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ON REVISION OPERATORS

P.D.WELCH

Abstract. We look at various notions of a class of definability operations that generalise inductive operations, and are characterised as "revision operations". More particularly we: (i) characterise the *revision theoretically definable* subsets of a countable acceptable structure; (ii) show that the categorical truth set of Belnap and Gupta's theory of truth over arithmetic using *fully varied revision* sequences yields a complete Π^1_3 set of integers; (iii) the set of *stably categorical* sentences using their revision operator ψ is similarly Π^1_3 and which is complete in Gödel's universe of constructible sets L; (iv) give an alternative account of a theory of truth - *realistic variance* that simplifies full variance, whilst at the same time arriving at Kripkean fixed points.

[...] no statement is immune to revision.
W. V. O. Quine, From a Logical Point of View, p 43.

§1. Introduction.

This paper concerns itself with some rather general forms of definability based on certain kinds of operator $\Phi : \mathcal{P}(\mathbb{N}) \longrightarrow \mathcal{P}(\mathbb{N})$. Unlike the more familiar forms of inductive definability where Φ is restricted to be given by some positive arithmetic formula, and hence be *monotone*, we shall not even restrict Φ to be *progressive* (that is $s \subseteq \Phi(s)$.) However we do think of Φ as an operator that is repeatedly applied to an initial starting set s_0 (which will not necessarily be the empty set).

A natural question then is what should be done at limit stages if we wish to repeat this process transfinitely. For monotone inductive definitions (or simply *progressive* ones) there is only one natural proposal, as $\alpha < \beta \longrightarrow \Phi^\alpha(s) \subseteq \Phi^\beta(s)$ we take unions at limit ordinal stages λ. At the level of generality mentioned so far it is not obvious that there should be any natural operation, and something will have to be devised as a *limit rule* Γ.

If this is to be at all a fruitful approach there will have to be restrictions in order that the concept is not just hopelessly general. Interestingly there are already several examples

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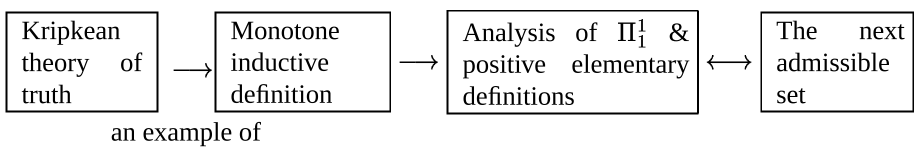
in the literature of such processes that one can classify as falling under this scheme, and which have arisen in unconnected fields.

A primary example version of this kind of definability has been worked out by Belnap and Gupta in their book [9]. Their theory of "*circular definitions*" based on "*revision sequences*" (whence the title of this paper) derives from their earlier account of a theory of truth where the Tarskian biconditionals are treated as being *definitional* of truth. Unlike the Kripkean theory of truth which provides a monotonic operator building up a set of truths and of falsehoods for a partially defined T predicate, they assume a total (and so inconsistent) extension of the T predicate over, for example, $\mathbb{N}$ the structure of arithmetic, which is then *revised* according to the Tarskian biconditionals. Latterly, they have sought to embed this in the aforementioned more general theory of circular definitions, which involve repeatedly revising the extension of some predicate and seeing which objects fall finally under the resulting extension no matter which starting hypothesis is used for its initial extension (or even which limit rule).

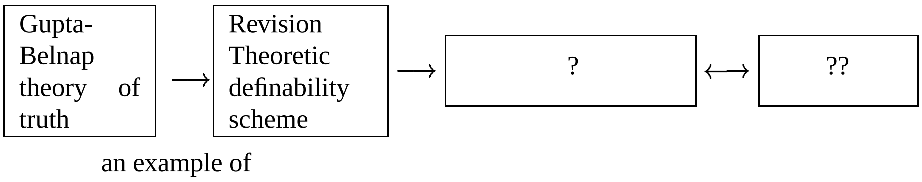
A further example (and was our own starting point) is afforded by the Infinite Time Turing Machines of [10]. A Turing machine acts on its tape, which *via* coding we view as a set of integers, continually "revising" in simple steps the tapes contents. If we adopt the formalism of [10], and suppose at limit stages of time λ that the value of a cell is the limsup of its previous values for $\alpha < \lambda$, we have thus a "limit rule". Such conceptual devices can "compute" surprisingly complicated sets.

Löwe first pointed out the similarity between Herzberger style revision sequences and Infinite Time Turing Machines in [15]. We wish to express our thanks to him here for his correspondence on these issues. In [15] he made the suggestion that a machine theoretic approach could help throw a new light on the various limit rules in Belnap and Gupta's definability theory. Whilst ultimately it is our viewpoint that it is Gödel's constructible hierarchy L that underlies both these phenomenon, (and that in the final analysis the choice of limit rule for Belnap and Gupta's theory *qua* a theory of definability, is almost irrelevant as the proof of Theorem 2.1 shows), it was his suggestion that led us to look at revision theoretic definitions. (We shall discuss later more precisely this connection, see Comment 3 below.) At this point we should also like to thank J. Burgess and D.A.Martin for some helpful comments on these topics, and in particular V. McGee for both illuminating conversations and correspondence as we were learning the revision theory of truth, and to A. Gupta for some extensive correspondence replying to our queries.

Belnap and Gupta, having formulated a general theory of circular definitions in [9] really are interested there in discussing truth, and we are unaware of any attempt in the literature to delineate what the class of definitions it is that they are defining. A primary motivation for us was to discover what this was, and a description is given at Theorem 2.2 below.



The above schematic diagram illustrates the relationship between Kripke's theory of truth, [14], using either of the Kleene 3-valued truth table schemes, or the van Fraassen supervaluation scheme, and the corresponding definability theory. Our original purpose had been to discover the corresponding relationships for the Gupta-Belnap theory, and thus find out the solutions for '?' and '??' below.



Briefly, the Gupta-Belnap theory raises the level of definability to Π^1_2, '?' thus being filled by an analysis of the Gupta-Belnap definitions and Π^1_2; and '??' by the companion structure of the "next stable set" - rather than the next admissible set at the Π^1_1-level. We were unaware of the paper of Burgess [3] at the time, which would have indicated what the answer might be. Burgess showed that the *"stable truth set"* (using Belnap's Limit Rule) over arithmetic was a complete Π^1_2 set. This can be obtained as a corollary to Theorem 2.1, where we show that the *revision theoretically definable* reals are precisely the Π^1_2 reals. We had previously shown a corresponding result that the set of indices of total infinite time turing computable functions formed a complete Π^1_2 set of integers. It also turned out that such machines are formally equivalent to the Herzberger Revision sequences of [12], [11].

There is thus an analogy with the theory of monotone inductive definitions over an *acceptable structure* (in the sense of Moschovakis [17]) and the next admissible set, and the theory of revision theoretic definability and the "next stable set" $\mathbb{S}_{\mathcal{M}}$ over a structure $\mathcal{M}$ with a suitable coding scheme. The latter theory forms the first part of the paper in §2. There one observes that the revision theoretically definable subsets of $\mathcal{M}$ are those that are $\Pi_1(\mathbb{S}_{\mathcal{M}})$. There is also an unfortunate disanalogy: Π^1_1 forms a Spector class, whilst Π^1_2 does not. Later sections visit various other aspects of Belnap and Gupta's theory of truth: in §3 we look at their notion of *fully varied* sequences which in fact essentially requires Π^1_3 notions: we show that the stable truth set over arithmetic is Π^1_3-hard. The motivation for this variety of revision theory seems to be that it allows certain "Gupta-puzzle"-like sets of sentences to be appropriately classified in all such revision sequences. §4 is devoted to our simplified version of full variance, which we dub "realistic variance", and which seems to achieve the same effect. It thus seems to us to have the advantages of full variance, but (i) it is simpler (it results in a stable truth set that is "only" Π^1_2-hard) and (ii) each revision sequence results in a stability pair set that is a Kripkean fixed point for the van Fraassen supervaluation scheme. Thus realistic variance can be considered an intermediate theory between full variance and the standard theory, or, as a stepping stone on the way to the notion of algorithmic variance alluded to below. The notion of *stable categoricity* (*cf.* [9] 6D.9) derived from arguments concerning strengthened liar paradoxes is seen in §5 to also lead to complicated sets: the stable categorical sentences form a Π^1_3 set, and at least in L, are a Π^1_3-hard set.

The results here are those announced in the first five sections of [21]. A further example of revision operator is suggested in the final section of that paper: making an "algorithmic" choice of a revision rule. Such a revision operator leads on individual sequences to sets at a simpler level of definability, and a theory of truth based on single processes, much as Kripke's. We leave discussion of this to [20]. We have made no attempt to discuss the implications philosophical or otherwise, of the results here for the theory of truth based on revision sequences (we leave this for [20] also), but use this opportunity to separate away from that discussion the mathematical proofs that are perhaps somewhat more technical than usual in papers on the semantical theories of truth. That the proofs do involve technical mathematics is a concomitant of the assertions in the theorems: revision theories of truth, of necessity, are complicated. We make some comments on this, but largely pass no judgement here on what that implies.

1.1. The general setting. We shall attempt to make the discussion as self-contained as possible; however we shall refer the reader to the relevant sections of [2](II.5), or [6] (II.1) for the basic definition of the Gödel constructible hierarchy L_α. We shall not use much of admissibility theory, but we shall be generalising the theory of monotone inductive definitions, and for these the books of [2] (Ch.VI) and [17] can be consulted. For κ an infinite cardinal, H_κ is the class of sets of hereditary cardinality less than κ: that is, those sets x whose transitive closure, $TC(x)$, satisfies card$TC(x) < \kappa$; HC denotes H_{ω_1}, the class of hereditarily countable sets. We identify $\mathbb{N}$ with ω.

Let $:\mathcal{P}(\omega) \longrightarrow \mathcal{P}(\omega)$ be an operator (frequently referred to in the sequel as a "revision operator" since we regard δ as "revising" the objects in its domain). Let Lim denote the class of limit ordinals. Let $\Gamma : {}^{Lim}\mathcal{P}(\omega) \longrightarrow \mathcal{P}\mathcal{P}(\omega)$ be a function defined on limit length sequences of subsets of ω. In the following definition, ∞ is to be taken, at least initially, as some appropriate uncountable cardinal, or even sometimes as On the class of all ordinals.

DEFINITION 1.1. *A* revision sequence based on δ, Γ *is a sequence* $\langle s_\alpha \mid \alpha < \infty \rangle$ *with:*
(i) $\forall \alpha \quad s_\alpha \subseteq \omega$;
(ii) $\forall \alpha \in On \quad s_{\alpha+1} = (s_\alpha)$;
(iii) $\forall \lambda \in On, Lim(\lambda), \quad s_\lambda \in \Gamma(\langle s_\alpha \mid \alpha < \lambda \rangle)$.

