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Post's and other problems of supertasks of higher type

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Abstract. We consider the theory of supertasks as implemented on infinite time Turing machines. We consider in particular computations in the type of sets of reals. We give some commentary on this and a number of open problems are raised.

1 Introduction

The theory of infinite time computations on sets of integers is now, at least, reasonably understood. We discuss here some open questions in both this theory but more particularly in that of the theory concerning

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computations in the next type up, that is using reals as inputs/outputs, and sets of reals as oracles. This higher type theory has not as yet received the same attention as the theory on integers. The aim of this article is to provide some commentary on what we feel this higher type theory should look like, and to ask a number of questions whose answers should help develop that theory, and place it in a context with other theories of reducibilities at this level.

In this section we give some of the basic definitions (from [HaLe00]), and mention an alternative machine architecture that yields the same class of computable functions on single tape machines. In the next section we state the corresponding definitions and facts for the higher type theory. We give no proofs of these latter results. Often they are simply the analogous ones for the theory on integers, and the reader can construct them for his or herself.

We refer the reader to [HaLe00] or to Joel Hamkins' article in this volume, for the basic structure, definitions, and results concerning such machines. We let P_n denote the n'th program in some fixed enumeration. In Section 2 we shall assume some familiarity with the notions of determinacy of two person perfect information games - see. *eg*, [Ka94]. For a pointclass Γ "$\mathrm{Det}(\Gamma)$" will denote the assertion that games with payoff sets $A \in \Gamma$ have winning strategies for one of the players.

It was noted in [HaSe01] that the machine architecture could be replaced by a single tape machine using 0 and 1 as an alphabet, but only as long as one considered computable functions $f : \mathbb{N} \to \mathbb{N}$. For functions $F : \mathbb{R} \to \mathbb{R}$ it turned out that a single tape is insufficient.

We first note here that a not unattractive one (or three) tape machine model is obtained which computes the same class of functions (on either integers or reals) as the original machine architecture, by allowing into the alphabet blank cells on the tape. (To be sure one now has a machine that in essence works on 3^ω instead of 2^ω, although for the theorem above we still consider $x \in \mathbb{R}$ input as an infinite sequence from $\{0, 1\}$ only. A function $F : \mathbb{R} \to \mathbb{R}$ is computable by such a machine, if for all real input strings, the output strings are also real strings, *i.e.*, without blanks.) The purpose of a "blank" or empty cell is to signify ambiguity. We adopt a new limit rule for specifying the contents of cells at limit stages: if a cell's value has varied cofinally often below λ, we set the value to the "ambiguous" value of a blank.

Ambiguity limit rule: *If λ is a limit ordinal, then the contents of the i'th cell on the tape at time λ, $C_i(\lambda)$, is given by: If $\exists\nu_0 < \lambda$ such that $\forall\nu < \lambda(\nu_0 < \nu \to C_i(\nu_0) = C_i(\nu))$ then set $C_i(\lambda) = C_i(\nu_0)$; otherwise set $C_i(\lambda)$ to be a blank.*

One then has:

Theorem 1 *Let $\mathcal{C}$ be the class of functions $F : \mathbb{R} \to \mathbb{R}$ computable by the Hamkins-Lewis machines of [HaLe00], and $\mathcal{C}'$ those of the one-tape machine just specified. Then $\mathcal{C} = \mathcal{C}'$.*

It is not hard to see that if we now define a one-tape machine using this ambiguity limit rule, then its action can be simulated on a $\{0, 1\}$-valued three tape [HaLe00]-machine. It was noted in [HaSe01] that the single tape machines with $\{0, 1\}$ values only computed a class of functions that was not closed under composition. This had the effect that the natural simulation of a 3 tape machine by a 1 tape machine could not realise that it had performed a final compression of the simulated output. (A *compression* is a map of the form $\langle x_0y_0z_0x_1y_1z_1\ldots\rangle \longrightarrow \langle z_0z_1z_2\ldots\rangle$; although clearly 1-tape computable on its own, one cannot compose it in a 1-tape computable fashion with the other steps of the simulation. One needs a flag or some device such as further scratch pad (see [HaSe01] Theorem 2.3) that is not part of the final output to allow us to know when the final compression is complete, and the 1-tape computation may halt.)

