Object restoration through dynamic polygons

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1 Introduction

A difficult task in the restoration of an image portraying objects is the recovery of their boundaries. This can be done at a low level (e.g. using an edge detector, see Rosenfeld and Kak [12]) or at a high level, trying to embed into the restoration process more complex information regarding the shape of objects, a form of pattern restoration.

In this article we will try to recover the boundary of a single object starting from a noisy version of it, relying on the prior information that the boundary is a closed and non-intersecting curve in the plane, that is, fitting a non-intersecting polygon to the object. In doing this, we can use more information than that required by a simple edge detector (such as the gradient of greylevel intensity, whose nature is essentially local), because a polygon, besides this information, conveys the idea of connectivity between the pairs of its vertices (not depending on their distance), and the global idea of closed curve.

A similar approach to this problem has been attempted by several authors, including Grenander *et al.* [8], Mardia and Qian [10], Qian *et al.* [11], and Storvik [13].

The deformable template model of Grenander is based upon the deformation of a template, a polygon with a fixed number of sides of variable length, to find the optimum fit to the object.

Mardia and Qian, and Qian *et al.* use a star-shaped polygon, that is, one for which the line segment joining each vertex to some fixed internal point lies entirely within the polygon. The main difference between these two papers is that in the first the restoration is chosen as the best possible fit to a known template belonging to a library of templates, while in the second the prior knowledge is contained in some smoothness properties of the object outline. In this respect the spirit of the second paper is nearer to ours.

Storvik reverses the problem, by considering a polygon with a variable number of sides with a fixed length, located on the pixel sides of the digitised image, thus obtaining a more flexible curve, but with a possibly unnecessarily high number of sides (such as stair-like configurations).

Our approach aims to reach flexibility in shape modelling and parsimony at the same time, by considering polygons with any number of sides of any length. The choice between polygons of different size can be viewed as a choice of different models, for which the reversible jump MCMC algorithm by Green [7] is particularly suitable.

Other papers in which the emphasis is on boundary detection are the papers by Gavin and Jennison [4], and by Helterbrand *et al.* [9], but their perspective is more in the stream of the line-site process proposed by Geman and Geman [5], with a constraint of connectedness imposed on the edges and no limitations on the number of objects. Gavin and Jennison's edges lie inside the pixels and are to be joined together to form closed curves, while Helterbrand *et al.* locate the edges on the pixel themselves and use the classical clique-based Markov random field models for lattices (see e.g. Besag [2]) to put zero probability on non connected configurations.

Finally, Clifford and Nicholls [3] developed a method for the restoration of polygonal images, based on a model for random polygonal patterns in the plane by Arak *et al.* [1] which will be sketched below.

In the following section we describe a statistical model that can produce polygons with any number of sides, and we show the connection between our model and the template model of Grenander *et al.* [8]. In section 3 we introduce the reversible jump algorithm developed by Green [7] which will be used for the restoration process and we show how to implement the algorithm in practice for this problem. Finally, in section 4 we describe the result of an experiment into the restoration of two synthetic pictures. A brief discussion concludes the paper.

2 The model

We will tackle the problem of the restoration of an object via the classical (in the sense of statistical image analysis) Bayesian approach (see e.g. Besag [2]), focussing our attention on the outline of the object only, so that we can also regard our procedure as a binary image restoration.

We are given a noisy image digitised into $M \times N$ square pixels, whose centers have coordinates (i, j), $i = 1, \ldots, M$, $j = 1, \ldots, N$. The rectangle containing the image can thus be defined as $D = [0.5, M + 0.5] \times [0.5, N + 0.5]$. The object splits D into two sets, O and O^c, the set of the points inside the object and its complement. If we decide to regard a pixel as being inside O when this is true for its central point, we obtain the corresponding partition for the pixels; note that because of the loss of information in discretization the same partition of the pixels can be produced by many different objects.

We assume that the observed image (data) has been produced by the following model:

$$z_{ij} = \mu_{ij}(O) + \epsilon_{ij} \qquad i = 1, \ldots, M, \quad j = 1, \ldots, N \tag{1}$$

where

$$\mu_{ij}(O) = \begin{cases} \mu_1 & \text{if} \quad (i,j) \in O \\ \mu_0 & \text{if} \quad (i,j) \notin O \end{cases} \qquad \epsilon_{ij} \; i.i.d. \; N(0, \sigma^2).$$

The values of μ_1, μ_0 and σ are assumed known in this article. This model has also been used in this form by Storvik [13] and in a more general form by Qian *et al.* [11], with two generic functions $f_1(i, j)$ and $f_0(i, j)$, whose value had to be estimated, in the place of μ_1 and μ_0.