Examples:(i) A monotone inductive definition over $\mathbb{N}$ via some positive arithmetical formula.
(ii) An arithmetic quasi-inductive definition (Burgess [3]).
(iii) Various other forms of non-monotonic inductive definitions. (See for example [18])
(iv) Sequences of computations of Infinite Time Turing Machines ([10]).
(v) Revision sequences of Belnap-Gupta.
(vi) Iteration of the Shoenfield Limit lemma along Kleene's $\mathcal{O}$ (Clote [5]).

In each of these examples ((i) being the most important and familiar) various processes are defined which have some clearly defined starting point and the process produces some object of interest.

We shall for the most part be interested in revision sequences with Δ^1_2 definable operators δ, and with a limit rule Γ which is $\Delta_1(HC, \in)$-definable. (When single valued

Γ - that is ran(Γ) consists only of singleton sets - it will have essentially a Δ^1_2 graph in codes for wellorders.) In such a case a countable length revision sequence will then simply be $\Delta_1(\langle HC, \in\rangle)$. All the examples will satisfy some *coherency condition* on Γ.

DEFINITION 1.2. *The* local set of stabilities at λ *is defined as follows.*
(i) For $Lim(\lambda)$ or $\lambda = \infty$, $s^+_{<\lambda} =_{df} \{n \mid \exists\beta < \lambda\forall\alpha(\beta \le \alpha < \lambda \longrightarrow n \in s_\alpha)\}$. $s^-_{<\lambda}$ is defined analogously with "$n \notin s_\alpha$" replacing "$n \in s_\alpha$."
(ii) We set $s_{<\lambda} =_{df} (s^+_{<\lambda}, s^-_{<\lambda})$.

We shall use the notation $\vec{s} \restriction \lambda$ for $\lambda \le \infty$ for the restriction of the sequence $\vec{s}$ to λ: $\vec{s} \restriction \lambda =_{df} \langle s_\alpha \mid \alpha < \lambda\rangle$. Then, with this notation, $\vec{s} \restriction \lambda_{<\infty} = s_{<\lambda}$.

DEFINITION 1.3. **(Coherence)** s coheres with $\vec{s} = \langle s_\alpha \mid \alpha < \lambda\rangle$ *(for λ a limit ordinal or ∞) if $s \cap s^-_{<\lambda} = \emptyset$ and $s \supseteq s^+_{<\lambda}$.*

We shall (with one exception in §5) always assume that
(i) Γ is coherent, in that for any limit $\lambda < lh(\vec{s})$

$$\forall s \in \Gamma(\langle s_\alpha \mid \alpha < \lambda\rangle)(s \text{ is coherent with } s_{<\lambda});$$

(ii) "$t \in \Gamma(\vec{s} \restriction \lambda)$" is $\Delta_1(HC, \in)$-definable.

DEFINITION 1.4. **(The stabilization pair of a sequence)** *Let $\vec{s}$ be a revision sequence.*
(i) Let $\sigma(\vec{s})$ = the least limit ordinal $\sigma < lh(\vec{s})$ so that

$$(a)\ s_{<\sigma(\vec{s})} = s_{<\infty}\,; \quad (b)\ \forall t \ge \sigma\ s_{<\sigma} \subseteq s_{<\tau}.$$

(ii) $\sigma(\vec{s})$ is called the stabilization point *and $s = (s^+_{<\infty)}, s^-_{<\infty})$ the* stabilization pair, *of $\vec{s}$.*

Note: (i) A closure argument shows that stabilization points and pairs must exist for any revision sequence if $lh(\vec{s}) = \infty$ is On or is a cardinal of uncountable cofinality: $\{\alpha < lh(\vec{s}) \mid s_{<\alpha} = s_{<\infty}\}$ is then closed and unbounded in $lh(\vec{s})$.

The following lemma is straightforward:

LEMMA 1.1. *Let $\vec{s}$ be a revision sequence, and let $\sigma(\vec{s})$ its stabilization point. Then there is a revision sequence $\vec{t}$ of countable length, with $t_{<\infty} = s_{<\infty}$, and $s_{\sigma(\vec{s})} = t_{\sigma(\vec{t})}$.*

PROOF: By Löwenheim-Skolem. Q.E.D.

§**2. Revision Theories of Truth and Definability.** We discuss the particular examples of limit rules of revision sequences arising from Revision Theories of truth.

Example (1) Herzberger Limit Rule, Γ_H (see [12],[11]).
Γ_H is single valued and takes as s_λ the smallest s coherent with $s_{<\lambda}$ for all $\lambda < lh(\vec{s})$. In some sense this is a "minimal" policy, only those integers in $s^+_{<\lambda}$ can be taken as in s_λ.
Example (2) Gupta Rule, Γ_G
is also single valued: Γ_G is defined by $s_\lambda = s^+_{<\lambda} \cup s_0 \backslash s^-_{<\lambda}$. The idea here is that we refer back to our original "hypothesis" $h = s_0$ to fill in for the ambiguous values in

$\omega \backslash (s^+_{<\lambda} \cup s^-_{<\lambda})$.
Example (3) Belnap Rule, Γ_B.
An example of a multi-valued limit operator: $\Gamma_B = \{s \mid s \text{ coheres with } s_{<\lambda}\}$. The Belnap rule thus places the least possible restriction on the choice at lmit stages, only coherence must be adhered to.

Originally these arose in discussions of notions of limit stage operators (called "boot-strapping policies") specifying how a sequence might be built from δ_τ, the Tarskian truth operator, at successor stages.

Corresponding to these rules are notions of definability. [9] expand their original revision theory of truth to cover a theory of circular definitions.

We consider the example of arithmetic and the structure of natural numbers $\mathbb{N}$ augmented with a predicate symbol $\dot{s}$. Let $\Phi(v_0)$ be a formula with one free variable in this language. Let δ_Φ be defined by

$$\delta_\Phi(s) = \{n \mid \langle \mathbb{N}, s\rangle \models \Phi(\dot{n})\}.$$

DEFINITION 2.1. $z \subseteq \mathbb{N}$ *is* S^*_Γ-definable *for some limit operator* Γ*, and some revision rule* δ_Φ *as above if*

$$z = \bigcap\{ s^+_{<\infty} \mid \vec{s} = \langle s_\alpha \mid \alpha < \infty\rangle \text{ is a revision sequence based on } \delta_\Phi, \Gamma \}.$$

Note that we could have specified the length $lh(\vec{s})$ of the revision sequence in the above definition to be always ω_1. Lemma 1.1 shows that we can find by Löwenheim-Skolem a countable revision sequence $\vec{t}$ of length τ with the same stability set as one of length On (or any regular uncountable cardinal). Suppose then $\vec{t}$ arises in this way. Set $\tau' = \sigma(\vec{t}), \quad t = t_{<\tau'} = t_{<\tau}$, and then t occurs cofinally as the local stability set below τ. Then we may simply append repeatedly ω_1 times the segment $\vec{t} \restriction [\tau', \tau)$ to obtain an appropriate revision sequence of length ω_1 with the same stability set of $\vec{s}$ and of $\vec{t}$.

The above notion is based on an earlier theory of truth suggested by Belnap. In their text they treat more of the following definition.

DEFINITION 2.2. $z \subseteq \mathbb{N}$ *is* $S^\#_\Gamma$-definable *for some limit operator* Γ*, and some revision rule* δ_Φ *as above if*

$$z = \bigcap\{ s^{\#+}_{<\infty} \mid \vec{s} = \langle s_\alpha \mid \alpha < \infty\rangle \text{ is a revision sequence based on } \delta_\Phi, \Gamma \}.$$

where

$$s^{\#+}_{<\infty} =_{df} \{k \mid \exists m < \omega \forall n < \omega \forall \beta > \sigma(\vec{s})(n > m \longrightarrow k \in s_{\beta+n})\}$$

The point of this latter definition is that it allows one to talk of "near stability": it is not that k must belong to *every* sufficiently large s_β: it is allowed to waver finitely often at and after limit levels s_λ, but only for a fixed number m of stages. We refer the reader to [9] 5.D.

Note Kremer ([13]) gives explicitly a set of definitions $\mathcal{D}$, giving rise to a revision rule $\delta_{\mathcal{D}}$ so that discrete linear orders with a first element have extensions of a current hypothesis that are isomorphic to $\mathbb{N}$ - so we choose to work over models of $\mathbb{N}$.

The question arises, what kinds of reals are S^*_Γ or $S^\#_\Gamma$ definable, and how do they depend on Γ? (see [15].)

Antonelli ([1]) gives explicit revision theoretic definitions for each of the complete Σ^0_n sets, *i.e.* for each level of the arithmetic hierarchy. Kremer (§*8 op.cit.*) gives an argument of Gupta showing that this result can be extended to the inductive sets:

(i) Inductively definable subsets of $\mathbb{N}$ are $S^{\#}_{\Gamma_B}$ and $S^{*}_{\Gamma_B}$-definable.

But also Kremer shows that:

(ii) For any Π^1_2 set of integers X, there is a finite set of positive definitions $\mathcal{D}$ so that X is recursively embeddable into $\vDash^{\mathcal{D}}_i$ - the set of sentences valid on $\mathcal{D}$ in S_i, where S_i is the semantic theory associated to truth-at-the-first-fixed-point of the positive inductive definitions $\mathcal{D}$.

As he shows that S^* and $S^{\#}$ extend S_i, it follows that the complexity of $\vDash^{\mathcal{D}}_*$ and $\vDash^{\mathcal{D}}_{\#}$ are also Π^1_2.

We show that strong definability extends throughout Δ^1_2 regardless of the choice of Limit Rule $\Gamma_H, \Gamma_G, \Gamma_B$.

THEOREM 2.1. *Let S be $S^{\#}_{\Gamma}$ or S^*_{Γ};*
(i) Let Γ be any limit rule whatsoever; let z be any Π^1_2 real. then z is S_Γ-definable. Hence any Δ^1_2-definable real is S_Γ-, co-S_Γ-definable.
(ii) Let Γ be any $\Delta_1(HC, \in)$-definable limit rule (this includes any of Γ_H, Γ_G, Γ_B). (a) The class of S_Γ-definable reals is precisely that of the Π^1_2 reals; (b) the class of Δ^1_2-definable reals coincides with the class of S_Γ-, co-S_Γ-definable reals.

REMARK: The S_Γ-, co-S_Γ-definable reals (for $\Gamma \in \Delta_1(HC, \in)$, *e.g.* $\Gamma \in \{\Gamma_H, \Gamma_G, \Gamma_B\}$) are thus the reals of the first transitive *stable set*, $\mathbb{S}_{\mathbb{N}}$, over the structure $\mathbb{N}$.