We have here an extra symbol - namely the blank - in the alphabet, and this gives us enough room to set flags and enable us to close up under composition, and thus calculate the same functions as the 3 tape model. (For readers familiar with [HaSe01] this is really an exercise.) However, the theorem above notwithstanding, we shall stick to the definitions and conventions of [HaLe00] for this paper. The rest of the definitions of this subsection are all taken from that paper. It should be noted that we deliberately elide the distinction between subsets of ω and their characteristic functions as elements of 2^ω and write "$n \in g$" interchangeably for "$g(n) = 1$" etc.

Definition 1 *$\varphi_n(x)$, for $x \in 2^\omega$, denotes the result of running P_n on input string x.*

For any n, x it is easily seen that there is a countable ordinal α so that the behaviour of $\varphi_n(x)$ has "settled" by stage α: either it has halted or it has entered a permanent looping cycle: notationally, either $\varphi_n(x)\downarrow y$ for some $y \in 2^\omega$, or else $\varphi_p(x)\uparrow$. For a machine with an oracle $f \in 2^\omega$ we allow the machine to receive $0/1$ answers to queries of the form "Does $f(n) = 1$"? We write "$\varphi_p^f(x)\uparrow y$" for "program p with input x and oracle f eventually has a settled y on its output tape"; in other words, there is a time ν, so that for all later times $\varphi_p^f(x)$ has y sitting on its output tape - although without requiring that $\varphi_p^f(x)\downarrow$. The relations "$\varphi_p^f(x)\uparrow y$", "$\varphi_p^f(x)\uparrow$", and "$\varphi_p^f(x)\downarrow y$" are Δ^1_2 relations of $f, x, (y)$. We let WO denote the set of reals in 2^ω that code wellorderings of ω.

Definition 2 $x \in \mathcal{W} \longleftrightarrow \exists p \in \omega \varphi_p(0)\downarrow x$ *($\mathcal{W}$ is the class of "writable" reals).*

Fact 1 *([HaLe00] 3.7, 8.3) The "writable" ordinals, (those ordinals α so that there is $x \in \mathcal{W} \cap WO$ with $||x|| = \alpha$) form an initial segment of the countable ordinals.*

Definition 3 *(i)* $\lambda =_{df} \sup\{||x|| : x \in WO \cap \mathcal{W}\}$*;*
(ii) $\gamma =_{df} \sup\{\alpha \mid \exists p \in \omega\ \varphi_p(0)\downarrow$ *and halts in exactly* α *steps* $\}$.

Definition 4 *(i)* $\mathcal{EW} =_{df} \{x \in 2^\omega \mid \exists p \varphi_p(0)\uparrow x\}$. *$\mathcal{EW}$ is the set of "eventually writable" reals. (ii)* $\zeta =_{df} \sup\{||x|| : x \in WO \cap \mathcal{EW}\}$.

Definition 5 *Let* $\Sigma =_{df} \sup\{||t|| : t \in \mathcal{AW} \cap WO\}$. *$\mathcal{AW}$ is the set of "accidentally writable" reals where:*
$\mathcal{AW} =_{df} \{x \in 2^\omega \mid \exists p\ x$ *appears on any tape at any time of the computation of* $\varphi_p(0)$ $\}$.

The accidentally writable reals are thus likely to be transient. (An equivalent class of reals is obtained by restricting the appearance of such reals to the output tape alone.) Clearly $\mathcal{AW} \supseteq \mathcal{EW} \supseteq \mathcal{W}$. We let HC denote the class of hereditarily countable sets.

Definition 6 *We write $H(\lambda)$ ($H(\zeta), H(\Sigma)$ respectively) for the class of sets $y \in HC$ so that $\exists \bar{y} \in \mathcal{W}$ ($\mathcal{EW}, \mathcal{AW}$ respectively) with $\bar{y}$ coding y.*

We use the notation $\mathcal{W}^f, \lambda^f \ldots, H^f(\lambda^f), \ldots$ for the notions relativised to a real f.

Definition 7 *(The weak jump) For* $f \in 2^\omega$, $f^\nabla =_{df} \{n \in \omega \mid \phi_n^f(0)\downarrow\}$.

It is easy to see that $f \to f^\nabla$ is a Δ^1_2 function. The notions of g *is a decidable (or semi-decidable) set of integers* is the natural one: g must have a totally (or partially) computable characteristic function:

Definition 8 $g \in 2^\omega$ *is* semi-decidable in the real f *if for some* e *we have*

$$\forall n \in \omega \quad n \in g \longleftrightarrow \varphi_e^f(n)\downarrow 1.$$

As usual g is decidable (in f) if both it and its complement are semi-decidable (in f).

