Truth and Turing: *Systems of Logic based on Ordinals*

We should like to link Turing's construction in *Systems of Logic based on Ordinals* on progressions of theories, with some recent similar looking progressions of axiomatisations of truth sets. However we first set the scene by sketching the original paper. It is interesting for two fundamental reasons. Firstly he introduces, in a rather understated fashion, the notion of a variant of his original Turing Machine, which was to be the *'o-machine'* for 'oracle-machine.' The latter is the well known version of the basic machine, the *'a-machine'*, introduced in his 1936 paper *'On computable numbers'*. The o-machine is of course the standard Turing machine equipped with an oracle tape. In the paper Turing describes rather a program that is allowed input at a stage of the computation when a special instruction is reached to ask for such input from the oracle tape. He envisaged then that in this way 'non-computable' functions could be introduced by calling for values. In the paper, after introducing this idea, he then repeats the argument that the halting problem was undecidable by such machines. He called this the 'circularity question': whether a particular TM $\mathcal{M}$ would eventually loop on a particular input. (I shall use TM to abbreviate Turing machines (with or without oracle tapes.)

Just as the o-machine became the standard model for a computer (in Turing's terms) so the o-machine has become for us the standard model for *relativised computability*: the notion that a set $A \subseteq \mathbb{N}$ can be computed 'relative to a set $B \subseteq \mathbb{N}$' is that membership questions as to whether $n \in A$ or not can be 'reduced' to finitely many similar queries of the set B, where we imagine the oracle tape of the machine to have the characteristic function of B written out as a series of 0's and 1's. We write nowadays in this case '$A \leq_T B$' for this relation. Sets A, B of numbers equivalent under $\leq_T$ are then declared to be in the same 'Turing degree' of incomputability. Thus the whole theory of such algorithmic degrees can be effected, using this model.

This however only occupies a page and a half. This is not what the paper is about. It is only a tool in his investigation of the second fundamental idea to emerge from the paper: the notion of an 'ordinal progression'. One has to admire the sweep of the paper: merely 8 years after Gödel's paper on the Incompleteness Phenomenon, and only 3 years after his own paper *On Computable Numbers* he attempted to grapple with the incompleteness phenomena of formal systems by systematically extending theories $T = T_0 \subseteq T_1 \subseteq \cdots$ by adding at each stage a *consistency* statement about the preceding theory. The assumption is that our acceptance of a theory T somehow also impels us to accept its consistency. Who would work in Peano Arithmetic (PA) if they believed Con(PA) was false? And of course it is the consistency statement 'Con(PA)' that Gödel showed was a statement unprovable in PA (assuming that it *was* itself consistent).

Martin Davis refers to the paper in his introduction in a volume of collected sources as "difficult" and in several ways it is: the ideas are not immediately transparent; the notation sticks with that of Church's λ-calculus (under whom Turing was at this time writing as his PhD thesis, which contained this research); the underlying extensions take place along a system of notations, related to one devised by Kleene. We now would use a system called 'Kleene's $\mathcal{O}$', but again the language is different: instead of asking whether a certain integer n can be seen to be in $\mathcal{O}$ Turing asks whether a certain formula is an 'ordinal formula': the latter are formulae used to name (what will be) constructive ordinals, and there is a list of 7 conditions in terms of λ-conversion for them. He also gives a definition of 'C(hurch)-K(leene) ordinal formulae' which contain in essence a definition (equivalent to that) of $\mathcal{O}$. Discussing this in today's notation we have the following definition (where $\mathrm{suc}(n)$ can be taken to be 2^n and $\lim(n)$ to be 3^n):

Definition 1 *By simultaneous recursion we define '$n \in \mathcal{O}$' and '$n <_{\mathcal{O}} m$' for $n, m \in \mathbb{N}$ together with an ordinal $|n|$ for each $n \in \mathcal{O}$:*

$0 \in \mathcal{O}$ *and* $|0| = 0$ *;*

If $n \in \mathcal{O}$, *then* $\mathrm{suc}(n) \in \mathcal{O}$, $n <_{\mathcal{O}} \mathrm{suc}(n)$ *and* $|\mathrm{suc}(n)| = n + 1$ *;*

If $\{e\}$ *is an index of a total recursive function, and* $\forall n(\{e\}(n) <_{\mathcal{O}} \{e\}(n+1))$ *then* $\lim(e) \in \mathcal{O}$, $\{e\}(n) <_{\mathcal{O}} \lim(e)$ *for every* n, *and* $|\lim(e)| = \sup\{|\{e\}(n)| : n \in \mathbb{N}\}$.

