Large Cardinals, Inner Models and Determinacy

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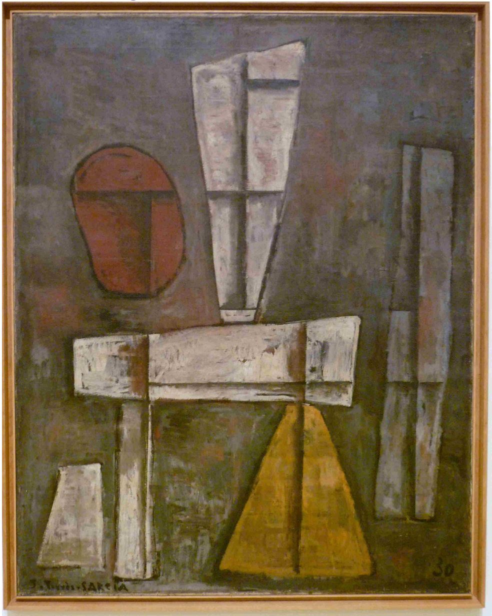

Overview

Part I: Inner Models, Elementary Embeddings and Covering Lemmas
Part II: Determinacy
Part III: Large Cardinals

Overarching Theorem

We use the following theorem to direct and anchor our discussion

Theorem

(ZF) TFAE
(a) $\exists j : L \longrightarrow_e L.$
(b) $\exists\gamma(\omega_2 \leq \gamma \in \mathrm{SingCard} \longrightarrow (\gamma \notin \mathrm{SingCard})^L)$*;*
(c) $\mathrm{Determinacy}(\Pi^1_1)$

(a) is an example of an *elementary embedding* of an *inner model*. This assertion is sometimes abbreviated as "$0^{\#}$ *exists*."
(b) is an example of the *negation* of a *Covering Lemma*, in this case over the inner model L; the negation would assert that the cardinality structure of V is deeply connected with that of L, in that every V-singular cardinal, is singular in L.
(c) is an assertion that two person perfect information games played on Baire space, ω^ω with (lightface) co-analytic payoff sets are determined.

This is a deep theorem. Why should these statements have anything to do with one another?
((a) $\leftrightarrow$ (b) is Jensen (b)$\rightarrow$ (c) is Martin, (b)$\leftarrow$ (c) is Harrington-Martin)

The Main Theorem: (a) $\leftrightarrow$ (b)

- We explore the background here.
- First we have to define some terms:

Definition (Inner Model of ZF)

$IM(M) \leftrightarrow \mathrm{Trans}(M) \wedge \mathrm{On} \subseteq M \wedge (ZF)^M$.

- $IM(M)$ actually has a first order formalisation: it is well known that

$$\mathrm{ZF} \vdash IM(M) \leftrightarrow \forall u \subseteq M \exists v \supseteq u(\mathrm{Trans}(v) \wedge \mathrm{Def}(\langle v, \in \rangle) \subseteq M).$$

Definition (Elementary Embeddings)

Let M, N be inner models of ZF*, $j : M \longrightarrow N$ is an* elementary embedding *if* the function *j takes elements $x \in M$ to elements $j(x) \in N$ is a 'truth preserving way': for any formula $\varphi(v_0, \ldots, v_{n-1})$ and any $\vec{x} = x_0, \ldots, x_{n-1} \in M$, then*

$$\varphi(\vec{x})^M \leftrightarrow \varphi(\overrightarrow{j(x)})^N$$

In this case we write: $j : M \longrightarrow_e N$ and $cp(j)$ for the critical point: the least ordinal α with $j(\alpha) > \alpha$, if it exists.
(ii) If the above holds, but with the formulae restricted to a certain class, eg. Σ_k formulae, then we write $j : M \longrightarrow_{\Sigma_k} N$.

• In the above scheme, we have assumed that the models M, N satisfy $IM(M), IM(N)$ above and are given by terms of our basic set theoretical language, and the same holds true for j. Our embeddings will all have critical points.