Our aim is to recover the object boundary by fitting to it a closed and non-intersecting polygon (a simple polygon). We can define the set of all k-side simple polygons belonging to D, for all $k \geq 3$, as

$$\begin{aligned} C_k &= \{\{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{k-1}, v_k\}, \{v_k, v_1\}\} \in D : \text{ joining all pairs} \\ &\quad (v_i, v_{(i \, mod \, k)+1}),\ i = 1, \ldots, k, \text{ we get a simple polygon belonging to} D\Big\}. \end{aligned}$$

The set af all polygons with any number of vertices belonging to D is then $C = \bigcup_{k=3}^{\infty} C_k$, and v is the generic element of C. Following a Bayesian approach, we will get a candidate for the restoration from the posterior distribution of v:

$$p(v \mid z) \propto p(z \mid v)\, \pi(v)$$

where $z = \{z_{ij},\ i = 1, \ldots, M,\ j = 1, \ldots, N\}$, $p(z \mid v)$ is the likelihood from model (1) when v is the true polygon, and $\pi(v)$ is the prior distribution of the polygon.

Now we want to define the prior distribution of v through a density with respect to some measure $\mu(\cdot)$ on the measurable space $(C, \mathcal{F})$, where $\mathcal{F}$ is a σ-algebra of subsets of C. For each $A \in \mathcal{F}$ we set

$$\mu(A) = \sum_{k=3}^{\infty} \int_{D^k} I(v \in A) \frac{dv_1 \ldots dv_k}{2\,k}.$$

The factor $2\,k$ appearing at the denominator counts the number of equivalent polygons obtained by a given vector $v_1, \ldots, v_k$ by changing the ordering of the vertices, dv_i, $i = 1, \ldots, k$, is the Lebesgue measure on $\mathcal{R}^2$ and $I()$ is the indicator function. We can now define our probability density of all polygons with respect to the measure $\mu(dv)$ as follows:

$$\pi(v) \propto \exp\{-\beta k^{\gamma} - \frac{\alpha}{k} \sum_{i=1}^{k} (\theta_i(v) - \pi)^2\} \tag{2}$$

where α and β are non-negative and $\theta_i(v)$ represents the angle interior to the i-th vertex of v in radians.

This density is invariant with respect to the position, the orientation and the size of the polygon, so it is a shape density. The term containing the angles penalises strongly irregular polygons and its minimum points, for fixed k, are all the regular polygons in D with k sides. It is also evident that the minimised value of this term tends to zero as $k \to \infty$, so, if it was the only term in the distribution, the upper limit points of the distribution would be all the circles in D. If γ was positive, the term βk^{γ} would increasingly penalise an increasing number of vertices, contributing to assigning a higher probability to "parsimonious" polygons.

We now have to look for some sufficient conditions on γ for the prior distribution to be proper, because otherwise the posterior distribution will be improper. Denoting the right member of (2) by $\exp\{-U(v)\}$, we have

$$\begin{aligned} &\int_C \exp\{-U(v)\}\, \mu(dv) \quad = \\ = \sum_{k=3}^{\infty} \int_{D^k} I(v \in C_k) \exp\{-U(v)\} \frac{dv_1 \ldots dv_k}{2k} &\leq \sum_{k=3}^{\infty} \int_{D^k} \max_{v \in D^k} \exp\{-U(v)\} \frac{dv_1 \ldots dv_k}{2k} \end{aligned}$$

$$\leq \sum_{k=3}^{\infty} \int_{D_k} \exp\{-\beta k^{\gamma} - \frac{\alpha 4\pi^2}{k^2}\} \frac{dv_1 \ldots dv_k}{2k} \quad \leq \quad \sum_{k=3}^{\infty} \left(\frac{MN \, \exp\{-\beta k^{\gamma-1}\}}{\sqrt[k]{k}} \right)^k \tag{3}$$

As $MN > 1$, the series (3) converges if and only if $\gamma \geq 1$, provided $\exp(\beta) > MN$ when $\gamma = 1$.