If $n <_{\mathcal{O}} m \wedge m <_{\mathcal{O}} p \longrightarrow n <_{\mathcal{O}} p$.

By this means *notations* can be assigned to any *constructive ordinal*: that is any ordinal less than the first non-recursive ordinal ω_1^{ck}, with $n <_{\mathcal{O}} m \longrightarrow |n| < |m|$ (but not conversely). However the relation '$n \in \mathcal{O}$' is complex being necessarily Π^1_1. A totally ordered subset of $\mathrm{Field}(<_{\mathcal{O}})$ is a *path* and the restriction of $<_{\mathcal{O}}$ to a path of the form $\{n : n <_{\mathcal{O}} m\}$ allows us to see that the latter set is actually recursively enumerable. Kleene's $\mathcal{O}$ then gives us a constructive framework to which we may attach objects, in this case *theories*.

Definition 2 *A* consistency progression based on a theory T *is a primitive recursive mapping* $n \longrightarrow \varphi_n$ *where* φ_n *is a* Σ_1 *formula such that* PA *proves:*

(i) $T_0 = T$*;* *(ii)* $\forall n\big(T_{\mathrm{suc}(n)} = T_n + \mathrm{Con}(\varphi_n)\big)$*;* *(iii)* $T_{\lim(n)} = \bigcup_m T_{\{n\}(m)}$.

Definition 3 *A* progressive (consistency) sequence *is then the restriction of a consistency progression to a path through* $\mathcal{O}$.

The existence of progressive sequences along paths has to be justified through the use of the Recursion Theorem. With these tools Turing proved a form of Completeness Theorem.

Theorem 1 *(Turing's Completeness Theorem) For any true* Π_1 *sentence of arithmetic,* σ, *there is an* $a = a(\sigma) \in \mathcal{O}$ *with* $|a| = \omega + 1$, *so that* $T_a \vdash \sigma$. *The map* $\sigma \rightarrowtail a(\sigma)$ *is given by a primitive recursive function.*

Thus we may for any true σ find a path of length $\omega + 1$, $T = T_0, T_1, \ldots, T_{\omega+1} = T_a$ with the last proving σ. At first glance it looks as if Turing's theorem is giving us an insight into mathematical knowledge, but this is illusory. There is a trick here: what one does is construct for *any* Π_1 sentence σ an extension $T_{a(\sigma)}$ proving σ with $|a(\sigma)| = \omega + 1$; then *if σ is true* we deduce that $T_{a(\sigma)}$ is a consistency extension. The set $\mathcal{O}$ is, as we've remarked, a complex set of numbers, and the argument draws on this complexity.

In the article Turing stated that he had tried to prove a theorem for statements at the level he called that of "number theoretic problems", which in effect are those expressible as Π_2 sentences. He expressed the hope that this might yet be proven. However it was not until Feferman extended this work much later in the fundamental paper [1], which used the somewhat strengthened Reflection Principles below, was it possible to prove a 'Completeness Theorem' in the above sense for Π_2 sentences.

There is the possibility of adding other statements than just consistency alone to progressions. The work of Feferman here has been far-reaching. Subsequent research of Beklemishev, Schmerl, Franzen and others have extended this, and no doubt will be commented on elsewhere in this volume.

It is possible to formalize the notion that: *"if $T \vdash \sigma$, where $\sigma \in \Sigma_n$ then σ is true"* and this '*n-reflection*' may be abbreviated REFL^n_T in that the theory T reflects the Σ_n truth of the matter. For T extending PA this can be expressed by a single Π_{n+1} formula. Full reflection for all Σ_n formulae, REFL_T is then the assertion of n-reflection for all n. Instead of consistency sequences it is possible to talk of n-, or *full-reflection progressions* and so forth. These turn out to have different properties from those of the simpler consistency statement studied by Turing, and the extensive study of these has been developed following [1], which, as mentioned, showed how there were Π_2, and indeed full, Completeness Theorems concerning paths through $\mathcal{O}$, of the kind that Turing discussed. (See, *e.g.* , the discussion of Kreisel on the subject of such putative paths delivering mathematical knowledge [7].) There is also a broad literature on the *kinds* of paths or progressions one have: *autonomous progressions* are those of a more self-justifying flavour. We shall not go into these details, but refer the reader to the excellent surveys of Franzen ([3], [4]).

