Large Cardinals, Inner Models and Determinacy

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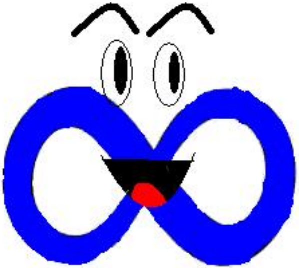

Determinacy: The Martin-Harrington Theorem

Definition (Games)

For $A \subseteq \omega^\omega$ (or more generally X^ω) the infinite perfect information game G_A is defined between two players I, II playing elements $n_i, m_i \in X$:

$$\begin{array}{lccccccccc} I & n_0 & & n_1 & & n_2 & & \cdots & & n_k & \\ II & & m_0 & & m_1 & & & \cdots & & & m_k \end{array}$$

together constructing $x = (n_0, m_0, \ldots, n_k, m_k, \ldots)$.

• We say that I *wins* iff $x \in A$; otherwise II wins. Notions of *strategy* and *winning strategy* are defined in the obvious fashion. Notice that for $A \subseteq \omega^\omega$ a strategy σ for, *e.g.* Player I is a map from $\bigcup_k {}^{2k}\omega \longrightarrow \omega$. Since there is a recursive bijection ${}^{<\omega}\omega \leftrightarrow \omega$, we can think of σ as essentially a subset of ω or as a real number.

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• We speak of a topology of open and closed sets of a space X^ω by letting a typical *basic open set* to be a neighbourhood N_s where $s \in \mathrm{Seq}_X$ is a finite sequence of elements of X:

$$N_s = \{x \in X^\omega : s \subset x\}.$$

Open Determinacy: The Gale-Stewart Theorem

Theorem (Gale-Stewart)

Let $A \subseteq X^\omega$ be an open set. Then G_A is determined*, that is one of the players has a winning strategy.*

Proof: Note that if I wins it is because he has manoeuvred the play so that there is a finite stage $(n_0, m_0, \ldots, n_k)$ so that $N_{(n_0, m_0, \ldots, n_k)} \subseteq A$. Essentially he has won by this stage as it matters not what n_l he plays, for $l > k$. II however, playing into the closed set which is $X^\omega \backslash A$ must be vigilant to the end. Suppose then I has no winning strategy. Then for every n_0 II has a reply m_0 so that I has no winning strategy in the game $G_{A/(n_0,m_0)}$ where $A/(n_0, m_0) =_{\text{df}} \{x \in A \mid x(0) = n_0, x(1) = m_0\}$. For, if there was an n_0 so that I did always have a winning strategy , σ say, in this latter game for whichever m_0 II played, then this would amount to a w.s. for him in G_A: first play n_0, wait for m_0 and then use σ. Thus given n_0 II should play m_0 so that I has no such strategy. But if she continues in this way, *this* is a winning strategy for II: always respond so that I has no w.s. from that point on. The resulting play x cannot be in A. Q.E.D.

• The motto here is that we can always in ZFC, prove Det(Open) for any space X^ω and so by taking complements Det(Closed) too. (AC is needed only to wellorder X if need be.) Many proofs of determinacy of complicated sets in X^ω, involve reducing the game to an open game in some larger space Y^ω. The latter are determined by Gale-Stewart. The difficulty arises in showing that the player with the winning strategy for the open set on the space Y^ω also has one for the related complicated set in X^ω.

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• The truth is that we may consider closed sets in X^ω as paths through trees using Seq_X, but are going to have to look further at tree representations of sets if we are to have some idea about how determinacy of more complicated sets is proven and why it is inevitably tied up with large cardinals.

Trees

Definition

(i) A *tree T on $\omega \times X$* (for $X \neq \varnothing$) is a set of sequences in $\bigcup_k {}^k\omega \times {}^kX$ where if (σ, u). and $k \leq |\sigma| = |u|$ then $(\sigma \restriction k, u \restriction k) \in T$. Similarly defines trees on ${}^n\omega \times X$.
(ii) For such a tree T we set $T_\sigma =_{\text{df}} \{u \mid (\sigma, u) \in T\}$ and $T^{\subseteq}_\sigma =_{\text{df}} \{u \mid (\sigma \restriction k, u) \in T, k \leq |\sigma|\}$.

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Definition

For T a tree:
(i) $[T] =_{\text{df}} \{(x,f) \mid \forall k(x \upharpoonright k, f \upharpoonright k) \in T\}$ - is the set of *branches through* T.
(ii) $p[T] =_{\text{df}} \{x \in {}^\omega\omega(\text{or } {}^k({}^\omega\omega)) \mid \exists f(x,f) \in [T]\}$ - the *projection* of T.
(iii) A set $A \subseteq {}^k({}^\omega\omega)$ is κ-*Suslin* (for $\kappa \geq \omega$, $\kappa \in \text{Card}$) if $A = p[T]$ for some tree on $\omega \times \kappa$.