Keeping in mind (1) and (2), the posterior distribution of the polygon turns out to be

$$p(v \mid z) \propto \exp\{-\beta k^{\gamma} - \frac{\alpha}{k} \sum_{i=1}^{k} (\theta_i(v) - \pi)^2 - \frac{1}{2\sigma^2} \sum_{i=1}^{M} \sum_{j=1}^{N} (z_{ij} - \mu_{ij}(v))^2\} \tag{4}$$

where we have substituted the actual object v to the generic object O in the argument of $\mu_{ij}(\cdot)$.

Our prior model appears to be related to that derived by the construction derived by Arak *el al.* [1]. In that case one simulates two independent Poisson point processes, the first in the interior of the image rectangle and the second on its perimeter, and creates a graph by connecting pairs of vertices subject to the constraint that V, Y, T and I-shaped vertices only are allowed. Provided an appropriate measure on all point patterns has been defined, it is possible to explicitly calculate the normalisation constant and some moments. Moreover, a consistency property holds: the process one observes in a window is defined on all $\mathcal{R}^2$.

The process we use is similar to one that could be defined as an Arak process conditioned on the fact that no points have been observed on the perimeter, V-vertices only are allowed and only the graphs with intersecting edges are admissible.

The density (2) can also be rewritten by extending the theoretical framework used by Grenander, Chow and Keenan [8] to define a probability distribution on the deformations of a given simple closed curve (template) belonging to a space of fixed dimension (such as C_k with k fixed). Every polygon belonging to a set C_k is obtainable by deforming a regular polygon $\bar{v}^{(k)}$ with k vertices and fixed labelling (template) by applying to each of its sides k functions $(s_1, \ldots, s_k)$ belonging to a group of trasformations S. If we predefine a sequence of templates $\{\bar{v}^{(k)}, k \geq 3\}$, we can define the prior distribution for the trasformations themselves instead that for the vertices, thus modifying the support of the prior distribution to the set $S = \bigcup_{k=3}^{\infty} S_k$, where $S_k = \{s \in S^k : s\,\bar{v}^{(k)}$ is a simple polygon$\}$, and $s\,\bar{v}^{(k)} = (s_1\,\bar{v_1}^{(k)}, \ldots, s_k\,\bar{v_k}^{(k)})$. We can then rewrite (2) as a product of *acceptor functions* depending on s as follows:

$$\pi(s; \bar{v}^{(k)}) \propto \exp\{-\beta k^{\gamma}\} \prod_{i=1}^{k} A(s_{i-1}, s_i; \bar{v}^{(k)}) \tag{5}$$

where

$$A(s_{i-1}, s_i; \bar{v}^{(k)}) = \exp\{-\frac{\alpha}{k} [\theta(s_{i-1}\,\bar{v}_{i-1}^{(k)},\; s_{i-1}\,\bar{v}_i^{(k)},\; s_i\,\bar{v}_{(i\,mod\,k)+1}^{(k)}) - \pi]^2\},$$

$s_0 = s_k$, $\bar{v}_0^{(k)} = \bar{v}_k^{(k)}$ and $\theta(\cdot)$ measures the interior angle formed by the vertex identifies by its three arguments. The densities (2) and (5) have the Markovian property that the distribution of a side (identified by its vertices in (2) and by the relative s function in (5)) conditioned to the other sides and to the number of sides depends only on its adjacent sides.

In what follows we will use neither this approach nor Clifford and Nicholls' for the synthesis and the restoration of objects, but we will use model (2), which is simpler and more immediate for the kind of application we are considering here.

3 Application of the reversible jump algorithm to object restoration

The posterior distribution (4) allows one to calculate different kinds of restoration, such as the point of maximum *a posteriori* (MAP) probability density

$$\max_v p(v \mid z)$$

or the posterior mean of the state of every pixel with respect to the polygon

$$\begin{aligned} &\sum_{k=3}^{\infty} \int_{D^k} \mu_{ij}(v)\ I(v \in C_k)\ p(v \mid z)\ \frac{dv_1 \ldots dv_k}{2k} \\ = \mu_1\ &\mathrm{prob}(v \ni i) + \mu_0\ \mathrm{prob}(v \not\ni i) \quad i = 1, \ldots, M,\ j = 1, \ldots, N, \end{aligned} \tag{6}$$

which corresponds to the posterior probability that the polygon includes a given pixel when $\mu_1 = 1$ and $\mu_0 = 0$. The posterior mean of v cannot be defined usefully, because it is impossible to average a variable which ranges over spaces of different dimension, and the MAP restoration does not carry a very meaningful information, so we have chosen to use (6) for our restoration, because it has the desirable feature of not being dependent on the dimension of the space in which the polygon lies.