The notion of such progressions can be used in a number of arenas, with rather differing levels of significance. I'd like to highlight one current area of work: iterated reflection principles in *truth theories.* In a truth theory one explicitly adds axioms concerning a truth predicate T say. One typically takes a base language of interest (and it is almost always Peano Arithmetic PA, since (i) mathematicians are very much interested in number theory and (ii) in PA the mechanisms of coding effectively given languages by numbers or 'gödel codes' is available. Let us call this language $\mathcal{L}$. To this is added a predicate symbol T and for numbers n that code sentences the intention is that '$T(n)$' is to be interpreted as the 'sentence coded by n is true'. Truth theorists discuss the interplay of notions of truth with various languages (for example we may extend $\mathcal{L}$ to $\mathcal{L}_T$ and allow n to range over codes of sentences not just of $\mathcal{L}$ but of $\mathcal{L}_T$); we may also consider *axiomatising truth*: we add a selection of axioms, axiom schemes, deduction rules *etc.*, *etc.* to the axioms of PA that express our beliefs about how the notion of 'truth' should behave. Depending on how this is done, theories of various types and strengths emerge. (One such is specified in more detail below.)

Just as Turing added consistency statements to make a progression of number theories, we may do the same for truth theories. (For example, *cf.* the recent [5].) We shall link this notion of progression with some current work in sequences of truth sets in a moment, but we point out that although superficially looking like Turing's progressions, the motivations are admittedly rather different.

Let $S_0 = \mathrm{PA}$ and S_1 be an axiom set of the kind just roughly described in the language $\mathcal{L}_1 =_{\mathrm{df}} \mathcal{L}_T$, using the new predicate symbol $T_0 =_{\mathrm{df}} T$ which is allowed into the induction scheme. S_1 is now a numerical theory, extending PA to which we can repeat this process: we add a new truth predicate T_1 so that $T_1(n)$ will be intepreted as saying that if n codes a sentence of $\mathcal{L}_1$, then that sentence is true. Again extend the axiom set to include the induction scheme for properties in the language with the new symbol T_1. At the limit stage ω we obtain a language $\mathcal{L}_\omega \supseteq \bigcup_{k\in\mathbb{N}} \mathcal{L}_k$, and again take the union of the previous axiom sets to obtain S_ω. We then continue with adding a truth predicate T_ω in the next language $\mathcal{L}_{\omega+1}$, and obtain thereafter $S_{\omega+1}, \ldots, S_\alpha, \ldots$ *etc.* up to some ordinal λ say. We ensure that the axioms of S_α are given by some Σ_1-arithmetic formula $\psi_\alpha(v_0)$ at each stage. With some care this can be effected in a way that ensures, inductively, that the theories S_α are *arithmetically sound*, that is assuming the axioms of $S = S_0$ are true, every theorem of S_α is true for $\alpha < \lambda$.

As a simple example of how this can work, we define the axioms of *Positive Friedman Sheard* which I shall call P for brevity. The first axiom set below is PA^T, Peano Arithmetic extended into a language $\mathcal{L}_T$ containing T, the formulae of which are allowed into the induction scheme.

1. PA^T;
2. $\forall$ atomic $\phi \in \mathcal{L}_{PA} : T(\phi)$ coincides with truth and $T(\neg\phi)$ with falsity;
3. $\forall \phi, \psi \in \mathcal{L}_T : T(\phi \wedge \psi) \leftrightarrow (T(\phi) \wedge T(\psi))$;
4. $\forall \phi, \psi \in \mathcal{L}_T : (T(\neg\phi) \vee T(\neg\psi)) \to T(\neg(\phi \wedge \psi))$;
5. $\forall \phi, \psi \in \mathcal{L}_T : (T(\phi) \wedge T(\neg\phi \vee \psi) \to T(\psi)$;
6. $\forall \phi(x) \in \mathcal{L}_T : T(\forall x \phi(x)) \leftrightarrow \forall x T(\phi(x))$; $\quad \exists x T(\neg\phi(x)) \to T(\neg\forall x \phi(x))$;
7. $CONS$: $\forall \phi \in \mathcal{L}_T : \neg(T(\phi) \wedge T(\neg\phi))$;

8. (Deduction Schemes): From A (respectively $T(A)$) deduce $T(A)$ (resp. A).

The axioms are dubbed *positive* because they only make claims as to which sentences are *in* the extension of T. Note there is no direct clause concerning simple negations. It is important for this axiomatisation that the Deduction Schemes are just that: schemes (the axiomatic versions would make the system inconsistent). '$CONS$' asserts consistency. $T^n(A)$ abbreviates n-fold $T(T(\cdots T(A)\cdots)$. The strength of this theory is known to be equivalent to infinitely many applications of arithmetical comprehension axioms: it is that of RA_ω the ω'th level of ramified analysis.