DEFINITION 2.3. *Let $\mathcal{M} = \langle M, R, \dots \rangle$ be any structure. Define:*

$$\mathbb{S}_{\mathcal{M}} =_{df} L_{\sigma_1^{\mathcal{M}}}(\langle M, R, \dots \rangle)$$

where the latter is the first level of the relative L-hierarchy built over M, using elements of M as urelemente, *wuth the property that $\mathbb{S}_{\mathcal{M}} \prec_{\Sigma_1} V$.*

Thus $\mathbb{S}_{\mathcal{M}}$ is correct about Σ_1 facts true in V of M. Note that as $\mathbb{S}_{\mathcal{M}}$ is an admissible structure, $\Delta_1(\mathbb{S}_{\mathcal{M}})$ subsets of M are in $\mathbb{S}_{\mathcal{M}}$. The notion of *acceptability* in the next theorem is that of Moschovakis [17].

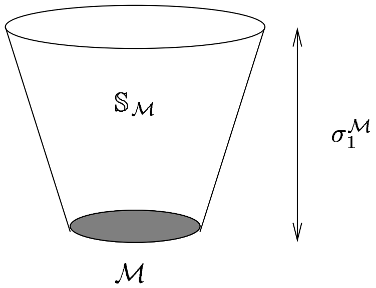
THEOREM 2.2. *Let S be $S^{\#}_{\Gamma}$ or S^*_{Γ}; let $\mathcal{M}$ be a countable acceptable structure.*
(i) Let Γ be any limit rule whatsoever; let z be any $\Pi_1(\mathbb{S}_{\mathcal{M}})$ subset of M. Then z is S_Γ-definable. Hence any $\Delta_1(\mathbb{S}_{\mathcal{M}})$-definable subset is S_Γ-, co-S_Γ-definable.
(ii) let Γ be any $\Delta_1^{(HC,\in)}(\{\mathcal{M}\})$-definable limit rule (this includes any of $\Gamma_H, \Gamma_G, \Gamma_B$). The class of S_Γ-definable subsets of $\mathcal{M}$ is precisely that of the $\Pi_1(\mathbb{S}_{\mathcal{M}})$ sets; the class of $\Delta_1(\mathbb{S}_{\mathcal{M}})$-definable sets coincides with the class of S_Γ-, co-S_Γ-definable subsets of M.

For $\Gamma \in \{\Gamma_H, \Gamma_G, \Gamma_B\}$ then, $\mathcal{P}(\omega) \cap \mathbb{S}_{\mathbb{N}}$ is precisely the class of S_Γ-, co-S_Γ sets of integers. In fact, by Levy-Shoenfield, it is thus the reals of the smallest model of Δ^1_2-Comprehension that is Σ^1_2-*correct*, that, any Σ^1_2 formula about real parameters from the model is true, if and only if it is true in the model.

REMARK: $\mathbb{S}_{\mathbb{N}}$ has domain that of L_σ where σ is the first stable ordinal. Note that many first order structures M will have $\mathbb{S}_{\mathcal{M}}$ the same. For example $\mathbb{S}_{\mathbb{N}}$ will have the same domain as $\mathbb{S}_{\mathcal{A}}$ where $\mathcal{A}$ is the least β-model of analysis.

The import of part (ii) of either theorem is that the choice of revision rule $\Gamma \in \{\Gamma_H, \Gamma_G, \Gamma_B\}$, or semantic scheme $S^\#$, S^* is immaterial, when calculating the class of definable subsets of the structure.

By way of analogy with admissibility theory and the hyperelementary sets over a structure, we view revision theoretic definability as building up for us the domain of $\mathbb{S}_{\mathcal{M}}$ over $\mathcal{M}$. Just as the inductive/co-inductive sets over $\mathcal{M}$ yield the hyperelementary sets of the "next admissible set" over $\mathcal{M}$, so the S_Γ-, co-S_Γ-definable sets yield the domain of the "next stable set over $\mathcal{M}$". The following diagram is supposed to be reminiscent of the analogous one from [2], p.43, of the "next admissible set" over $\mathcal{M}$.



- $\mathbb{S}_{\mathcal{M}} =_{df} L_{\sigma_1^{\mathcal{M}}}[\mathcal{M}]$ and is Π^1_2-absolute.
- $X \subseteq \mathcal{M}$ is S_Γ-definable over $\mathcal{M}$ $\Longleftrightarrow X$ is $\Pi_1(\mathbb{S}_{\mathcal{M}})$.
- $X \in \mathbb{S}_{\mathcal{M}} \cap \mathcal{P}(\mathcal{M}) \Longleftrightarrow X$ is both S_Γ- and co-S_Γ-definable over $\mathcal{M}$.

FIGURE 1. $\mathbb{S}_{\mathcal{M}}$: the next stable set over $\mathcal{M}$

We content ourselves with proving Theorem 2.1 as the route to extending such a theory to acceptable structures has been charted in [17]; indeed one could strengthen Theorem 2.2 by requiring only that $\mathcal{M}$ have a *strongly definable* coding scheme.

PROOF: of Theorem 2.1. We let $S = S^*$. (The proof for $S^\#$ is immediate from what follows.) We describe informally an arithmetically definable operation $\delta : \mathcal{P}(\omega) \longrightarrow \mathcal{P}(\omega)$ so that $\delta(s) = \{n \mid \langle \mathbb{N}, s\rangle \models \Phi(n, \dot{x})\}$ for some Φ. It would be an invidious task to explicitly give all the coding required, but give only the important steps; we leave it to the reader to provide the actual details.

Let $B_0 = \Pi_1$-$Th(L_{\omega_1})$ - that is the complete Π_1 (- in the language $\mathcal{L}_{\{\dot\in, \dot=\}}$) theory of $\langle L_{\omega_1}, \in\rangle$. It is well known that B_0 is a complete Π^1_2 set. Without loss of generality we shall assume that this theory is coded by even integers. Say $2n \in B_0 \Longleftrightarrow \langle L_{\omega_1}, \in\rangle \models \psi_{2n}$ where $\langle \psi_{2k} \mid 2k < \omega\rangle$ is a recursive enumeration of the Π_1-sentences. Note we may assume we have also a recursive enumeration $\langle \varphi_{2k} \mid 2k < \omega\rangle$ of the Σ_1 sentences, with a recursive $g : Evens \longrightarrow Evens$ so that $\varphi_{g(k)}$ is logically equivalent to $\neg\psi_k$. If $A_0 = g``Evens \backslash B_0$, then A_0 is recognizable as, essentially, a code for Σ_1 truth over L_{ω_1}. Let $s \subseteq \mathbb{N}$. Steps 1 - 3 below describe the transition, or revision step, $\delta(s) = s'$.

Step 1 of δ: look at $s \cap$ Evens and ask of $A = \{g(2k) \mid 2k \notin s \cap \text{Evens}\}$, whether

$$\mathcal{A} = \langle \omega, E_A\rangle \models \text{“}V = L\text{”}$$

In other words, if elements $2n, 2m$ of A code parameter free Σ_1 terms t_n, t_m, and if the $\in$-relation E_A is defined by $\langle m, n\rangle \in E_A \longleftrightarrow \ulcorner t_m \in t_n \urcorner \in A$, then we ask if $\langle \omega, E_A\rangle$ is an ω-model of "$\forall x \exists \alpha \in On\, x \in L_\alpha$."

If the answer is YES to this question the procedure goes to *Step 2*.

If NO then it sets $s' = \mathbb{N}$ and EXITS. Otherwise:

Step 2.

If $s \cap$ Odds $\neq \emptyset$ then it proceeds to Step 3. If $s \cap$ Odds $= \emptyset$ then it "writes" a copy of the field of the linear ordering isomorphic to the $\mathcal{A}$-ordinals on to the Odds. To say something specific, it defines a set of odd integers coding this field: $\{2k + 1 | 2k \in \mathcal{A}$ codes a term $t_k \wedge \mathcal{A} \models$ "$t_k \in On$"$\}$. Then it proceeds to:

Step 3i) The process checks that every odd integer in the current s arises indeed from the Field($E_A \restriction On^{\mathcal{A}}$) (which remains coded on the Evens): if not then it sets $s' = \mathbb{N}$ and EXITS.
Step 3ii) It looks for the $\mathcal{A}$-least ordinal in this Odd copy of Field($E_A \restriction On^A$), by comparing the contents of the Odds with E_A. If it succeeds in finding it, let us suppose it is t_{k_0}, it then erases $2k_0 + 1$ from this copy. It thus sets $s' = (s \cap$ Evens$) \cup (s \cap$ Odds$\backslash\{2k_0 + 1\})$ and EXITS. Suppose it failed to find such a least ordinal. Then this is because the ordering written is illfounded, indeed has no minimal element even (note it is not empty). It sets $s' = \mathbb{N}$ and EXITS.

This completes the description of the successor operation $s \to= s'$.

REMARK: $s \cap$ Evens is left untouched. We let the reader verify the following Claim.

Claim 1 There is an arithmetic formula Φ so that $\delta_\Phi(s) = s'$ in the above sense. Q.E.D.

At limit stages we are free to apply any Γ limit rule whatsoever. (Note that once the initial "write" at Step 2 of the Field(E_A), integers are only removed in the transition from s_α to $s_{\alpha+1} = \delta(s_\alpha)$, or else $s_{\alpha+1} = \mathbb{N}$. Consequently at limit stages of length less than the ordertype of the wellfounded part of E^A *all* integers are either stably in or stably out. Therefore the choice of limit rule is not relevant at these stages, as we have no call to appeal to it.

But we claim it is not relevant at any limit stage.

Claim 2 Let $\vec{s}$ be any revision sequence based on δ_Φ and any limit rule Γ. Then one of two kinds of sets of stabilities occur: either (i) $s^+_{<\infty} = \mathbb{N}$ or (ii) $s_0 = s^+_{<\infty}$ is Π_1-$Th(L_\mu)$ for some ordinal $\mu < \omega_1$.

PROOF: The point is that in "most" cases we have to default to setting $s_{\alpha+1} = \mathbb{N}$. This happens if a) $s_0 \cap$ Evens does not code correctly an ω-model of "V = L"; if b) the process discovers an instance of illfoundedness in the Field(E_A); or if c) $s_0 \cap Odds \neq \emptyset$ but is also not a subfield of Field(E_A). (Note the first stage at which we would need to appeal to a limit rule to assign values, is when we have a good s_0 passing the test affirmatively at Stage 1, and whose ordering is wellfounded, and we have cycled through the whole verification process (a multiple of) ω times; then and only then can the bootstrapper/limit rule write something (and then only to the subset of those odds with $2k + 1 \in Field(E_A)$); it can at best thus only write in a set of odd integers that essentially code a sub-ordering of E_A. In any case it is impossible for the limit rule to write an ill-founded ordering to Field(E_A) $= s_\lambda \cap Odds \backslash (s^+_{<\lambda} \cup s^-_{<\lambda})$, since at this stage we have established that E_A contains no illfounded-suborderings!