• Clearly a tree is wellfounded if $[T] = \varnothing$.
• Fact: For a $C \subseteq {}^\omega X$ a closed set, there is a tree T on X with $C = [T]$. In particular for $X = \omega$, if C has a recursively open complement, then we may take T as recursive set of sequences from Seq_ω.

• Fact: If $A \subseteq \omega^\omega$ is Σ^1_1 then $A = p[T]$ for a tree on $\omega \times \omega$ - thus such sets are *projections of closed sets*, and conforms to the idea that

$$x \in A \leftrightarrow \exists y \in \omega^\omega (x, y) \in [T].$$

This classical result (due to Suslin) is sometimes stated that analytic (= $\boldsymbol{\Sigma^1_1}$) sets are "ω-Suslin". Many of the classical properties of analytic sets as studied by analysts can be attributed to this (and similar) representations.

Let then $A \in \Pi^1_1$ be a (lightface) co-analytic set. Then by the above there is a recursive tree T on $\omega \times \omega$ with:

$$\forall x (x \in A \leftrightarrow T_x \text{ is wellfounded}) .$$

We may 'linearise' the tree in the above so that there is a linear ordering $<_x$ for any $x \in \omega^\omega$ with the property $x \in A \leftrightarrow <_x \in WO$. We consider partial maps of the linear orderings $<_x$ arising into On:
$x \in A \leftrightarrow \exists g : \langle \omega, <_x \rangle \to \langle \omega_1, < \rangle$.

$$x \in A \leftrightarrow \exists g \in {}^\omega\omega_1((x, g) \in [T^*])$$

$$\leftrightarrow x \in p[\hat{T}].$$

Consequently: $x \in A \leftrightarrow \exists g(g : T(x) \longrightarrow \omega_1$ in an order preserving way)
That is we may define a tree $\hat{T}$ on $\omega \times \omega_1$ as follows:

$$\hat{T} = \{(\tau_k, u) \mid \forall i, j < |\tau_k| : \tau_i \supset \tau_j \wedge (\tau_k \upharpoonright |\sigma_i|, u) \in T \longrightarrow u(i) < u(j)\}.$$

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• (i) $T^* \in L$ and by the absoluteness of well founded relations on ω to ZF^--models containing all countable ordinals $[T^*] \neq \varnothing \leftrightarrow ([T] \neq \varnothing)^L$.
(ii) If the underlying set is a $\Pi^1_1(a)$ set for some real parameter a then

Now the argument can be stepped up to Σ^1_2 sets: suppose

$$x \in B \leftrightarrow \exists y(x,y) \in A \text{ with } A \in \Pi^1_1$$

and hence there is a tree T^*_A on $(\omega \times \omega) \times \omega_1$ with

$$(x,y) \in A \leftrightarrow \exists g \in {}^{\omega}\omega_1((x,y),g) \in [T^*_A].$$

However there are Δ^{ZF}_0 definable bijections

$$(\omega \times \omega) \times \omega_1 \leftrightarrow \omega \times (\omega \times \omega_1) \leftrightarrow \omega \times \omega_1,$$

thus re-defining the tree T^*_A as $\overline{\overline{T}}$ but on different sequences, so that

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Corollary

Let B be any Σ^1_2 relation. The $\exists x B(x) \leftrightarrow (\exists x B(x))^L$. In particular Σ^1_2 sentences are absolute between L and the universe V. Moreover if $A \subseteq \mathbb{N}$ is Σ^1_2 then $A \in L$.

Cleaning up

In fact there is a useful wellordering, the *Kleene-Brouwer* ordering of Seq_{On} so that in the previous analysis for a Π^1_1 set A, there was a tree T on $\omega \times \omega$ with:

$$\forall x (x \in A \leftrightarrow T_x \text{ is wellfounded}) .$$

We can extend this equivalence to say that:

$$\forall x (x \in A \leftrightarrow T_x \text{ is ordered under } <_{KB})$$

(a) $\rightarrow$ (c)

Theorem (Martin)

$\exists j : L \longrightarrow_e L$ *implies* $Det(\Pi^1_1)$.

Proof: *Idea:* Let $A \in \Pi^1_1$. The usual game G_A involves integer moves but has a complicated payoff set. We replace the space ω^ω with a larger one X^ω for some X to be defined, and relate A to some *closed* $A^* \subseteq X^\omega$. By Gale-Stewart, this is determined. We have to show that winning strategies in G_{A^*} can be translated to winning strategies for the same player in G_A.