For the ordinary notion of Turing computability one has the equivalence between the semi-decidable, *i.e.*, the recursively enumerable, sets of integers as those that are the domain of some computable function, with, as alternative, those that are the range of such. One can establish this by observing that from the universality of Kleene's T-predicate one may simply test integers to see if they code a whole course of computation of the relevant function, and seeing if they terminate in a desired number output. The point is that the course of computation can be coded as something within the domain of the Turing machine. But is that the case for infinite time Turing machines? Or do we have in the phraseology of Sacks [Sa80], a "violation of parity" (between the type of the object in the domain of the computation, and that of the computation itself)? In essence we are asking whether (a code for) a halting computation can also be the result of a halting output of another computation. For this it is enough to have a real code for the *length* of any halting computation to be a potential output of some halting computation. Once we have such a code to hand we can then write a program simulating the original course of computation along that ordinal, and so output a code for the course of that computation. In the terms we have defined, we are asking whether $\gamma \leq \lambda$. This had been the principal open problem left from [HaLe00]. The answer is affirmative.

Theorem 2 *([We00a] Thm.1.1)* $\lambda = \gamma$*, and by relativisation for all* $f \in 2^\omega \quad \lambda^f = \gamma^f$.

Once we have this result we can answer a number of questions, as to what the decidable sets of integers are, what is complexity of the weak jump operator, what are the sets $H(\lambda)$ etc.

Theorem 3 *([We00a] Cors. 3.1,3.3,3.5) For* $\theta \in \{\lambda, \zeta, \Sigma\}$ $H(\theta) = L_\theta$*, and for any* $f \in 2^\omega \quad H^f(\theta^f) = L_{\theta^f}[f]$.

Corollary 4 *The decidable sets of integers are precisely those of* $\mathcal{P}(\omega) \cap L_\lambda$.

The relationship between the various $H(\theta)$ sets is given by:

Theorem 5 *([We00] The "*(λ, ζ, Σ) *Theorem") (i)* $L_\lambda \prec_{\Sigma_1} L_\zeta \prec_{\Sigma_2} L_\Sigma$. *(ii)* (λ, ζ, Σ) *is the lexicographically least triple of ordinals satisfying this relation.*

This theorem implies that we may consider the definable sets of integers as those Σ_1-definable inside the least model of KP with a transitive Σ_2-end extension. It further implies that L_λ, L_ζ are admissible sets satisfying strong reflection properties. For example, there are unboundedly many α below any of the ordinals λ, ζ, Σ where $L_\alpha \models \Sigma_2$-$KP$ (meaning Kripke-Platek set theory with Σ_2-Collection and Δ_2-Comprehension axioms).

By appealing to Theorem 2 as $\gamma^f = \lambda^f$, Theorem 3, and by considering the running of such machines inside $L_{\lambda^f}[f]$ one sees:

Corollary 6 $f^\nabla \in \Sigma_1(L_{\lambda^f}[f])$.

As f^∇ is not an f-decidable set of integers, it cannot lie inside $L_{\lambda^f}[f]$. It is thus perhaps unsurprisingly a Σ_1-mastercode for L_λ.

Corollary 7 *([We00])* f^∇ *is (ordinary) Turing isomorphic to* Σ_1-$Th(L_\lambda)$*, the complete* Σ_1 *theory of* L_λ.

There is the natural *reducibility relation* associated to reals:

Definition 9 *For* $x, y \in 2^\omega$*:*

$$x \leq_\infty y \longleftrightarrow \exists n \forall k \in \omega\, [\varphi_n^y(k){\downarrow}1 \leftrightarrow x(k) = 1 \wedge \varphi_n^y(k){\downarrow}0 \leftrightarrow x(k) = 0]$$

Corollary 6 enables one to see that the assignment $f \mapsto \lambda^f$ satisfies the usual *Spector Criterion*:

Lemma 8 *(i)* $x \leq_\infty y \longleftrightarrow x \in L_{\lambda^y}[y]$*; thus (ii)* $x \leq_\infty y \longrightarrow \lambda^x \leq \lambda^y$*; (iii) The assignment* $x \longrightarrow \lambda^x$ *then satisfies a Spector criterion:*
$x \leq_\infty y \longrightarrow (x^\nabla \leq_\infty y \longleftrightarrow \lambda^x < \lambda^y)$.