If we do not default to $s_\alpha = \mathbb{N}$ (in which case (i) of the claim is thereby ruled out) the only possibility is for the process to repeat successfully: but this only works if $s_0 \cap Evens$ codes a suitable Π_1-Theory of some wellfounded countable model of $V = L$. Such a model is isomorphic to some L_μ for a limit $\mu < \omega_1$. As $s_0 \cap Evens$ is untouched by the revision process, whereas no odd integers are stably in s_α throughout, we conclude that $s^+_{<\infty} = s_0 \cap Evens$.

Q.E.D.(CLAIM 2)

Claim 3 B_0 is S^*_Γ-definable.

PROOF: We need to show:

$$B_0 = \bigcap \{ s^+_{<\infty} \mid \vec{s} = \langle s_\alpha \mid \alpha < \infty \rangle \text{ is a revision sequence based on } \Gamma, \delta_\Phi \}$$

($\longrightarrow$) If $n \in B_0$, then $\forall \mu < \omega_1\ L_\mu \models \psi_n$, and so for all s_0, if $A = g``Evens \backslash s_0$ codes a wellfounded ω-model $\mathcal{A}$ satisfying $V = L$, then $\mathcal{A} \models \psi_n$. This direction then follows from Claim 2.

($\Leftarrow$). If $n \notin B_0$, then for some least limit $\mu < \sigma\, L_\mu \models \neg\psi_n$. By the leastness of μ one may show that cofinally in μ new Σ_1 sentences become true. In particular, L_μ is the Σ_1-Skolem Hull inside itself of the integers. (Thus every element of L_μ is given by some Σ_1 term.) Hence, setting $A = g``Evens \backslash s_0$, $\mathcal{A} = \langle \omega, E_A \rangle \cong L_\mu$. Let $h = s_0$ code the Π_1 theory of L_μ. (Thus $s_0 \cap Odds = \emptyset$ and we have a clean start.) As E_A is wellfounded, the process successfully recycles and $s_0 = s^+_{<\infty}$. But $n \notin s_0$.

Q.E.D.(CLAIM 3)

If the complete Π^1_2 set of integers is S-definable, then it is easy to modify the above to obtain the same for any other Π^1_2 set z, as any set of this type can be obtained as a recursive preimage of the complete one. If u is any Δ^1_2 set of integers, then u is both Π^1_2 and co-Π^1_2. Q.E.D.(THEOREM 2.1)

By suitably encoding the operator δ_Φ into the Tarskian operator δ_τ, this method yields a previous result of Burgess [2] that for the language of arithmetic with a T-predicate symbol, the stable truth set over the standard model of arithmetic, using the Belnap Rule Γ_B, forms a complete Π^1_2 set.

Comment 1: It would seem then that the Gupta-Belnap theories of truth $\boldsymbol{T}^\#$ and $\boldsymbol{T}^*$ whilst absolutely defined over the natural numbers $\mathbb{N}$, if one considers the truth sets themselves, it is not hard to ask questions that are sensitive to the axioms of set theory, or equivalently, to the model of set theory one may be working in. For example one can consider $T_0 = T^*_{\mathbb{N}}$ (the latter the $\boldsymbol{T}^*$ truth set over $\mathbb{N}$); $T_{k+1} = T^*_{\langle \mathbb{N}, T_k \rangle}$ for $k < \omega$. One proceeds fairly quickly to the equivalent questions concerning truth sets bounding all the T_k. Since each T_{k+1} is essentially a code of the Δ^1_2 jump of T_k, one has the variations on the questions concerning minimal upper bounds in the Δ^1_2 degrees, which have different answers depending if $V = L$ or not ... see [7]. Thus *as a theory of truth* it must yield to the inevitable fact that answers to simple questions concerning truth sets are just not absolute. The author had made this observation (which was subsequently discussed in

[16]) before turning to fully varied sequences and the notion of stable categoricity [9] 6D. Here it transpired that the problem is even more acute: when considering truth sets for fully varied sequences (see §3 below) the set of integers coding the truth set over $\mathbb{N}$ is itself not absolute between different models of set theory. The phenomenon occurs again (Theorem 5.1) when considering the notion of stable categoricity (concerning which the T_k sequence mentioned above could now be considered as the first ω steps of a putative "categoricity operator sequence"). Revision theory must accommodate itself to this lack of absoluteness.

Comment 2: In view of Comment 1, it might throw extra ammunition into the debate as to what revision theories of *truth* actually are. However as a theory of *definability* revision theoretic definability seems unexceptionable: it yields the next stable set over a structure. But sensitive questions in the definition of which limit rule to choose are simply overwhelmed by the outer universal quantifier: it allows all possible hypotheses (and hence in the above proof all possible wellfounded relations) to be considered. Although one ostensibly has the potential for quantifying over the power set of the integers by using the Belnap Rule Γ_B at limit stages, what Theorems 2.1 and 2.2 also imply is that this is not the point: it is the outer universal quantifier that inevitably raises this complexity to the level of Π^1_2.

Comment 3: Löwe has raised the suggestion that Infinite Time Turing Machines (ITTMs) may help with sorting out some of these questions. Indeed in [15] Theorem 4.1 he shows explicitly how the action of a revision sequence $\vec{s} = \langle s_\alpha \mid \alpha < \beta \rangle$ constructed according to the Herzberger minimal rule Γ_H, with successor revision step given by some arithmetic δ_Φ, with Infinite Time Writable length β, can be modelled on a certain ITTM $M_{\Phi,\beta}$. Conversely, inputting s_0 into $M_{\Phi,\beta}$, the machine will 'eventually write' the whole sequence $\vec{s}$ to its output tape. ('Eventually write' has a technical meaning here: that (a code for) the sequence will eventually be residing on the output tape of such a machine if it is allowed to run for long enough - although the machine process itself may not halt.) In fact the restriction on the lengths of the sequences $\vec{s}$ being writable is not necessary here. One can also devise a machine program that runs on input $h = s_0$, and if h is recurrent according to S_{Γ_H} and δ_Φ, will halt with h on its output tape. This answers one question from [15]: how large can $Refl_{\Phi,\Gamma}(h)$ (essentially the length of the period between h's occurrence and its reappearance in a revision sequence) be? For $\Gamma = \Gamma_H$ (or $\Gamma = \Gamma_G$), and for any arithmetic Φ, (indeed for a broad class of revision operations δ, and limit rules) it is less than λ^h - the supremum of the lengths of time of any halting computation, with 0 on the input tape, with oracle h. With this in mind one can write a "program" that halts precisely on the reappearance of recurrent hypotheses h. But in fact, conversely, one can view any sequence of ITTM computations with input y as some sort of "slow moving" Herzberger revision sequence using the Tarskian revision operator $\delta\tau$ over the structure of arithmetic augmented by a predicate for y: $\langle \mathbb{N}, \ldots y \rangle$: one simply has to code into a suitable diagonalising formula the specification of the machine program.

Comment 4: Similar considerations to that of the theorem show that if L is the language of Arithmetic (or any recursive language that has acceptable models), then V^*_L (and $V^{\#}_L$) have complexity also precisely that of a complete Π^1_2 set. (Problem 32 of [13]).

Comment 5: We saw in Theorem 2.1 that essentially any reasonable limit rule Γ would define for us the same class of reals. We can generalise this in another direction: we do not even need a theory of circular *totally defined* definitions in order to do this: instead of requiring that at each stage hypotheses are assumed for the extensions of the predicates being defined that are *total*, that is totally specify both the extension and the anti-extension of the predicate, it is possible to develop a theory of circular definitions of *partial hypotheses*, where we have some information as to what is in the extension of a predicate G, and what is not. An attractive feature of such a theory is that it dispenses entirely with limit rules Γ: we should only take the local stability pair set as giving us the current extension and anti-extension: those values that had alternated cofinally in the limit ordinal would then simply be undefined at the limit stage. We should thus have truth-value gaps. We should perform our revisions δ_Φ for some formula Φ as before but using some suitable 3-valued logic. However it turns out that exactly the same set of $z \subseteq |\mathcal{M}|$ would be definable as the intersection of all stable (partial) extensions arising in this way: namely the same $\Pi_1(\mathbb{S}_\mathcal{M})$-sets as before.

(To make this plausible, consider the process of the proof of Theorem 2.1: under a regimen as just outlined (using say Kleene's Strong 3-valued logic), this can be simply taken over. The process will review a "good" hypothesis that contains a true Π_1 theory in the same way. As mentioned above after *Claim 1*, no appeal is made to any particular limit rule. In short the set B_0 will still be definable here. For the converse direction, one may argue that no new sets can be defined. Such kinds of revision sequences based on partial extension/anti-extensions can be constructed using a defining formula Φ in a suitable language containing a predicate symbol $\dot{X}$, which can be partially interpreted as (X^+, X^-), just as is done using a partially interpreted $\dot{T}$ symbol. It is simply that we are not restricting our starting hypotheses to be *sound* for the operator δ_Φ (as is done in the Kripkean theory when one wants to build up to a fixed point using δ_τ). It is a straightforward matter to see that such revision sequences can be simulated, for example, on ITTM's. The latters' action can in turn be coded into Herzberger revision sequences *via* some formula Φ' - which can be effectively determined from Φ. (See the last sentence of Comment 3 above.))

This shows that as a theory of definitions the Belnap and Gupta theory is not of necessity tied to revising totally defined hypotheses: the same class of definable sets arises in the way just described. Again, it is the power of the "outer quantifier" that allows all this to happen.

§**3. Fully varied sequences.** We consider Yaqūb sequences [22] and the *fully varied (fv)* sequences of Chapuis [4]. Both authors are only considering sequences formed using the revision function τ - the evaluation of sentences containing a T-predicate, based on the Tarskian biconditionals. We generalise this to arbitrary arithmetic revision operators δ_Φ, before specialising it again at Corollary 3.5.

DEFINITION 3.1. *A revision sequence (based on a revision rule δ) $\vec{s} = \langle s_\alpha \mid \alpha < \infty\rangle$ is* fully varied (fv) *if any real $r \subseteq \omega$ that is coherent with the whole sequence $\vec{s}$, has actually been applied as a bootstrapper to assign values cofinally in ∞.*

REMARK: In general then a fv-sequence must have length at least $\mathfrak{c} = 2^{\aleph_0}$, that of the continuum - at least *prima facie* - although it is easy to see by a Löwenheim-Skolem argument, that proper initial segments of sequences determine the set of stabilities; moreover for any fv-sequence $\vec{s}$ there is another fv-sequence $\vec{r}$ with the same set of stabilities, that in fact is determined by a countable initial segment of $\vec{r}$, as follows.

Note: If $\vec{s}$ is fv, then as $s^+_{<\infty}$ is trivially coherent with $s_{<\infty}$,$\{\alpha < lh(\vec{s}) \mid s_\alpha = s^+_{<\infty}\}$ is cofinal in $lh(\vec{s})$.

Then the stabilization point of a revision sequence is destined to repeat cofinally with the positive truth values those of the stably true integers of the whole sequence.

