*_n $\Sigma_1^{(m'-1)}$-singularised over M^*_n.*

(b) *M^*_n is normally iterable above β^*_n.*

Proof : (a) The Pseudo-ultrapower Theorem 2.5 (with $k = m-1$) shows the right degree of elementarity of π^*_n, *i.e.* that it is $\Sigma_0^{(m-1)}$ preserving. It further states that the map is cofinal and thus $\Sigma_1^{(m-1)}$-preserving, and that it yields that β^*_n is $\Sigma_1^{(m-1)}$-singularised over M^*_n, whilst β^*_n is $\Sigma_1^{(m'-1)}$-regular over M^*_n for any $m' < m$ (if $m > 1$). For (b) this is a standard argument about canonical extensions defined from pseudo-ultrapowers using the fact that $\mathrm{cf}(\beta_n) > \omega$. See [18] Lemma 5.6.5. Q.E.D.(7)

(8) *M^*_n is an initial segment of K.*

Proof: Note that by construction $M^*_n \restriction \beta^*_n = K \restriction \beta^*_n$. By 7(a) $\rho^m_{M^*_n} \leq \beta^*_n$; again the pseudo-ultrapower construction shows M^*_n is sound above β^*_n and hence is coded by a $\Sigma_1^{(m-1)}(M^*_n)$ subset of β^*_n, A say. An elementary iteration and comparison argument shows that, when K is compared with M^*_n, to models N_η, M^*_η then A is $\Sigma_1^{(m-1)}$ definable over N_η, and thus is in K itself. As M^*_n is a mouse in K, its soundness above β^*_n implies that after any supposedly necessary coiteration, we must have $N_\eta = M^*_\eta$ and hence $\mathrm{core}(N_1) = \mathrm{core}(N_\eta) = \mathrm{core}(M^*_\eta) = M^*_n$. Hence M^*_n is an initial segment of K. Q.E.D.(8)

(9)(a) $s^*_n = \langle \beta^*_n, M^*_n \rangle \in S^+$;

(b) *M^*_n is the assigned K-singularising structure for β^*_n; hence in K, $C_{\beta^*_n}$ is defined over M^*_n, that is $C_{\beta^*_n} =_{\mathrm{df}} C_{s^*_n}$.*

Proof: For (a), by (7)(a) M^*_n singularises appropriately, it is sound above β^*_n, and by (8) it is a mouse. For (b) we have shown that M^*_n is an initial segment of K, and thus conforms to the definition of the segment chosen to define the canonical C-sequence associated to β^*_n in K. Q.E.D.(9)

We thus conclude:

(10) *For relevant $n \geq k_0$ $\mathrm{ot}(C_{\beta^*_n}) < \pi(\beta_{k_0}) = \omega_{k_0}$.*

Proof: Let n be relevant. By (6) $\mathrm{ot}(C_{s_n}) < \beta_{k_0}$ because $\tilde{h}_{s_n}(\delta, p(s_n))$ is cofinal in $\omega\rho_{s_n} =$

$\omega\rho^{m-1}_{M_{i_n}}$. Set $\delta' = \pi^*_n(\delta)$. By the $\Sigma_1^{(m-1)}$-elementarity of π^*_n we shall have that

$$\pi^*_n \text{ ``} \, \tilde{h}_{s_n}(\delta, p(s_n)) \subset \tilde{h}_{s^*_n}(\delta', p(s^*_n)).$$

As $\pi^*_n \restriction \omega\rho_{s_n}$ is cofinal into $\omega\rho_{s^*_n}$, we deduce that $\rho(f_{(\delta',0,s^*_n)}) = \omega\rho_{s^*_n}$. By Lemma 3.12 this ensures that $\lambda(f_{(\delta',0,s^*_n)}) = \nu_{s^*_n} = \beta^*_n$. This in turn implies by Lemma 3.38, that $\mathrm{ot}(C_{s^*_n})$ is less than the least p.r. closed ordinal greater than δ'. However $\pi^*_n \restriction \beta_n$ extends $\pi \restriction \beta_n$, and thus this ordinal is less than $\pi(\beta_{k_0})$.

Now (10) yields the final contradiction, as for relevant n, S_n was chosen to consist of points β where $\mathrm{ot}(C_\beta) \geq \omega_{n-3}$, whereas (10) establishes an ultimate bound on such order types of ω_{k_0}.
Q.E.D.(Theorem 4.2)

We finally remark that the Corollary 1.5 is immediate: after shifting our attention to cardinals above $\aleph_k$ we still use the same hypothesis concerning sufficient singular ordinals in K in order to establish the stationarity of the T_n now contained in $\mathrm{Cof}(\omega_k)$. We take $\omega_k \subseteq X$ and now the analogues of the ordinals β_n have cofinality ω_k; H is correct about the cofinality of any ordinal whose V-cofinality is less than ω_k. The proof of (3) now shows that there is no closed ω_k subsequence of critical points κ_i unbounded in such a β_n, as the map π is now continuous at points of cofinality less than ω_k. Hence we can deduce (4) that the iterates are indeed singularising structures for the β_n as required.

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