• It is an easy consequence of the ZF axioms (using the definition of the rank function, the V_α hierarchy, and Replacement) that if $j : M \longrightarrow_{\Sigma_1} N$, then by a (meta-theoretic) induction on k we may prove $j : M \longrightarrow_{\Sigma_k} N$ for any $k \in \omega$.

Extracting ultrafilters from embeddings

If $\exists j : L \longrightarrow_e L$ then we may define a measure $U = U_j$ on $\kappa = \text{cp}(j)$ and U_j is a normal measure on $\mathcal{P}(\kappa)^L$ as follows: we set

$$X \in U_j \leftrightarrow X \in \mathcal{P}(\kappa)^L \wedge \kappa \in j(X).$$

Much large cardinal theory is about which ultrafilters can or do exist on (large) sets, in particular when those large sets are the power set of some cardinal of some inner model, then there is usually an equivalent formulation in terms of elementary embeddings of that inner model such as at (a).

More generally: Suppose we have $j : M \longrightarrow_e N$; $\text{cp}(j) = \kappa$, then we may define $U = U_j$:

$$X \in U_j \leftrightarrow X \in \mathcal{P}(\kappa)^M \wedge \kappa \in j(X).$$

Measures (often) yield embeddings

• Given a measure (a non-principle ultrafilter) U on $\mathcal{P}(\kappa)^M$ for some $IM(M)$ the ultrapower construction yields a map $i : M \to \mathrm{Ult}(M, U)$ where

$$|\,\mathrm{Ult}(M, U)| = \{[f]_\sim \; : f \in^\kappa M \cap M\}$$

and we define an equivalence as a pseudo-identity:

$$f \sim g \leftrightarrow \{\alpha \; : f(\alpha) = g(\alpha)\} \in U$$

and on which we can define a pseudo-epsilon relation:

$$f\,E\,g \leftrightarrow \{\alpha \; : f(\alpha) \in g(\alpha)\} \in U.$$

If $U = U_j$ has come from an embedding, we are guaranteed E is wellfounded on $\mathrm{Ult}(M, U)$ and in fact we have: [Diag.]

- We can, and often do, have $(M, \in) = (V, \in)$.
- Also *starting* from a κ-complete U on $\mathcal{P}(\kappa)^M$, $(\kappa > \omega)$, then we can define $\mathrm{Ult}(M, U)$ as above. For $M = V$, we can then prove outright that $\mathrm{Ult}(V, U)$ will be wellfounded. In wellfounded cases we may define by recursion along E a transitivising collapse map $\pi : \mathrm{Ult}(M, U) \longrightarrow (\overline{N}, \in)$ isomorphism. By composition we then,

Theorem

(ZFC) TFAE: Let $\kappa > \omega$: (a)There is a κ-complete non-principle ultrafilter on $\mathcal{P}(\kappa)$.
(b) $\exists j : V \longrightarrow_e M$ with $cp(j) = \kappa$.

An early result on L

Theorem (Scott)

(Scott) (ZF) $\exists\kappa(\kappa$ *a measurable cardinal*$)\longrightarrow V \neq L$.

Proof: If $V = L$, let κ be the least such MC, form the ultrapower and so the embedding above. Then from $j : V \longrightarrow_e N$, and elementarity we have: $(V = L)^V \longrightarrow (V = L)^N$; so (Trans(N)), and so $N = V = L$. But

"κ is the least MC"$\longrightarrow$ "$(j(\kappa)$ is the least MC$)^N$".

But $N = V \wedge j(\kappa) > \kappa$! Q.E.D.

- The assumption implies $\exists j : V \longrightarrow N$, but by Gödel $L^N = L$, so $j \upharpoonright L : L \longrightarrow_e L$. Note that no first order formula $\varphi(v_0)$ can differentiate between κ and $j(\kappa)$: $\varphi(\kappa)^L \leftrightarrow \varphi(j(\kappa))^L$. Ther are *indiscernible.*
- Work of Kunen shows that if $\exists j : L \longrightarrow_\varepsilon L$, then a number of consequences follow:

Consequences of $j : L \to L, j \neq id$

(i) Then there is such a j: $L \longrightarrow L$ with $\mathrm{cp}(j) < \omega_1$. Moreover defining U_0 from such a j with critical point κ_0 least, we are guaranteed wellfoundedness of *iterated ultrapowers*: that is we may define $j_{01} : L \longrightarrow_e L$ by taking the ultrapower of L by U_0; define U_1 on $\kappa_1 =_{\mathrm{df}} j_{01}(\kappa_0)$, and then $\mathrm{Ult}(L, U_1)$ will also be wellfounded. We thus may take its transitive isomorph and then have the ultrapower map $j_{12} : L \longrightarrow L$ with critical point κ_1. The process may be iterated without breaking down, forming a directed system $\langle\langle M_\alpha\rangle, j_{\alpha\beta}, \kappa_a, U_\alpha\rangle_{\alpha\leq\beta\in\mathrm{On}}$ with (in this case) all $M_\alpha = L$ and elementary maps into direct limits at limit stages λ, and the κ_a forming a class C of L-inaccessibles, which is closed and unbounded below any uncountable cardinal.

(ii) The iteration points of such ultrapowers enjoy full-blooded indiscernibility properties in L: if $\varphi(v_0, \ldots, v_n)$ is any formula of $\mathcal{L}$ and $\vec{\gamma}$,$\vec{\delta}$ any two ascending sequences from $[C]^{n+1}$ then $(\varphi(\vec{\gamma}) \leftrightarrow \varphi(\vec{\delta}))^L$.

• NB in (i) really all the action of $j_{\alpha,\alpha+1}$ takes place where the subsets of κ_α are in L: we don't need the whole of L to make sense of this, only the $L_{\kappa_\alpha^{+L}}$. This leads to:

Mice!

Definition (Dodd-Jensen)

Let $j_{\alpha\beta}$ etc. be as above. Let $M_0 = \langle L_{\kappa_0^{+L}}, \in, U_0 \rangle$. This is called the "$0^{\#}$-mouse" which itself has iterated ultrapowers using maps that are the restrictions of the

$$j_{\alpha\beta} : M_\alpha \longrightarrow M_\beta \quad \text{where } M_\alpha = \langle L_{\kappa_\alpha^{+L}}, \in, U_\alpha \rangle \text{ etc.}$$

The viewpoint is shifted to that of the *mouse* (M_0) *generating the model* (in this case L). All of this is a paradigm for *generalised constructible* inner models K - the core models.

By these means we argue for

Theorem (Kunen (a) $\rightarrow$ (b))

(ZF) If $\exists j : L \longrightarrow L$ *then* $\forall\gamma((\gamma \in \text{SingCard}) \longrightarrow (\gamma\, \text{Inacc})^L)$.

Proof: The above implies that $C \cap \gamma$ is unbounded below γ. But C is closed, so $\gamma \in C$. Each $\gamma \in C$ is inaccessible in L. Q.E.D.

Weak Covering over L

Theorem (Jensen)

(ZF) Suppose $\gamma \in \mathrm{SingCard}$ *but* $(\gamma \in \mathrm{Reg})^L$. *Then* $\exists j : L \longrightarrow L$, *with* $j \neq \mathrm{id}$.

Proof: Suppose $\neg \exists j : L \longrightarrow L$, but γ is chosen least with $\gamma \in \mathrm{SingCard}$ but $(\gamma \in \mathrm{Reg})^L$. Without loss of interest, we shall assume that (i) $\mathrm{cf}(\gamma) > \omega$ (ii) $\delta < \gamma \longrightarrow \delta^\omega < \gamma$. Let $\tau = \mathrm{cf}(\gamma)$. By assumption then $\tau < \gamma$ and so we may choose $X_0 \subseteq \gamma$ with $|X_0| = \tau$ but X_0 unbounded in γ. By (ii) we'll assume also that for some $X \supset X_0$ we have (a) $\gamma \in X \prec L_{\gamma^{+L}}$ (b) ${}^\omega X \subseteq X$ (c) $|X| = \tau^\omega < \gamma$.
Let $\pi : \langle X, \in \rangle \longrightarrow \langle L_{\bar{\delta}}, \in \rangle$ with $\pi(\gamma) = \delta$ say.