LEMMA 3.1. *If $\vec{s}$ is fv, $\sigma = \sigma(\vec{s})$ and $\vec{t}$ any revision sequence (not necessarily fv) with $\vec{t} \restriction \sigma = \vec{s} \restriction \sigma$, then for any limit $\tau \geq \sigma \quad t_{<\tau} \supseteq s_{<\tau} \supseteq s_{<\sigma}$.*

PROOF: The second containment follows from the definition of σ. Suppose for a contradiction the first fails, and let $\tau = \gamma + \omega$ be the least limit ordinal greater than σ for which this fails, for some limit γ. By induction $t_{<\gamma} \supseteq s_{<\gamma}$, and consequently t_γ is coherent with $s_{<\sigma} = s_{<\infty}$. As $\vec{s}$ is fv, $t_\gamma = s_\delta$ for cofinally many $\delta > \sigma$. But then $s_{\delta+\omega} = t_{<\tau} \supseteq s_{<\sigma}$. Contradiction! Q.E.D.

COROLLARY 3.2. *If $\vec{s}, \vec{t}$ are as above, then $s_{<\infty} \subseteq t_{<\infty}$.*

For later use we make the definition:

DEFINITION 3.2. *Let $\mathcal{S}$ denote the class of* static *pairs; where $s = (s^+, s^-)$, a disjoint pair, is* static *iff for all revision sequences $\vec{t}$ (t_0 coherent with $s \longrightarrow t_{<\infty} \supseteq s$).*
(ii) $s \in \mathcal{S}$ is "maximal static pair" if no $t \in \mathcal{S}$ properly contains s,

Thus if $\vec{s}$ is fv, and σ is its stabilization point, then the last lemma implies that $s_{<\sigma} \in \mathcal{S}$. Note also that if $\tau > \sigma$ and $s_{<\tau} \not\subseteq s_{<\sigma}$ then $s_{<\tau}$ is not static: because there is a later $\eta > \tau$ with $s_{<\eta} = s_{<\sigma}$.
Note: (i) "$s \in \mathcal{S}$" can be expressed as a $\Pi^1_1(\delta)$ predicate: $s \in \mathcal{S} \longleftrightarrow \forall t(t \supseteq s \longrightarrow \delta(t) \supseteq s)$.
(ii) Being a maximal static pair similarly is $\Pi^1_2(\delta)$.

LEMMA 3.3. *s is a stabilization pair of a fv-revision sequence using a revision operator δ iff s is maximally static.*

PROOF: Let $\vec{s}$ be fv, and $s = (s^+_{<\sigma}, s^-_{<\sigma})$ its stabilization pair. s is clearly in $\mathcal{S}$, but must also be maximal: if $t \in \mathcal{S}$ with $t \supseteq s$ then t^+ occurs on the $\vec{s}$ sequence cofinally. But so does $s^+_{<\sigma}$. Hence we must have $s = t$. Conversely if $s \in \mathcal{S}$ is maximal, consider any $\vec{s}$ fully varied with $s_0 = s^+$. Note that we must have $s = s_{<\sigma(\vec{s})}$. Q.E.D.

Yaqūb has a different definition of revision sequence to enforce variability of limit rule assignments. He makes an implicit use of the Axiom of Choice to wellorder the continuum, and his sequences are (at best) members of $H_{\mathfrak{c}^{++}}$. We omit his definition. But it is a result of Chapuis and Gupta ([4] Theorem 3.1, proven for the Tarskian revision rule τ (but which works in this more general setting) that if s is the stabilization pair of a fully varied sequence, then there is a a Yaqūb sequence, with s as its stabilization pair..

We consider then the set of integers that are stably in all fv-sequences. The analogy with S^*_Γ is:

DEFINITION 3.3. *(i) $z \in \widetilde{S}^*$ iff there is an arithmetic Φ so that:*

$$z = \bigcap \{ s^+_{<\infty} \mid \vec{s} = \langle s_\alpha \mid \alpha < \infty \rangle \textit{ is a fv- revision sequence based on } \delta_\Phi \}$$

(ii) For a particular operator δ (not necessarily arithmetic) by $\widetilde{S}^*_\delta$ we mean that particular z determined as in (i) but by using δ for δ_Φ.

REMARK: In (i) above, the very same class $\widetilde{S}^*$ will result if we restrict δ_Φ with Φ varying over just recursive operations, or widen it to all functions $\delta \in \Delta^1_2$. We have not given the # form of the definition, as it turns out that the same classes are definable with either formulation.

It looks, again *prima facie*, that $\widetilde{S}^*$ is $\Pi_1(H_{\mathfrak{c}^+})$ definable. But it is simpler than that.

THEOREM 3.4. *(i) For any operator δ, every $z \in \widetilde{S}^*_\delta$ is a $\Pi^1_3(\delta)$ set of integers.*
(ii) Conversely: $\widetilde{S^*}$ contains a complete Π^1_3 set of integers.

If we specialise the result to the Tarskian rule τ for partially defined truth predicates we obtain:

COROLLARY 3.5. *Let $\widetilde{V}_\mathbb{N}$ be the truth set over the standard model of arithmetic, using $\widetilde{\boldsymbol{T}}$, the theory of truth for fully varied revision sequences. (That is*

$$\widetilde{V}_\mathbb{N} = \bigcap \{ s^+_{<\infty} \mid \vec{s} = \langle s_\alpha \mid \alpha < \infty \rangle \textit{ is a fv revision sequence based on } \delta_\tau \}.)$$

Then $\widetilde{V}_\mathbb{N}$ can be construed as a complete Π^1_3 set.

PROOF: of theorem (for $\widetilde{S}^*$):

(i) Given an arithmetic operator δ_Φ it is straightforward to write out the corresponding $z \in \widetilde{S}^*$ in a Π^1_3 way. (See the $(\geq_1)$ direction below.) We concentrate on the converse (ii): Let T be a complete Π^1_3 set of integers. We can think of T as represented by $k \in T \iff \forall x \exists y \forall z \forall n R(x, y, z, n, k)$ for some recursive R.

LEMMA 3.6. *There is an arithmetic Φ so that $T \equiv_1 \widetilde{S}^*_{\delta_\Phi}$.*

PROOF: Again we informally describe a revision operator $\delta_\Phi : \mathcal{P}(\omega) \longrightarrow \mathcal{P}(\omega)$ so that $k \in T$ iff for all fv-sequences $\vec{s}$, $f(k)$ is stably true in $\vec{s}$ (*i.e.* $f(k) \in s^+_{<\infty}$), for some total (1-1) recursive f. This will show $T \leq_1 \widetilde{S}^*_{\delta_\Phi}$.

We recursively split up ω into infinitely many infinite blocks $B_k = \{m \mid r(m) \in \{k\} \times \omega\}$ where $r : \omega \leftrightarrow \omega \times \omega$ is some recursive bijection. We reserve the blocks $B_{3k}, B_{3k+1}, B_{3k+2}$ for "thinking about" whether $k \in T$. For any $s \in \mathcal{P}(\omega)$ we view it ambiguously as a function in 2^ω and write interchangeably "$t \in s$" or "$s(t) = 1$." Given an $s = s_\alpha \in \mathcal{P}(\omega)$, we split it recursively into infinitely many other subsets as follows:

$m \in x_k \longleftrightarrow \exists t \in s\, (r(t) = \langle 3k, m+1 \rangle)$
$m \in y_k \longleftrightarrow \exists t \in s\, (r(t) = \langle 3k+1, m \rangle)$
$m \in z_k \longleftrightarrow \exists t \in s\, (r(t) = \langle 3k+2, m \rangle)$

Step 1 We ask:
Q. Does $\forall n R(x_k, y_k, z_k, n, k)$ hold?

If NO:
(i) we 'reverse' all the digits of z_k, or in other words ensure that it is replaced by its complement in forming $s_{\alpha+1}$. (So, by "reversing" we mean the obvious thing: if $r(t) = \langle 3k+2, m \rangle$ then t is either removed from, or put into, s_α to form $s_{\alpha+1}$ depending on whether m was, or was not, in z_k . Thus we shall have, for such t, $t \in s_\alpha \iff t \notin s_{\alpha+1}$.)

(ii) We write a "0" in $\langle 3k, 0 \rangle$, (that is, ensure that $s_{\alpha+1}(t) = 0$ where $r(t) = \langle 3k, 0 \rangle$), and similarly "reverse" all the digits of y_k.

We then go back to the beginning of *Step 1*.

If YES, we proceed to *Step 2*.

One must bear in mind that Q is actually infinitely many questions, one for each k.

Step 2

(i) We ensure that a "1" is written in $\langle 3k, 0 \rangle$ (*i.e.* change s, if need be, so that now $s(t) = 1$, where $r(t) = \langle 3k, 0 \rangle$).
(ii) We reverse all the digits of z_k, and return to *Step 1*.

This completes the part of the revision process for dealing with k. Of course here we are performing the argument simultaneously as one step for all k, as an arithmetic revision. Claims 1 and 2 below will show that the (1-1) total recursive function f defined by $f(k) = s(r^{-1}(\langle 3k, 0 \rangle))$ witnesses that $T \leq_1 \widetilde{S}^*_{\delta_\Phi}$.

Claim 1 Suppose $\vec{s}$ is a fv-revision sequence. If $k \in T$ then $s(r^{-1}(\langle 3k, 0 \rangle))$ stabilizes on 1.

PROOF: Note that from the start of $\vec{s}$, for each k, x_k is unvaried. The process searches for a y_k so that the answer to Q is YES for all z_k. If at any stage z_k fails to be suitable it is "erased" by reversing all its digits; if this happens cofinally in some limit stage, the values of B_{3k+2} can receive input by the limit rule assignment, and a new z_k inspected. We set $\langle 3k, 0 \rangle$ according to the answer to Q. If the answer is NO here, it is because we have not yet found a correct y for y_k. So we ensured that y_k was erased by reversing all of its digits. Because the sequence $\vec{s}$ is fully varied, at some stage, if $k \in T$, there will be a y_k read in that will work for all z_k. From this point on $\langle 3k, 0 \rangle$ stabilizes at 1.
Q.E.D.(CLAIM 1)

Claim 2 If $k \notin T$ then there is a fv-revision sequence $\vec{s}$ so that $s(r^{-1}(\langle 3k, 0 \rangle))$ does not stabilize on 1.

PROOF: If $k \notin T$ pick $x = x_k$ so that

$$\forall y \exists z \exists n \neg R(x_k, y, z, n, k) \quad (*)$$

Let $\vec{s}$ be any fv-sequence with $m \in x_k \longleftrightarrow s_0(r^{-1}(\langle 3k, m+1 \rangle)) = 1$ (*i.e.* with our choice of x_k written to the appropriate block of the starting hypothesis). Then whatever y_k may appear at any stage, however long it lasts on B_{3k+1}, as $\vec{s}$ is fv, some z_k will be read in witnessing $(*)$, and thus that $s(r^{-1}(\langle 3k, 0 \rangle)) = 0$, at some stage.