Definition

G_{A^*} is defined between two players *I*, *II* playing as follows:

$$\begin{array}{llllllllllll} I & (n_0,\xi_0) & & (n_1,\xi_1) & & (n_2,\xi_2) & & \cdots & & (n_k,\xi_k) & \\ II & & m_0 & & m_1 & & & \cdots & & & m_k \end{array}$$

together constructing

$x = (\ (n_0,\xi_0), m_0,\ (n_1,\xi_1), m_1, (n_2,\xi_2), m_{2,} \cdots \qquad (n_k,\xi_k), , \ldots).$

The *Rules* are that $n_i, m_i \in \omega$, $\xi_i \in \omega_1$. We think of the ξ_i as laying out a function $g : \omega \longrightarrow \omega_1$ with $g(i) = \xi_i$, and the integers yielding $x = (n_0, m_0, n_1, \ldots)$

Winning Conditions: I wins iff $(x, g) \in T^*_A$. g then witnesses that T_x is wellfounded, by giving an order preserving map from $<_x$ into ω_1. Note that the game is closed: if I loses he does so at some finite stage. Note that T^* is defined L, and so by the G-S theorem, is determined in L.

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Claim: If σ^* is a winning strategy for I (resp. II) in G_{A^*} in L, then there is a w.s. for I (resp. II) in G_A.

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Claim: If σ^* is a winning strategy for I (resp. II) in G_{A^*} in L, then there is a w.s. for I (resp. II) in G_A.

Proof of *Claim.* If $\sigma^* \in L$ is a w.s. for I he can use it to play in G_A by suppressing the ordinal moves, and clearly wins (both in L and V). So suppose then $\sigma^* \in L$ is, in L, a w.s. for II.
This is where we use the indiscernibles for L.

continuing

Idea: *II* simulates a run of G_{A^*} by using indiscernibles from $C_0 =_{\mathrm{df}} C \cap \omega_1$ as 'typical' ordinal moves for *I*. She defines a strategy $\sigma : \bigcup_n {}^{2n+1}\omega \longrightarrow \omega$ as follows; fix $n < \omega$ and consider the formula $\varphi(\omega_1, T^*_A, \sigma^*, \tau, \vec{\xi}, m)$ which defines a term $t(\omega_1, T^*_A, \sigma^*, \tau, \vec{\xi}) = m$:

$$\vec{\xi} \in {}^{n+1}\omega_1 \wedge \sigma^*(\tau \restriction n+1, \vec{\xi}) \in T^* \wedge \sigma^*(\tau, \vec{\xi}) = m.$$

As $\sigma^*, T^* \in L$ the term t will have a fixed value m for any $\vec{\xi} \in^{n+1} C_0$ she chooses since the latter are indiscernibles for the formula φ. So she sets (in V):

$$\sigma(\tau) = m = t(\omega_1, T^*, \sigma^*, \tau, \vec{\xi}) \textit{ for any } \vec{\xi} \in {}^{n+1}C_0.$$

Now argue that were $x \in A$ even though *II* followed this strategy, then we'd have that there is an order preserving embedding $g : (\omega, <_x) \longrightarrow (C_0, <)$ (as C_0 is uncountable). But that corresponds to a run of the game G_{A^*} where *II* has used σ^*, which was supposed to be winning for her! Contradiction!

Q.E.D.

Stepping Up

Definition (Homogeneous Trees and Sets)

Let T be a tree on $\omega \times X$ and $\kappa > \omega$ be a cardinal.
*(A) Then T is a κ-*homogeneously Suslin tree *iff* $\exists \langle U_\tau \mid \tau \in \mathrm{Seq} \rangle$ *where*
(i) Each U_τ is a κ-complete measure on T_τ ;
(ii) For any $\tau \supset \sigma$ U_τ projects to U_σ: i.e.

$$u \in U_\sigma \leftrightarrow \{v \in T_\tau : v \restriction |\sigma| \in u\} \in U_\tau .$$

(iii) The U_τ form a countably complete tower*: if $\sigma_i \in \mathrm{Seq}$, $Z_i \in U_{\sigma_i}$ are such that $i < j \longrightarrow \sigma_i \subset \sigma_j$ then there is a $g \in {}^{\omega}X$ so that: $\forall i (g \restriction i \in Z_i)$.*

*(B) We say $A \subseteq \omega^\omega$ is κ-*homogeneously Suslin *if $A = p[T]$ for a κ-homogeneously Suslin tree T.*

- Notice that one way to phrase *(iii)* is to say that if $x \in p[T]$ then $\langle U_{x \restriction i} : i < \omega \rangle$ form a countably complete tower.
- (Martin) If there exist a measurable cardinal κ then any co-analytic set is $p[T]$ for a κ-homog. Suslin tree on $\omega \times \kappa$.

Theorem

Suppose A is κ-hom.Suslin for some κ. Then G_A is determined.

Proof: Suppose $A = p[T]$ with T κ-hom. Suslin on $\omega \times \lambda$ (some $\lambda \geq \kappa$).