The analogy of $\leq_\infty$ with ordinary Turing reducibility is explored in [HaLe00]. We take Lemma 8 as indicating that the proper analogy here is rather with something intermediate between hyperdegrees (with the assignment $x \mapsto \omega_1^{x}$[1] and that of Δ^1_2-degrees (with assignment $x \mapsto \Delta^1_2(x)$).

The principle open question we can formulate at this point is the following.

Question 1 *Let D be a countable set of infinite time degrees. Does D have a minimal upper bound?*

This question remains unresolved for hyperdegrees. For Δ^1_2-degrees there are two differing but complete pictures, both with affirmative answers, depending on whether $\mathbb{R} \subseteq L$ or not (see [Fr74]). There are some partial results on Question 1 in [We99], but they are very partial, and the problem looks difficult[2].

1.1 Eventually infinite time degrees

There is one sense in which one could argue that the ordinal ζ is perhaps more fundamental than λ. It is after all the point by which the behaviour of the machine on a computation of the form $\varphi_e(k)$ is determined: either it has halted, or it has begun to loop permanently at some ordinal $\bar{\zeta} \leq \zeta$. (Moreover there is e so that this upper bound is attained here. It would take us too far afield here, but it is essentially the pattern of the universal machine φ_U's output tape at stage ζ that is recursively isomorphic to certain sets of interest occurring in the revision theory of truth based on Herzberger sequences.)

We could thus define a reducibility relation "x is eventually writable from y".

Definition 10 $x \leq_{e\infty} y \longleftrightarrow x \in \mathcal{EW}^y$.

We define a jump:

Definition 11 *$\tilde{x}$ is the set of indices $\{p \in \omega \mid \exists y \in \mathcal{EW}^x \;\varphi_p^x(0){\uparrow}y\}$.*

[1] We write ω_1^x for the least ordinal not recursive in x

[2] [We99] uses some unexplained notation which we here clarify: $F(e, f) = \beta$ abbreviates "$\varphi_e^f(0){\downarrow}$ in exactly β steps"

We have parallel to the above:

Lemma 9 *(i)* $x \leq_{e\infty} y \longleftrightarrow x \in L_{\zeta^y}[y]$; *thus (ii)* $x \leq_{e\infty} y \longrightarrow \zeta^x \leq \zeta^y$; *(iii) The assignment* $x \longrightarrow \zeta^x$ *then satisfies the Spector criterion:*
$x \leq_{e\infty} y \longrightarrow (\tilde{x} \leq_{e\infty} y \longleftrightarrow \zeta^x < \zeta^y)$.

Lemma 10 $\tilde{x}$ *is Turing isomorphic to the complete* $\Sigma_2(L_{\zeta^x}[x])$ *set of integers.*

Again the variant of Question 1 arises.

Question 2 *Let* D *be a countable set of eventually infinite time degrees. Does* D *have a* $\leq_{e\infty}$ *minimal upper bound?*

Definition 12 *Let* $F = \{x \in 2^\omega \mid x \in L_{\zeta^x}\}$.

We note that it makes no difference whether we write ζ^x or λ^x in this definition, since there is a program that given an x as input, searches for its L-rank. Once found we can ensure the program halts. Hence if $x \in F$ then $x \in L_{\lambda^x}$. There is an analogy here with Q, (see [Ke75] or [Mo80, 4F.4] for the basic properties and structure of Q) the largest thin Π^1_1 set of *quickly constructible* reals: $Q =_{df} \{x | x \in L_{\omega_1^x}\}$.

We may define a natural hierarchy of eventually infinite degrees through F in the spirit of [Ke75] (where this is done for Q and hyperdegrees) by $\boldsymbol{e_0} = [0]_{\sim}$; $\boldsymbol{e_{\alpha+1}} = [\tilde{e}_\alpha]_{\sim}$; $\boldsymbol{e_\mu} \simeq \mathrm{lub}\{\, \boldsymbol{e_\alpha} \mid \alpha < \mu\}$ for $\mathrm{Lim}(\mu)$ if defined. (Here "lub" abbreviates "least upper bound" in the degree ordering.)

Question 3 *What is the natural length of this hierarchy? That is, what is the least* ρ *so that* $\boldsymbol{e_\rho}$ *is undefined?*

(The version of this question for the infinite time degrees has a known answer: the natural hierarchy is undefined even at the ω'th stage.)