It is possible to iterate such theories: set A_0 to be '$0 = 0$', and P_0 to be P.

Definition 4 *Set: (i)* P_δ *to be* $P \cup \{A_\beta \mid \beta < \delta\}$*; (ii)* $A_\delta \equiv \forall \phi \in \mathcal{L}_T[Prov_{P_\delta}(\phi) \to T(\phi)]$.

As one can see by the subscripts to the predicate expressing provability in a recursively given axiom system S, $Prov_S$, we are considering extensions of the system P by adding iterations of "S-provability implying truth." We have left vague what we mean by the ordinals there, or what the statements A_α actually are. Also, although superficially resembling systems of axioms of increasing strength in order to form the *reflexive closure* of a theory, we are not doing this so as to form, as in that process, a theory encapsulating all of our committments to the theory P. Rather we can use it to axiomatise some truth sets, those that arise as various levels of a so-called *Herzberger truth sequence*. Set $H_0 = \emptyset$:

$$H_{\alpha+1} = \{\phi \in \mathcal{L}_T \mid \langle \mathbb{N}, H_\alpha\rangle \models \phi\}. \text{ For limit } \lambda : H_\lambda = \{\phi \mid \exists \alpha < \lambda \forall \beta \in (\alpha, \lambda)\ \phi \in H_\alpha\}.$$

Here each H_α is the extension of the T predicate of each model in turn. Note the 'liminf' rule for limit stages: φ is put in the λ'th set if from some point α onwards it is in. Such limit rules have been used by a variety of philosophical logicians to build truth sets. Field in [2] constructs a similar hierarchy $\langle F_\alpha \rangle$. It would go too far into the theory to discuss these here, but essentially these hierarchies run up to some ordinal ζ. The question has been asked: can we axiomatise in some way the sets H_λ for $\lambda \leq \zeta$? On general grounds a simple first order axiomatisation is ruled out, but it might be possible to do so on an initial segment, or in some larger language. It turns out that for $\lambda < \zeta$ some iterations of the theories $P + A_\lambda$ axiomatise H_λ in that they become true first at H_λ and no earlier H_μ (And we may do the same for the F_λ.)

In view of the previous comments about building hierarchies of Reflection Principle theories T_a for $a \in \mathbb{N}$, where we thought of a as a notation, the reader may wonder as how one can precisely do this, as the 'α' *etc.* above are not part of the language, (as they were not for Turing) but here they are very much larger than the constructive ordinals, and were left vague. There are two possible answers here: one can show that *within the* *system* of building up the H- or F-hierarchies for any $\alpha < \zeta$ there are certain sentences B_α that can themselves be construed as notations for those ordinals, and wo we may use these as devices for referring indirectly to them, and incorporate these somehow into our iterated truth theories. The other possibility is to extend Kleene's $\mathcal{O}$ itself to a system $\widetilde{\mathcal{O}} \supset \mathcal{O}$. To do this we extend the notion of 'computability': whereas $\mathcal{O}$ is a system of notation for the computable ordinals using ordinary Turing machines we now allow them to run transfinitely and thus we have a new notion of 'decidable' corresponding to 'having some fixed output, 0 or 1 from some point on'. The beauty of this is that we don't even have to change Definition 1 at all beyond replacing the word 'recursive' by 'transfinitely computable' in the above sense. The Turing machines programs are not altered; the finite computations are just a special subclass of the transfinite ones, and the resulting system subsumes $\mathcal{O}$ and then stretches out precisely to ζ. If we are willing to indulge in this use of 'decidable', we can use these members of $\widetilde{\mathcal{O}}$ as notations applicable for our theories T_a. Of course we no longer have the possibility of $\{a \mid a <_{\widetilde{\mathcal{O}}} b\}$ being *c.e.* in the ordinary sense any more, for $b \in \widetilde{\mathcal{O}}$, but this set has to be '*c.e.*' in this new, wider sense. This would mean that any pursuit of analogies to the Turing/Feferman theorems would have to leave behind the notion of theories being (ordinarily) computably axiomatised. However for the analysis of the truth sets H_α, F_α in [6], through Turing-style iterations of the Positive Friedman Sheard theory, these kind of notations look good enough.

Time will tell whether this kind of approach (or indeed the underlying truth set constructions) will prove to be of any value.

References

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