Further this value of 0 will occur cofinally often in $lh(\vec{s})$ if it is truly fully varied. Q.E.D.(CLAIM 2 AND $(\leq_1)$)

$(\geq_1)$ We shall just show that for any revision operator δ "$n \in \widetilde{S}^*_\delta$" is $\Pi^1_3(\delta)$. Thus for our arithmetic δ_Φ, this is simply Π^1_3 and so is $(1-1)$-reducible to T. Let $\mathcal{S}$ be the class of static pairs arising from sequences built according to the revision rule δ. The result follows from:

$$n \in \widetilde{S}^*_\delta \longleftrightarrow \forall s(s \in \mathcal{S} \wedge s \text{ maximal } \longrightarrow n \in s^+)$$

by appealing to Lemma 3.3, and the Note (ii) on maximally static pairs immediately preceding it.

Q.E.D.(LEMMA & THEOREM).

§**4. Realistically Varied Sequences.** The following notion of "realistic variance" is supposed to capture some of the advantages of full variance from the last section, without the increase of complexity involved. There is motivation and some discussion of this notion in [21].

DEFINITION 4.1. *A revision sequence $\vec{s}$ is* realistically varied *if for all limit $l < \infty$ s_λ is chosen as a coherent extension of s in the following fashion: let $s = (s^+_{<\lambda}, s^-_{<\lambda})$ be the local pair of stabilities at λ.*
(i) Either s_λ is recursive in s or in some s_α for an $\alpha < \lambda$, and s_λ has not been used as an application of the limit rule cofinally in λ;
(ii) Or, if at stage λ there is no s_λ that satisfies clause (i), then s_λ may be chosen arbitrarily.

The maxim here is "use the simple ones first" when it comes to formulating limit rules Γ. Note that this takes the form of a local definition and does not have the global nature of the fully varied sequences.

THEOREM 4.1. *(i) If $\vec{s}$ is a realistically varied revision sequence, and if we let $s_{<\infty} = t = (t^+, t^-)$ be the final stability pair set then any s that coheres with t, and is recursive in t, has been used cofinally often in $lh(\vec{s})$.*
(ii) t forms a Kripkean fixed point for the supervaluation jump.
(iii) The truth set (over $\mathbb{N}$) for the set of all stably true sentences (quantifying over all realistically varied revision sequences), is complete Π^1_2.

PROOF: Let $s, t, \vec{s}$ be as hypothesised. (i) holds by a simple pressing down argument: Let C be the closed and unbounded in ∞ class of points μ where $s_{<\mu} = t$ (such exists by the Note (i) following Def.1.4). If s were a counterexample, we should have that on a stationary class, $E \subseteq C$, of limit ordinals below ∞ either some other fixed $s' \leq_T t$ has been used, or some fixed $s' \leq_T s_\gamma$ has been used for some constant γ. However, considering $\gamma \in E$ which is also a limit point of E, either alternative contradicts the specification of realistically varied. We turn to (ii).

We first prove this under the working assumption that (i) holds for any s with "recursive in t" replaced by "hyperarithmetic in t", and then show how to reduce this to recursive.

We shall let j_{vF} denote the jump operator for the (regular) van Fraassen supervaluational scheme.

Claim (1): Let $u = (u^+, u^-) =_{df} j_{vF}(t)$. Then $u = t$.

PROOF: By definition of j_{vF} we have:

$$u = \Big(\bigcap_{\substack{\overline{u} \supseteq t^+ \\ \overline{u} \cap t^- = \emptyset}} j_T(\langle \mathbb{N}, \overline{u} \rangle), \bigcap_{\substack{\overline{u} \supseteq t^+ \\ \overline{u} \cap t^- = \emptyset}} \mathbb{N} \backslash j_T(\langle \mathbb{N}, \overline{u} \rangle) \Big)$$

(Here $j_T(\langle \mathbb{N}, \overline{u} \rangle)$ denotes the set of (codes of) true sentences over $\langle \mathbb{N}, \overline{u} \rangle$ with the T predicate symbol interpreted as $\overline{u}$.) Essentially "$\varphi \in u^+$" and "$\varphi \in u^-$" are both Π^1_1. For example, if $\exists \overline{u} \supseteq t^+, \overline{u} \cap t^- = \emptyset$ with $\langle \mathbb{N}, \overline{u} \rangle \not\models \varphi$ we can find such a $\overline{u}$ hyperarithmetic in t. [Look at the tree of attempts to assign "true" to some set of sentences $\overline{t} \supseteq t^+$, with $\overline{t} \cap t^- = \emptyset$, with the additional property of having $\langle \mathbb{N}, \overline{t} \rangle \models \neg\varphi$. This tree is hyperarithmetic in t, has a branch in V if $\varphi \notin u^+$, and thus by absoluteness has a branch hyperarithmetic in t.]

Consequently to calculate u it is only necessary to take that intersection above over $\overline{u}$ that are hyperarithmetic in t. Using our working assumption, we may further assume, without loss of generality, that all such $\overline{u}$ appear cofinally below any α in a closed and unbounded class $C_1 \subseteq C$. It is then routine to check for such $\alpha \in C_1$, $t_{<\alpha} = t = j_{vF}(t)$. Q.E.D.(1)

We turn now to the necessary reduction of the hyperarithmetic case to the recursive one. We need to show that any s that coheres with, and is hyperarithmetic in t, has in fact appeared cofinally in ∞, for then we can appeal to the argument just given.

By the definition of realistically varied, again by a closure argument, there is a cub $D \subseteq C$ so that for any $\mu \in D$, any $r \leq_T t$, or $r \leq_T s_\alpha$ for an $\alpha < \mu$, has been used cofinally in μ.

Idea: To find a recursive subset A of $\omega \backslash (t^+ \cup t^-)$ in which to code a copy of t^+, and where we may calculate (copies of) the following sets of integers. (There seem to be many ways of doing this, with the same effect: we just want to guarantee that sufficiently complicated sets do after all, appear sufficiently often, and the following is just one method.) We choose (uniformly for all reals t) an index e for the e'th function recursive in t that is a code for a linear ordering, which has a wellfounded initial segment of length ω^t_{1ck} - the least ordinal not recursive in t. (The existence of such an index e is demonstrated in [8].) We may then set $t^1 = t^+ \oplus t^{+\prime}$, $t^2 = t^1 \cap t^{1\prime}, \ldots t^{\alpha+1} = t^\alpha \cap t^{\alpha\prime} \ldots t^\lambda = \cap_{\alpha<\lambda} t^\alpha$ (where x' denotes the Turing jump of $x \subseteq \omega$) repeating the Turing jump operation along the linear ordering, taking intersections at limits. At some point we discover that the linear ordering is not a wellordering, and we simply abort our process at that stage. We now appeal to the fact that any set $v \subseteq \omega$ which is hyperarithmetic in t is actually recursive in some t^α (for an $\alpha < \omega^t_{1ck}$) arising along one of these wellorderings (see [19] Exercise 16.93). As each t^α will be coded onto some recursive subset of ω at some stage β, by some stage $\mu \in D$ we shall have used any possible limit rule value recursive in it unboundedly in μ. The argument of Claim (1) will then yield the full result of (ii).

We now look at the routine details of embedding an arithmetic operator δ into the operations of the δ_τ operator. We follow Burgess' ([2] p678) notation and method here. Let T be the truth predicate symbol of the language, we assume a coding of the language, and adopt the convention that $\overline{n}$ is the numeral for the number n; we let $\#\varphi$ be the number that codes φ, and let 'φ' be its numeral. We assume that $\mathbb{N}$ is enriched with a symbol for each primitive recursive function. We let $Sub(u, v)$ be the substitution function, for the code number resulting when the term with number v is substituted for the term, or into the formula, numbered u. If '$k \in \delta(X)$' is defined by the arithmetic formula $\Psi(k, X)$ we let $\psi(k, x)$ be the formula obyained by translation from Ψ by replacing each atomic instance of '$k \in X$' by $T(\overline{Sub}(x, k))$. By appealing to the Diagonal Lemma, we may find a formula $\varphi(k)$ such that $\varphi[\overline{n}]$ 'says' that $\psi[\overline{n}, `\varphi(k)']$.

We intend to describe an arithmetical operator, η, and then use the above translation of the defining formula for '$k \in \eta(X)$'. If the defining formula is $\Psi(k, X)$, and $\psi_0(k, x)$ is this translate, we shall consider

$$\psi(k, x) \Longleftrightarrow \psi_0(k, x) \vee [\forall v \neg T(\overline{Sub}(x, v)) \wedge k = \overline{\langle 0, m\rangle} \wedge T(m)].$$

Our particular diagonalising formula of ψ then reads:

$$\varphi[\overline{n}] \Longleftrightarrow \psi_0[\overline{n}, `\varphi(k)'] \vee [\forall v \neg T(\overline{Sub}(`\varphi(k)', v)) \wedge \overline{n} = \overline{\langle 0, \overline{m}\rangle} \wedge T(\overline{m})].$$

Interpreted this is to mean that n is in the current revision $\eta(X)$, or else if we are at a stage γ with $X = \emptyset$ (as it will be for cofinally many $\gamma < \infty$) then we intend to put into X the codes of all pairs $\langle 0, m\rangle$ of zero, with codes m of all sentences currently assigned 'True' at stage γ, *i.e.* those elements of s_γ. This set will be the starting point of a 'sub-revision' of our arithmetic operator η.

We describe the stages of the operation η, and leave it again to the reader to verify that everything can be arithmetically coded.

DEFINITION 4.2. $E_i =_{df} \{\langle i, k\rangle \mid k \in \omega\}$

If $E_1 \cap X = \emptyset$ then we set $\eta(X) = X \cup \{\langle 1, \langle k, l\rangle\rangle \mid \{e\}^X(\langle k, l\rangle) \downarrow = 1\}$.

If $E_1 \cap X \neq \emptyset$ and $E_1 \cap X$ contains a code of a linear ordering L of ω, then if k is the L-least integer so that $E_{k+2} \cap X = \emptyset$ and either:

Case (1) if k is the L-successor of j, and $W_j =_{df} \{m | \langle j + 2, m\rangle \in X\}$, we set $y_k = W_j \cap W_j'$ (where W_j' is the Turing jump of W_j).

Case (2) if k is an L-limit then we set

$$y_k = \bigcap_{jLk} W_j.$$

In Cases (1) & (2) we set $\eta(X) = X \cup \{\langle k + 2, m\rangle \mid m \in y_k\}$.

If $E_1 \cap X \neq \emptyset$ but does not code a linear ordering, or if there is no such k as in the two cases above, then set $\eta(X) = \emptyset$.

If we set

$$\eta^0(X) = X; \eta^{\alpha+1}(X) = \eta(\eta^\alpha(X))$$
$$\eta^{<l}(X) = \liminf_{\alpha \to \lambda} \eta^\alpha(X) \quad \text{if} \quad Lim(\lambda).$$

$$\eta^{\lambda}(X) = \eta(\eta^{<\lambda}(X))$$

then:

LEMMA 4.2. $\forall X \, \exists \alpha \; \eta^{\alpha}(X) = \emptyset$.