2 Higher Type Computation

We briefly review again the relevant definitions from [HaLe00].

For $A \subseteq 2^\omega$ $\varphi^A_p(x)$ represents the function defined by the p'th program in a recursive listing of programs allowing queries of the set A, as to whether a real currently on the scratch tape is, or is not, in A (and receiving a simple $0/1$ answer), using real input x.

Definition 13 *([HaLe00] The strong jump)*
$0^{\blacktriangledown} =_{df} \{(n,x) \in \omega \times 2^\omega \mid \varphi_n(x)\downarrow\}$.
For $A \subseteq 2^\omega$ *define* $A^{\blacktriangledown} =_{df} \{(n,x) \in \omega \times 2^\omega \mid \varphi_n^A(x)\downarrow\}$.

There is an associated *reducibility relation* associated to sets of reals:

Definition 14

$$A \leq_\infty B \longleftrightarrow \exists n \forall x \in 2^\omega \left[\varphi_n^B(x)\downarrow 1 \longleftrightarrow x \in A \ \wedge \ \varphi_n^B(x)\downarrow 0 \longleftrightarrow x \notin A\right]$$

We make specifically semi-decidability into a definition:

Definition 15 *A set of reals A is* semi-decidable *in a set of reals B if and only if:*

$$\exists n \forall x \in 2^\omega \left[\varphi_n^B(x)\downarrow 1 \longleftrightarrow x \in A\ \right]$$

(ii) A set of reals A is semi-decidable in a set of reals B and a real $y \in 2^\omega$ *if and only if:*

$$\exists n \forall x \in 2^\omega \left[\varphi_n^{B,y}(x)\downarrow 1 \longleftrightarrow x \in A\ \right]$$

We want a notion of "semi-decidability" that results in a pointclass of sets closed under continuous preimages. Thus (when sufficient determinacy) it shall have a Wadge rank, which in turn gives a measure of complexity to the notion of semi-decidability we have defined. (Wadge defined the following ordering on sets of reals: $A \leq_W B$ if there is a continuous reduction of A to B: *i.e.* there is a continuous $f : 2^\omega \to 2^\omega$ so that $x \in A \leftrightarrow f(x) \in B$. Then $\leq_W$ is easily seen to be reflexive and transitive. This will give us a notion of *Wadge rank* of the sets of reals in any given pointclass $\mathbf{\Gamma}$ provided we know that $\leq_W$ restricted to $\mathbf{\Gamma}$ is wellfounded. Assuming sufficient determinacy Wadge's Lemma (*cf.* [Mo80, 7D.3]) asserts that $\leq_W$ so restricted is (virtually) a linear ordering, and a theorem of Martin (*cf.* [Mo80, 7D.14]) asserts - still assuming sufficient determinacy, that it is wellfounded. For $\mathbf{\Gamma}$ equal to the class of Borel sets no determinacy is required (indeed, unlike Borel Determinacy, these facts can be proven in second order arithmetic). Assuming the full axiom of determinacy, AD, then the Wadge hierarchy so obtained has rank $\Theta =_{df} \sup\{\gamma \mid \exists g : \mathbb{R} \to \gamma, g \text{ onto }\}$.)

The notion *"A is decidable in B" (respectively "in a real and B")* is as usual: this relation holds if both A and $\neg A$ are semi-decidable in B (respectively in a real and B).

Definition 16 $A \leq_{r,\infty} B$ *denotes that* A *is decidable in* B *and a real.*

For the rest of this section "semi-decidability" will abbreviate "semi-decidability in a real" - the boldface notion.

Definition 17 *(i)* $\mathbf{\Gamma}_0 =_{df} \{A \subseteq 2^\omega \mid A$ *is semi-decidable from a real* $\}$
(ii) $\mathbf{\Delta_0} =_{df} \mathbf{\Gamma}_0 \cap \neg \mathbf{\Gamma}_0$.

Theorem 11 *(Hamkins-Lewis)[HaLe00]* $\mathbf{\Delta} \subseteq \mathbf{\Delta}^1_2$ *is a* σ*-algebra, closed under Suslin's operation* $\mathcal{A}$*, properly containing the Selivanovski*[3] C*-sets.*

It is easy to verify that the semi-decidable in z sets of reals are provably $\Delta^1_2(z)$ and thus, by a result of Solovay (a proof is given in [FeNo74]), are all absolutely measurable, and enjoy the property of Baire.