Definition (G_{A^*})

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together constructing $x = ((n_0, \xi_0), m_0, \ (n_1, \xi_1), m_1, (n_2, \xi_2), m_{2,} \ \cdots \ )$.
Rules: $n_i, m_i \in \omega$, $\xi_i \in \lambda$. *Winning Conditions:* as before I wins iff $(x, g) \in [T]$ where $g(i) = \xi_i$.
Again this is a closed game, and so there is a w.s. for one of the players, now in $L[T]$. To show that if II has a w.s. in G_A if she has a w.s. σ^*in G_{A^*}, we use the ω_1-completeness of the measures rather than indiscernibility.
She defines a strategy σ as follows; for $n < \omega$ and then for $\tau \in {}^{2n+1}\omega, u \in {}^{n+1}\lambda$, define
$\overline{\sigma}(\tau, u) = \sigma^*(\tau(0), u(0), \cdots, \tau(2n), u(2n)) \in \omega$.
Define $Z_{\tau,k} =_{\mathrm{df}} \{u \in T(\tau \restriction n+1) : \overline{\sigma}(\tau, u) = k\}$. Since $U_{\tau \restriction n+1}$ is a measure on $T_{\tau \restriction n+1}$, by its ω_1-completeness, for precisely one value of k is $Z_{\tau,k} \in U_{\tau \restriction n+1}$. So let that value of k_0 be the response given by the strategy:

$\sigma(\tau) = k_0$ where k_0 is the unique value of k_0 with $Z_{\tau,k_0} \in U_{\tau \restriction n+1}$.

Finish as before. Q.E.D.

Hence determinacy for a class of sets would follow if we could establish homogeneity properties for trees projecting to those sets.

Definition

$A \subseteq \omega^\omega$ is *weakly homogeneously Suslin* if $A = pB$ where $B \subseteq (\omega^\omega)^2$ is hom. Suslin.

• In fact this is not the real def. of w.h. S. which defines "*weakly homogenous trees*" and is in terms of towers of measures.
• W.hom.Suslin sets (defined as the projections of w.homog. trees) are of interest in their own right: we have seen that if there is a measurable cardinal, then Σ^1_2 sets are w.h.Suslin.

Theorem

If a set A is w.hom.Suslin, then it has the regularity properties (LM, BP, PSP).

Unfortunately A w.hom Sus. $\not\longrightarrow$ A hom. Sus.
They also have complements defined as projections of trees $\tilde{T}$ with the latter definable from their w. homog. tree T:

Lemma

Let A be $p[T]$ with T w. homog.. Then there is a tree $\tilde{T}$ with $\omega^\omega \backslash A = p[\tilde{T}]$.

But on its own this is no help. The breakthrough was:

Theorem (Martin-Steel)

Suppose that λ is a Woodin cardinal and T is λ^+ w. hom. tree. then for $\gamma < \lambda$, the $\tilde{T}$ above is γ-hom.Suslin.

Theorem

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Then $\mathrm{Det}(\Pi^1_2)$.
Further if $\lambda_{n-1} < \lambda_{n-2} < \cdots < \lambda_0$ *are* $n-1$ *further Woodin cardinals, then* $\mathrm{Det}(\Pi^1_{n+1})$.
Thus:
$\mathrm{ZFC} +$ *"there exist infinitely many Woodin cardinals"* $\vdash \mathrm{PD}$.

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Thus:
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Proof: Let $A \subseteq \omega^\omega$ be Π^1_2. Then A is the complement of a Σ^1_2 set on ω^ω which is itself the projection of a Π^1_1 set $B \subseteq (\omega^\omega)^2$. B is κ-hom.Suslin for some hom. tree T, as κ is measurable, and *a fortiori* is also λ_0^+-hom. Suslin. By the Martin-Steel Theorem we have a γ-homog. tree $\tilde{T}$ projecting to A with γ-complete measures. By a previous theorem we have G_A is determined. The last two sentences follow by repetition of the argument.

Q.E.D.

Woodin's Equiconsistency

• The exact consistency strength of the assumption above is slightly stronger, with the $(\Longleftarrow)$ being very involved:

Theorem (Woodin)

$Con(ZFC + \exists^{\infty}(\textit{Woodin Cardinals}) \Longleftrightarrow Con(ZFC + \mathrm{AD}^{L(\mathbb{R})})$.

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Woodin cardinals fix the analytical truths:

Theorem

(Woodin) Suppose there is a proper class of Woodin cardinals; then $\mathrm{Th}(L(\mathbb{R}))$ *is absolute with respect to set forcing. Thus if* $V[G]$ *is a set generic extension:*

$$(L(\mathbb{R}))^{V} \equiv (L(\mathbb{R}))^{V[G]}.$$