PROOF: Even in the "best case scenario" when $X \subseteq E_0$, the successive operation of η adds more Turing jumps to the sets E_{k+2} along the wellfounded part of the linear ordering L given by $\{e\}^X$. For limit points λ of the operation then, we have that membership in the sets $\eta^{\alpha}(X)$ (for $\alpha < \lambda$) is stable below λ (elements are only put *in*.) The only time elements are removed is to set $\eta^{\alpha+1}(X) = \emptyset$. This will eventually happen at stage $\alpha = \omega_{1ck}^{X}$ when we encounter the first instance of illfoundedness along $\{e\}^X$. In other scenarios, it is easy to see at some point one of the clauses will be unfulfilled and we default to the empty set. Q.E.D.

If $\Psi(k, X)$ is this arithmetical formula describing $k \in \eta(X)$ and we "translate" the formula using the method above, and if we assume X is input as s_{γ} say, with

$$X = \{\langle 0, n\rangle \mid n \in s_{\gamma}\},$$

it will not affect us here what choice of s_{λ} is made for limit $\lambda \in (\gamma, \omega_{1ck}^{s_{\gamma}}]$ as such s_{λ} must cohere with $s_{<\lambda}$, and hence will respect the placing of elements into $\eta^{\alpha}(X)$ for $\alpha < \lambda$.

LEMMA 4.3. *With* X, s_{γ} *as above,* $\eta^{\alpha}(X) \leq_T s_{\gamma+\alpha+1}$ *for* $\alpha < \omega_{1ck}^{s_{\gamma}}$.

PROOF: By induction on α. For $\alpha \geq 0$, $\eta^{\alpha}(X) = \{n \mid \#\varphi[\overline{n}] \in s_{\gamma+\alpha+1}\}$. Q.E.D.

LEMMA 4.4. *With* X, s_{γ} *as above, if* y *is hyperarithmetic in* s_{γ} *then* $\exists \alpha < \omega_{1ck}^{s_{\gamma}} \, (y \leq_T s_{\gamma+\alpha})$.

PROOF: By Shoenfield (see Rogers, [19] Exercise 16.93) every $y \subseteq \omega$ hyperarithmetic in X is Turing recursive in some z_k for a $k < \omega_{1ck}^{X}$ where $z_0 = X$; $z_{\beta+1} = z_{\beta} \cap z'_{\beta}$ and $z_{\lambda} = \cap_{\alpha<\lambda} z_{\alpha}$ (for $Lim(\lambda)$). Q.E.D.

But for unboundedly many $\gamma \in D$, we have that $s_{\gamma} = s^{+}_{<\gamma} = t^{+}$ (that is the liminf rule was used at these stages). Hence for such γ we have $\forall p \in \omega \, Sub(\text{`}\varphi(k)\text{'}, p) \notin s_{\gamma} = t^{+}$. But then by definition of ψ and φ every y hyperarithmetic in $s_{\gamma} = t^{+}$ is recursive in some s_{β} for a $\beta \geq \gamma$. Hence every possible hyperarithmetic in t limit rule has been used unboundely often. As indicated at the "Idea" stage above, this completes the proof of (ii) of the Theorem. The proof of (iii) is just as for the truth set using any of the limit rules Γ_B etc. Q.E.D.(THEOREM 4.1)

Comment 6: The proof of 4.1(ii) uses only the property shown to hold of realistically varied sequences in (i). Thus we could have taken 4.1(i) as the defining property of realistically varied and obtained the same results. We have chosen to define the notion as we have done to emphasise its non-globalarity.

§**5. Categoricity.** Belnap and Gupta consider an augmented language to that (here) of arithmetic, $\mathcal{L}'$, containing a new predicate symbol $\dot{K}$ to be interpreted as the current hypothesis concerning the categorical (or stably true) sentences. The basic model of arithmetic is enlarged to a model $\mathbb{N}^1 =_{df} \langle \mathbb{N}, T, K \rangle$ with the displayed predicates being the obvious interpretation. They wish to consider revisions of a hypothesis concerning now what will ultimately be the set of stably true sentences, K, in exactly the same way that revisions were used to create new approximations to truth. One thus takes a hypothesis concerning the categorical sentences, call it $K = k_0$, and uses this extension in the expanded model above. One then finds the stably true sentences relative to the new model in the new language, keeping the extension of $\dot{K}$ fixed as we perform the revision process on extensions of $\dot{T}$ until we have the stably true sentences with this language. The latter yields a new set of sentences as a revised hypothesis for $\psi_{\mathbb{N}}(k_0) = k_1$.

DEFINITION 5.1. $\psi_{\mathbb{N}}(k) = \{n | n$ *is the gn of a sentence of* $\mathcal{L}'$ *that is stably true under the revision process for* $\dot{T}$*, over the expanded model* $\mathbb{N}^1\}$.

For $\vec{k}$ a revision sequence based on $\psi_{\mathbb{N}}$ and, say the limit rule Γ_B, let $\tilde{k}^+_\infty$ be the set of integers *almost stably in* $\vec{k}$, according to the definition of Gupta & Belnap 6D.9.

Thus the set we are interested in is:

$$z = \bigcap \{ \tilde{k}^+_{<\infty} \mid \vec{k} = \langle k_\alpha \mid \alpha < \infty \rangle \text{ is a revision sequence based on } \psi_{\mathbb{N}}, S^{\#}_{\Gamma_B} \}.$$

In the above, using $\psi^n_{\mathbb{N}}$ for the n-fold iteration of the function $\psi_{\mathbb{N}}$:

$$\tilde{k}^+_{<\infty} =_{df} \{ m \mid \exists \text{ recurrent } k \text{ so that } m \in \psi^n_{\mathbb{N}}(k) \}.$$

The authors are attempting to capture the notion of *almost stability*, that is those m that are almost in $k^+_{<\infty}$: they only fail to be in this because they may not be in k_λ for a limit λ, and for a finite number of revisions thereafter but there is a fixed n so that for all $n \leq n' < \omega\, m \in k_{\lambda+n'}$ for any recurrent k_λ. (We make no attempt to justify this definition but refer the reader to [9] 6D.)

We have also chosen just to take the set of stable truths as opposed to the stability set (of all stable truths and falsehoods) but this makes no difference as far as the complexity computation below is concerned: the stable falsehoods are anyway also a complete Π^1_2-set, thus together they form again such a set.

We shall call the set z above the *stably categorical* set. We then have the following theorem:

THEOREM 5.1. *(i) The stably categorical (over* $\mathbb{N}$*) set of sentences form a* Π^1_3 *set.*
(ii) In Gödel's constructible universe L*, this set is (recursively isomorphic to) a complete* Π^1_3 *set.*

PROOF: We have seen how for $\langle \mathbb{N}, T, \ldots \rangle$ the set of sentences stably in T over all revision sequences is a Π^1_2 complete set. We are essentially then trying to calculate the set of all sentences that are stably categorical in the categoricity operator $\psi_{\mathbb{N}}$ as a revision operator. We thus have here an operator that has a Π^1_2 description.

It is easy to compute the complexity of the above definition of z as Π^1_3, using the fact that the graph of $\psi_{\mathbb{N}}$ is Δ^1_3. It is then (1-1) reducible to the complete Π^1_3 set P, say. The hard part is to show that $P \leq_1 z$.

DEFINITION 5.2. *For $x \subseteq \omega$ let σ_1^x be the least x-stable ordinal. That is, σ_1^x is the least σ so that $L_\sigma[x] \prec_{\Sigma_1} V$.*

Thus σ_1^x is $On \cap \mathbb{S}_{\langle \mathbb{N}, x\rangle}$ in our earlier notation, and we shall abbreviate the latter structure as $\mathbb{S}_x =_{df} \langle L_{\sigma_1^x}[x], \in, x\rangle$. We have shown that the Π_1-Theory of $\mathbb{S}_x$ is recursively isomorphic to the set of categorically true sentences of $\langle \mathbb{N}, x\rangle$. Let us call this Π_1-theory T_x. Then T_x is definable over $\mathbb{S}_x$ and so is a set in $L_{\sigma_1^x+1}[x]$. We shall consider it coded in a recursive way, as a set of integers uniformly in x.

DEFINITION 5.3. $\psi(x) =_{df} T_x$

Then *via* this coding, the set z defined above is then recursively isomorphic to the set defined in the same way with ψ replacing $\psi_{\mathbb{N}}$.

DEFINITION 5.4. $C =_{df} \{\alpha < \omega_1^L \mid L_\alpha \prec_{\Sigma_1} L_{\omega_1}\}$.

Then C is the set of countable Σ_1-stable ordinals. It is well known that adding the predicate C gives the effect of a Σ_1-mastercode. Namely:

FACT 5.1. *In L, $\Pi_{n+1}(\langle L_{\omega_1}, \in\rangle)$ relations are precisely those that are $\Pi_n(\langle L_{\omega_1}, \in, C\rangle)$.*

By Levy-Shoenfield, then, if we assume $V = L$, Π^1_3 relations are $\Pi_2\langle L_{\omega_1}, \in\rangle)$ and so $\Pi_1(\langle L_{\omega_1}, \in, C\rangle)$.

DEFINITION 5.5. *For a constructible set $x \in L$ we let $\rho_L(x)$ be its L-*rank. *That is $\rho_L(x) =_{df}$ the least α so that $x \in L_{\alpha+1}$.*

Now let $x = k_0$ be a hypothesis concerning the categorical sentences of $\mathbb{N}$, that is an initial starting hypothesis for a revision sequence using ψ. We let $k_\nu = k_\nu(x)$ be the set of integers arising at the ν'th stage of this process - we are thus using the letters k_ν rather than s_ν to represent the revisions of the categorical hypothesis x.

From our discussion above we see that the rank of $k_{\nu+1}$ in the $L[k_\nu]$-hierarchy, is just $\sigma_1^{k_\nu}$.

LEMMA 5.2. *(i) If $k_\nu \in L$ then $\rho_L(k_{\nu+1}) = \sigma_1^{k_\nu}$.*
(ii) For any $x \in L$, σ_1^x is the least element of $C > \rho_L(x)$. Hence if $V = L$, any ψ-revision sequence satisfies: $\forall \nu\, \sigma_1^{k_\nu} < \sigma_1^{k_{\nu+1}}$.

PROOF: (i) If $k_\nu \in L$ then so is $k_{\nu+1}$ by Shoenfield's Absoluteness theorem, and is $\Pi_1(\langle L_{\sigma_1^{k_\nu}}[k_\nu]\rangle)$. As $k_{\nu+1}$ is a complete Π_1 theory of this model, it is not a member of $L_{\sigma_1^{k_\nu}}[k_\nu]$. Hence $\rho_L(k_{\nu+1}) \geq \sigma_1^{k_\nu}$.