There are two questions related to hierarchies with a classical flavour (as opposed to the descriptive set-theoretic viewpoint based on the later effective theory of Moschovakis and others). The first is to ask how the semi-decidable sets sit with relation to the Kolmogorov R-sets.

Question 4 *Classify the semi-decidable sets of reals within the hierarchy of the* R*-sets.*

We conjecture that they sit very low down in this hierarchy. The reader may consult [Hi78] $V.4$ and $V.5$ for an account of classical hierarchies through $\mathbf{\Delta}^1_2$. There is another notion of generalised computation, due to Blackwell [Bl78], the "*Borel programmable functions*" - derived from the theory of dynamic programming in probability theory. For this, let X, Y be perfect polish spaces, and 2^ω be Cantor space. A *program* is a function $p : 2^\omega \rightarrow 2^\omega$ so that $\forall n\, x(n) \leq p(x)(n)$ (thus we are really thinking, *via* characteristic functions, of p as operating on a subset x of ω and returning a possibly larger set $p(x)$). We iterate p in an obvious way, and let p_α be the α'th iterate, with $p_\lambda(x)(n) = \sup_{\alpha<\lambda} p_\alpha(n)$ for all n at limits λ. Let p_Ω be the fixed point function attained at some $\Omega < \omega_1$. A Borel function $e : X \rightarrow 2^\omega$ is called an *encoder*, a Borel $d : 2^\omega \rightarrow Y$ is a *decoder*. Let c_i be the constant function on ω with value i.

[3] The Selivanovski C-sets are the smallest σ-algebra of sets containing the Borel sets, and closed under operation $\mathcal{A}$.

Definition 18 *A function $f : X \to Y$ between perfect polish spaces X, Y is* Borel programmable *if it is of the form $d \circ p_\Omega \circ e$ for some d, p, e of the above form. A set $A \subseteq X$ is* Borel programmable *if its characteristic function $\rho_A : X \to 2^\omega$ is (where $\rho_A(x) = c_1$ if $x \in A$ and $\rho_A(x) = c_0$ if $x \notin A$.)*

Question 5 *Are the infinite time Turing machine decidable sets all Borel programmable sets?*

It is easy to see from the definition of Borel programmable functions, that their computation could be effected on an infinite time Turing machine (equipped with oracles for the requisite Borel codes of the functions concerned). The question concerns the reverse inclusion. The question *prima facie* is not unreasonable, since by a result of Burgess and Lockhart [BuLo83] the Borel programmable sets also properly include the C-sets. However we conjecture the answer is negative.[4]

In [HaLe??] a positive solution to Post's problem (whether there can be A with $0 <_\infty A <_\infty 0^\blacktriangledown$) is proposed. In fact two *countable* semi-decidable sets $A, B \subseteq 2^\omega$, which could be construed as constructed definably over L_λ, are seen to be $\leq_\infty$ -incomparable *via* a Friedberg-Muchnik type construction. Since the sets constructed are countable, they are obviously both of degree 0 when considering the boldface reducibility of decidability in reals. One can "remedy" that as follows

Lemma 12 $(V = L)$. *The set F of fast reals satisfies: $0 <_\infty F <_\infty 0^\blacktriangledown$. Moreover F is not decidable in a real, and $0^\blacktriangledown$ is not decidable in any real and F; that is $0 <_{r,\infty} F <_{r,\infty} 0^\blacktriangledown$.*

F is thus, in L, a (boldface) strictly intermediate set. That some kind of set theoretic hypothesis is necessary to build intermediate sets will be seen below.

We take the view here that one tends not to consider the relation of "A is Δ^1_n in B" between sets of reals (although of course this makes perfect sense for sets of integers). We should like to generalise the notion of boldface semi-decidability that is afforded by *Kleene degrees* (see [HrSi80] for a discussion of this notion). Briefly, for sets of reals A, B one writes $A \leq_K B$ ("A is Kleene recursive in B") iff there is a real

[4] We are grateful to John Burgess for bringing this example to our attention - albeit in a rather different context.

y so that the characteristics function χ_A of A is recursive (in the sense of Kleene [Kl59]) in y, χ_B and the existential integer quantifier ${}^2E$. The computational model here differs from the infinite time Turing machines, but one can view it as a machine equipped with ability to quiz an oracle for B and y, with a countably infinite memory, and an ability to search that memory in a *finite* amount of time.