(1) $\mathrm{cf}(\delta) = \tau$ also, with $\overline{|\delta}| = |L_{\overline{\delta}}| = |X| < \gamma$.
Suppose we had $\mathcal{P}(\delta)^{L_{\overline{\delta}}} = \mathcal{P}(\delta)^{L}$. then we could define derive a measure from π^{-1} let $\alpha = \mathrm{crit}(\pi^{-1})$ and define U as usual by

$$X \in U \Longleftrightarrow X \in \mathcal{P}(\delta)^{M} \wedge \alpha \in \pi^{-1}(X).$$

Then X would be a *countably complete* ultrafilter on $\mathcal{P}(\delta)^{L}$ (that is why we chose ${}^{\omega}X \subseteq X$ as this implies ${}^{\omega}M \subseteq M$), which implies that $\mathrm{Ult}(L, U)$ is wellfounded. But that implies $\exists j : L \longrightarrow L$.
Hence we must have: $\mathcal{P}(\delta)^{M} \subsetneq \mathcal{P}(\delta)^{L}$. So:

(2) $\exists \beta \geq \overline{\delta}(\mathrm{Def}(L_{\beta}) \cap \mathcal{P}(\delta)) \not\subseteq L_{\overline{\delta}}$.
Choose β least so that (2) holds. By Fine Structural methods Jensen showed how there is a superstructure L_{η} for some $\eta > \gamma^{+L}$ and a sufficiently elementary map $\tilde{\pi} \supset \pi^{-1}$, $\tilde{\pi} : L_{\beta} \longrightarrow L_{\eta}$, and definably over L_{η} there is also a 'new' subset of $\gamma = \pi^{-1}(\delta)$. But this is absurd as by L's construction $(\mathcal{P}(\gamma) \subset L_{\gamma^{+}})^{L}$. Q.E.D.

The assumptions (i) and (ii) can be dropped, but not without some difficulty, and the format of the argument remains roughly the same.

Strong Covering Lemma

Theorem (Jensen)

($ZF + \neg 0^{\#}$) For any $X \subseteq \mathrm{On}$, if $|X| > \omega$ then there exists $Y \in L$ with (a) $|Y| = |X|$ and (b) $Y \supseteq X$.

Corollary

($ZF + \neg 0^{\#}$) (a) Let $(\tau \in \mathrm{Reg})^L$ with $\tau \geq \omega_2$, then $\mathrm{cf}(\tau) = |\tau|$.
(b) Let $\tau \in \mathrm{SingCard}$. Then $\tau^+ = \tau^{+L}$.

• The Corollary above is sometimes called WCL the Weak Covering Lemma. For other inner models M we may have $\mathrm{WCL}(M)$ provable (obtained by replacing L by M in the Corollary's statements) whilst the strong $\mathrm{CL}(M)$ is not.

Generalizations

If $0^{\#} = M_0$ exists as above, perhaps there is no non-trivial $j : L[0^{\#}] \longrightarrow_e L[0^{\#}]$ and then we have a $\mathrm{CL}(L[0^{\#}])$? This is indeed the case; if the assumption fails then we have "$(0^{\#})^{\#}$". We then get a theorem along the lines of $\mathrm{CL}(L[0^{\#}])$ iff $\neg j : L[0^{\#}] \longrightarrow_e L[0^{\#}]$. $(0^{\#})^{\#}$ is again a countable object and we can repeat this process. After we have done this uncountably often our $\#$-like mouse objects are no longer countable and we have to resort to uncountable M.

Theorem (Dodd-Jensen)

(ZF) if there is no IM with a measurable cardinal, then there is an inner model K^{DJ}, so that there is no non-trivial embedding $j; K^{\mathrm{DJ}} \longrightarrow K^{\mathrm{DJ}}$ and $\mathrm{CL}(K^{\mathrm{DJ}})$.

This was the first core model to go beyond L (if one discounts the models $L[0^{\#}]$ etc.)

Theorem (Steel)

(ZFC) If there is no IM for a Woodin cardinal, then there is a model K^{Steel}, which is again rigid, and over which $\mathrm{WCL}(K^{\mathrm{Steel}})$ *holds.*