In L, every set of integers x has $\rho_L(x)$ less than σ_1^x by Σ_1-elementarity applied to "$\exists \xi\; x \in L_\xi$". In particular $\rho_L(k_\nu) < \sigma_1^{k_\nu}$. Since for any $y \in L, L_{\sigma_1^y}[y] = L_{\sigma_1^y}$, by applying this to k_ν, we have that $k_{\nu+1} \in L_{\sigma_1^{k_\nu}+1}$. Hence $\rho_L(k_{\nu+1}) \leq \sigma_1^{k_\nu}$ also. This proves (i), and (ii) is straightforward, using the definition of C and $x \in L \longrightarrow L_{\sigma_1^x}[x] = L_{\sigma_1^x}$. Q.E.D.

Assume for the rest of the proof that $V = L$.

However, now, no such equality as in (i) of the last lemma is true in general for limit λ: we may have (and in fact in many places, must have) $\rho_L(k_\lambda) < \sup_{\nu<\lambda}\{\sigma_1^{k_\nu}\}$.

LEMMA 5.3. *There is a (1-1) recursive $f : \omega \longrightarrow \omega$, so that for all $x = k_0$, and for all γ*

$$\tau \in \Pi_2\text{-}Th(L_{\rho(k_\gamma(x))}) \longleftrightarrow f(\tau) \in k_{\gamma+1}(x).$$

PROOF: Fix an $x = k_0$ and for brevity set $k_\gamma = k_\gamma(x)$ etc. Recall that we are identifying $k_{\gamma+1}$ as the the complete $\Pi_2^{1\,k_\gamma}$ set of integers which is recursively isomorphic to Π_1-Th$(\mathbb{S}_{k_\gamma})$.

Claim "$r = \rho(k_\gamma)$", "$l = L_{\rho(k_\gamma)}$", and "$S = \Pi_2$-Th$(L_{\rho(k_\gamma)})$" are all $\Delta_1(\mathbb{S}_{k_\gamma})$.

PROOF: of Claim. In general "$u = L_v$" is Δ_1 over any transitive rudimentary closed model. Hence it suffices to observe:

$r = \rho(k_\gamma) \longleftrightarrow$

$$\forall y(y = L_{r+1} \longrightarrow k_\gamma \in L_{r+1}\backslash L_r) \longleftrightarrow \exists y(y = L_{r+1} \wedge k_\gamma \in L_{r+1}\backslash L_r).$$

The other statements are then immediate. This is all uniform in γ and independent of the choice of x. Hence (assuming there is a sensible coding of languages by integers) there is (1-1) recursive f with $\tau \in \Pi_2$-Th$(L_{\rho(k_\gamma)}) \Longleftrightarrow f(\tau) \in \Pi_1$-Th$(\mathbb{S}_{k_\gamma}) \equiv k_{\gamma+1}$.
Q.E.D.

Let $S_\gamma =_{df} \Pi_2$-Th$(L_{\rho(k_\gamma)})$; let $T_\gamma =_{df} \Pi_1$-Th$(L_{\sigma_1^{k_\gamma}}[k_\gamma])$. In the notation of the last lemma then there is a fixed (1-1) recursive $f : \omega \longrightarrow \omega$ so that for any revision sequence $\vec{k}$, and for all γ $S_\gamma = f^{-1}{}''T_\gamma$.

Let Ω be least so that $L_\Omega \prec_{\Sigma_2} L_{\omega_1}$. Then as we are assuming $V = L$, we may identify a complete Π_3^1-set, P, say, with the complete Π_2-theory of $\langle L_{\omega_1}, \in\rangle$, which is thus Π_2-definable over $\langle L_\Omega, \in\rangle$.

We shall have:

$$(1)\quad P = \bigcap_{\alpha\in C} \Pi_2\text{-Th}(L_\alpha) = \bigcap_{\alpha\in C\cap\Omega} \Pi_2\text{-Th}(L_\alpha) = \bigcap_{x\in\mathbb{R}\cap L_\Omega} \Pi_2\text{-Th}(L_{\sigma_1^x}).$$

The first equality holds by definition of C (note for $\alpha \in C$, Π_2 sentences are downwards absolute from L_{ω_1} down to L_α), the second by that of Ω, and the third by 5.2 that $\sigma_1^x \in C$ for any $x \in L_{\omega_1}$, and the fact that it suffices to take the middle intersection over all successor elements of $C \cap \Omega$ only.

We note that by the leastness of Ω in its definition, unboundedly below it there are new Σ_2 facts becoming true: that is for unboundedly many $\sigma < \sigma' \in C \cap \Omega$, Π_2-Th$(L_\sigma) \supsetneq \Pi_2$-Th$(L_{\sigma'}) \supsetneq \Pi_2$-Th$(L_{\omega_1})$.

We intend to show:

Claim $P \leq_1 z$ via *the recursive function* f.

First suppose that $\tau \notin P$. We shall define a revision sequence $\vec{k}$ with $f(\tau)$ not in its 'almost stability' set. As $\tau \notin P$ there is a least $\alpha_0 \in C \cup \{0\}$ so that if $\alpha \in C$ is least greater than α_0, we have that $\tau \in \Pi_2\text{-}Th(L_{\alpha_0}) \backslash \Pi_2\text{-}Th(L_\alpha)$. Let $k_0 = x$ where $x \subseteq \omega$ codes α_0; to be specific, pick $x \in WO$, $||x|| = \alpha_0$, of least L-rank; then $\rho_L(x) < \alpha$. (Let $x = \omega$ if $\alpha_0 = 0$.)

(2) There is a revision sequence $\vec{k}$ so that $\forall\delta\exists\gamma > \delta\, \rho_L(k_\gamma) > \alpha_0$.

This then implies, as $\sigma_1^{k_\gamma} = \rho_L(k_{\gamma+1}) \in C$, together with our supposition that $L_\alpha \models \neg\tau$, that for all such γ, $f(\tau) \notin k_{\gamma+2}$. Then $f(\tau) \notin z$ as required.

PROOF: k_γ is defined by induction on γ. $k_0 = x$ is already defined, and for all δ $k_{\delta+1} = \psi(k_\delta)$. Then $\rho_L(k_{\delta+1}) = \sigma_1^{k_\delta} > \rho_L(k_\delta)$. Suppose then k_γ is defined for all $\gamma < \lambda$ and consider the local stability pair set at λ, $(k^+_{<\lambda}, k^-_{<\lambda})$. Suppose that $k^0 =_{df} k^0_{<\lambda} = \omega \backslash (k^+_{<\lambda} \cup k^-_{<\lambda})$ is finite, or that $\rho_L(k^+_{<\lambda}) \geq \alpha_0$, then set $k_\lambda = k^+_{<\lambda}$ (thus we use the liminf rule for such a λ). Otherwise, we shall "fill in" k^0 with a suitably complicated set of integers $A \subseteq k^0$ so that, setting $k_\lambda = k^+_{<\lambda} \cup A$, we shall have $\rho_L(k_{\lambda+1}) > \rho_L(k_\lambda) \geq \alpha_0$. We are thus describing an application of a Belnap style ruling at this limit stage, to ensure a sufficiently complicated set. To be specific, let $\rho^0 =_{df} \rho_L(k^0)$. It is easy to see that $\rho^0 < \alpha_0$ also.

Let $g : k^0 \longleftrightarrow \omega$ be an order preserving bijection recursive in k^0. Let $A = g^{-1}\text{"}x$ (where $x \in WO$ coded α_0). There is thus a wellorder of rank $\alpha_0 \in C$ recursive in k^0 and A. As any element of C is an admissible ordinal, any such wellorder, and thus A itself, must have L-rank $\geq \alpha_0$. We then set $k_\lambda = k^+_{<\lambda} \cup A$. Then check easily that $\rho_L(k_\lambda) \geq \alpha_0$.

This completes the construction of $\vec{k}$. The final point to check before claiming that (2) is proven is that for arbitrarily large λ we do in fact have that $k^0_{<\lambda}$ is infinite, and thus we have an opportunity to "fill in" with an appropriate A to raise ranks, if needed. But this is immediate for any revision sequence based on ψ (or any non-trivial revision rule):

(3) $\forall\gamma\, \exists\lambda > \gamma\, Lim(\lambda) \wedge k^0_{<\lambda}$ is infinite.

PROOF: Suppose not, and that γ_0 is such that for all limit $\lambda > \gamma_0$ $k^0_{<\lambda}$ were finite. But then it is easy to see that this implies that there is $\overline{\gamma}$ with $\forall\delta > \overline{\gamma}, k_\delta$ only differs finitely from $k_{\overline{\gamma}}$. But this is absurd, as, *e.g.*, $k_{\delta+1}$ is the complete Π_2^{1,k_δ} set of integers.

Q.E.D.(3) & (2)

We now suppose that $\tau \in P$. We claim that $f(\tau)$ is in the "almost stability" set of any revision sequence $\vec{k}$, *i.e.*, $f(\tau) \in z$. Suppose for a contradiction $\vec{k}$ is a $\psi_{\mathbb{N}}$-revision sequence, and $k = k_\beta$ is a cofinally recurrent hypothesis such that $\forall m \in \omega\exists p \in \omega(m \leq p \wedge f(\tau) \notin k_{\beta+p})$. But this implies that for some $n \geq 1$ there is $\sigma = \rho_L(k_{\beta+n}) \in C$ (recalling that any $\rho_L(k_{\delta+1})$ is in C), with $\tau \notin \Pi_2\text{-Th}(L_{\rho_L(k_{\beta+n})})$. However τ is in P, hence $\tau \in \Pi_2\text{-Th}(L_\sigma)$ for any $\sigma \in C$. Contradiction!

We thus have a (1-1) recursive function f witnessing that $P \leq_1 z$.

Q.E.D.(THEOREM 5.1)

CONJECTURE: The theorem is true using the Herzberger Rule Γ_H everywhere rather then the Belnap Rule Γ_B.

We leave this as an open problem: if true the proof then becomes significantly more complicated, as we have to argue about the ranks $\rho_L(k_\lambda)$ without having the advantage of artificially raising them by using the trick above. Similarly we should be surprised if the theorem were not true if S^* replaced $S^\#$ in the above. As the purpose of this section is to demonstrate the complexity of the sets involved in this concept, we satisfy ourselves with the version based on the official definition of categoricity in [9] above.

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DEPT. OF MATHEMATICS,
UNIVERSITY OF BRISTOL.
BRISTOL BS8 1TW, ENGLAND.
and
GRADUATE SCHOOL OF SCIENCE & TECHNOLOGY,
KOBE UNIVERSITY,
ROKKO-DAI, NADA-KU
KOBE 657, JAPAN.

E-mail: p.welch@bristol.ac.uk

Current address: Mathematischesinstitut der Universität Bonn,, Beringstr. 6, D-53115 Bonn,, Germany.