The Kleene recursive sets are then the Borel sets, and the Kleene semi-recursive sets are the coanalytic sets. Solovay had shown ([So71]) that under AD, the axiom of determinacy, the Kleene degrees are well-ordered. The complete semi-recursive set is WO, the Π^1_1 set of reals coding wellorders. The nature of the reducibility ordering $\leq_K$ depends on one's universe of discourse: contrasting with Solovay's AD result already mentioned, in L, or in set generic extensions thereof, there are $2^{\aleph_0}$ $\leq_K$-incomparable semi-recursive sets below the degree of WO ([HrSi80]). A sharper result than Solovay's (no pun intended) is the fact that if $\mathbf{\Pi^1_1}$-determinacy holds then all Kleene semi-recursive, non-recursive sets have the same Kleene degree. This is a result of Steel [St80]. The converse also holds by Harrington [Ha78, Theorem 4.4]. We should expect an entirely analogous picture to obtain for the degrees of the semi-decidable sets defined above.

In the following we use freely the natural generalisations of the definitions and notations from the first section. We thus, for example, write "λ^B" for the first ordinal that is not (coded by) the output of any halting computation with oracle B, on input 0.

It follows from a version of the "λ, ζ, Σ-Theorem" (Theorem 5 above) that the sets of reals semi-decidable in $x \in 2^\omega$ are those sets A expressible as follows: let $\psi_0(v_0, v_1)$ be a Σ_1 formula, then let

$$y \in A \longleftrightarrow L_{\lambda^{\langle x,y\rangle}}[x, y] \models \psi_0(x, y).$$

A is *semi-decidable in* B *(*$A, B \subseteq 2^\omega$*) and* $x \in 2^\omega$ can be seen to be given similarly as:

$$y \in A \longleftrightarrow L_{\lambda^{\langle x,y,B\rangle}}[x, y, B] \models \psi_0(x, y, B).$$

In the latter equivalence, we should have precisely the notion of A being Kleene semi-recursive to B, if one simply replaced $\lambda^{\langle x,y,B\rangle}$ by $\omega_1^{\langle x,y,B\rangle}$. However we are not simply mimicking a (version equivalent to) Kleene's definition here. Suppose we are given any function $f : \mathcal{D} \to On$

of ordinary Turing degrees to countable ordinals that is definable *via* a Σ_1 formula $\psi(v_0, v_1)$ so that for any (ordinary) Turing degree $[y]_T$ we have $f([y]_T) = \alpha$ iff $L[y] \models \psi(y, \alpha)$; then we may define a slice through Δ^1_2 by defining a lightface pointclass Γ_0 as follows: $A \in \Gamma_0$ if and only if for some formula $\theta(v_0, v_1)$ we have $x \in A \longleftrightarrow L_{f(x)}[x] \models \theta(x)$. (A boldface definition would add in a real parameter.) How high a rank f has in ${}^{\mathcal{D}}\aleph_1$ modulo the Martin measure (*cf* [Ka94] p386), assuming say $\mathrm{Det}(\boldsymbol{\Delta}^1_2)$, determines the complexity of the class Γ_0. By some measures λ^x is a large ordinal, and we are dealing with a complex class.

We may then, to be specific, ask questions about the semi-decidable sets below $0^\blacktriangledown$ as follows:
1) Is there $A \subseteq 2^\omega$, $A \in \boldsymbol{\Gamma}_0$, A not decidable in a real, but satisfying that $0^\blacktriangledown$ is not decidable in a real and A?
2) Are there $A, B \in \boldsymbol{\Gamma}_0$ neither of which is decidable in a real from the other?

The outcome, just as for Kleene degrees, depends on the set theoretical universe one inhabits.

Theorem 13 *In L, the constructible universe, (or in set generic extensions thereof) there are A, B as in 2).*

We have not checked, but fully expect that, there are $2^{\aleph_0}$ many such incomparable sets A. We list a number of problems, which are supposed to elucidate the boldface reducibility ordering in the constructible universe. We expect their answer (and solution) to be similar to that for Kleene degrees.

Question 6 *Assume $V = L$.*
(i) Can there be a semi-decidable set A of minimal $\leq_{r,\infty}$-degree?
(ii) Can there be infinite descending sequences of $\leq_{r,\infty}$-degrees?
(iii) Are the $\leq_{r,\infty}$-degrees of semi-decidable sets dense between 0 and $0^\blacktriangledown$?

(We conjecture the answer to (iii) to be affirmative, thus solving the other two parts.) In the following we let for $\boldsymbol{\Gamma}$ (any pointclass) $\mathrm{Boolean}(\boldsymbol{\Gamma})$ denote the pointclass obtained from the sets of $\boldsymbol{\Gamma}$ and closing up under complementation and finite unions.

Theorem 14 $\mathrm{Det}(\mathrm{Boolean}(\boldsymbol{\Gamma}_0)) \Longrightarrow$ *any $A \in \boldsymbol{\Gamma}_0 \setminus \boldsymbol{\Delta}_0$ satisfies: $0^\blacktriangledown$ and A are mutually decidable in a real.*

Thus sufficient determinacy ensures the answers to both questions is negative. In essence all one has to do for this latter theorem is to quote a result of Steel ([St80]) that for any boldface pointclass $\boldsymbol{\Gamma}$ the determinacy of sets in $\mathrm{Boolean}(\boldsymbol{\Gamma})$ ensures that any set not $\boldsymbol{\Gamma}$-self-dual is continuously reducible to another such set.

There remains the question of how to measure the complexity of $\boldsymbol{\Gamma}_0$. One method is to ascertain how much determinacy $\mathrm{Det}(\mathrm{Boolean}(\boldsymbol{\Gamma}_0))$ actually is. Can one find an equivalence in terms of inner models of large cardinal axioms? We formulate this as follows:

Question 7 *Determine the strength of* $\mathrm{Det}(\mathrm{Boolean}(\boldsymbol{\Gamma}_0))$. *Can the sharp of an inner model of some large cardinal axiom be found which is equivalent to this?*

In general the direction from a sharp to the determinacy is difficult. For the theory of Kleene degrees we have already remarked that there is a precise answer: $\mathrm{Det}(\mathrm{Boolean}(\boldsymbol{\Pi}^1_1))$ yields, by the above comments, that all non-Borel co-analytic sets are Kleene mutually reducible to the complete Π^1_1 set. This conclusion had been obtained by Steel assuming just $\mathrm{Det}(\boldsymbol{\Pi}^1_1)$. (Of course Harrington showed that $\mathrm{Det}(\boldsymbol{\Pi}^1_1)$ implied the existence of $x^{\#}$ for any real x, and this was known by work of Martin to imply $\mathrm{Det}(\mathrm{Boolean}(\boldsymbol{\Pi}^1_1))$).

We can appeal to methods of Steel [St82], using Friedman-style games to show that $\mathrm{Det}(\boldsymbol{\Gamma_0})$ implies reasonable large cardinal strength. We first have:

Theorem 15 *The following is a (lightface) decidable relation*

> "$\langle x, y\rangle \in 2^\omega$ *codes countable premice* $M, N <_* 0^{\P}$ *and* $M \leq_* N$ *in the (pre)mouse ordering.*"

In the above the (pre)mouse orderings $<_*$ and $\leq_*$ are the canonical ones. For premice $M \leq_* N$ is to be interpreted as "In the comparison coiteration of M with N to models M_θ, N_θ then either (i) θ is least so that one of the models is illfounded, and that model is M_θ or (ii) M_θ is an initial segment of N_θ." The notation $0^{\P}$ denotes the sharp for an inner model of a strong cardinal (if it exists). Using the fact that $\mathrm{Det}(\boldsymbol{\Gamma}_0))$ implies $\mathrm{Det}(\boldsymbol{\Pi}^1_1)$ we may use the Σ^1_3-correctness of the core model built assuming there is no inner model with a strong cardinal ([StWe94]) and obtain:

Theorem 16 $\mathrm{Det}(\mathbf{\Gamma}_0)$ *implies* $0^{¶}$ *exists.*

This latter embedding property is known as $0^{¶}$ (read as "0-pistol") and in fact one has $x^{¶}$ for any real x. But this seems far from an optimal result. Indeed to ask a question as to lower bounds:

Question 8 *Does the mutual decidability in reals of all* $A, B \in \mathbf{\Gamma}_0 \setminus \mathbf{\Delta}_0$ *imply* $x^{¶}$ *exists for all* $x \in \mathbb{R}$*? Does it imply* $x^{\#}$ *exists?*

We expect the answers should be affirmative.

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