Since we cannot calculate (6) directly, it is necessary to simulate a Markov chain with state space C and with (4) as equilibrium distribution, and then calculate

$$\frac{1}{niter - n_0} \sum_{n=(n_0+1)}^{niter} \mu_{ij}(v(n)) \tag{7}$$

as an approximation of (6). The sequence $\{v(n)\}$ is the sequence of polygons $v(n)$ obtained during the simulation, $niter$ is the number of iteration, and n_0 is the length of the *burn-in* period.

The problem we have just stated cannot be solved with ordinary MCMC methods (e.g. the Metropolis algorithm, see Tierney [14]), because the chain must visit spaces of different dimension and switch from one to another. The reversible jump algorithm proposed by Green [7], gives a solution by suitably modifying the Metropolis-Hastings algorithm. We give here one way of exploring the space C through a Markov chain, which will also show in detail how our problem fits into Green's framework.

Assuming that the current state is $v \in C_k$, we move to different states only through slight changes, so that we can easily check whether the move has led the chain to an admissible state. We do this by choosing among two *move types* for moving to $v' \in C_{k'}$.

1. Move type 1 ("move"): for all $k \geq 3$, with probability m_k we propose a new position for a randomly chosen vertex v_i of v, $i = 1, \ldots, k$, sampling it from the proposal density $q_1(\cdot; v_i, v)$, with support on $\mathcal{R}^2$; we accept the proposed move with probability $\alpha_1^{(k)}(v, v')$, so that the probability of moving to $A \in \mathcal{F}$ is defined as

$$\begin{aligned} \textstyle\int_A Q_1^{(k)}(v, dv')\ \alpha_1^{(k)}(v, v') \quad &= \quad \textstyle\frac{m_k}{k} \sum_{i=1}^{k} \int q_1(v_i'; v_i, v)\ \alpha_1^{(k)}(v, v(v_i' \to v_i)) \\ &\qquad I[v(v_i' \to v_i) \in A \cap C_k]\ dv_i', \end{aligned}$$

where $v(v_i' \to v_i)$ is the polygon obtained by substituting into v the new coordinate v_i' to v_i.

2. Move type 2 ("change space"). For this move type we need some additional notation: for any given vertex v_i, $i = 1, \ldots, k$, we denote by v_i^- and v_i^+ the vertices preceding and following v_i. Move type 2 works as follows:

   (a) for all $k \geq 3$, with probability b_k we add a new vertex ξ between two randomly chosen consecutive vertices v_i and v_i^+, $i = 1, \ldots, k$, sampling its position from the proposal density $q_2(\xi; v_i, v)$ with support on $\mathcal{R}^2$; we accept the new configuration with probability $\alpha_2^{(k)}(v, v')$, so that the probability of moving to A is defined as

$$\int_A Q_2^{(k)}(v, dv')\, \alpha_2^{(k)}(v, v') \;=\; \tfrac{b_k}{k} \textstyle\sum_{i=1}^k \int q_2(\xi; v_i, v)\, \alpha_2^{(k)}(v, v(\xi, v_i))$$
$$I[v(\xi, v_i) \in A \cap C_{k+1}]\; d\xi,$$

   where $v(\xi, v_i)$ is the configuration obtained from v by joining to ξ v_i^- and v_i^+ and deleting the segment between them;

   (b) for all $k \geq 4$, with probability d_k we cancel a randomly chosen vertex and join its two adjacent vertices; we accept the move with probability $\alpha_2^{(k)}(v, v')$, so that we would move to A with probability

$$\int_A Q_2^{(k)}(v, dv')\, \alpha_2^{(k)}(v, v') = \frac{d_k}{k} \sum_{i=1}^{k} \alpha_2^{(k)}(v, v(-v_i))\; I[v(-v_i) \in A \cap C_{k-1}],$$

   where $v(-v_i)$ is the configuration obtained by deleting vertex v_i. If $k = 3$ we just resample the move type.

Naturally $m_k + b_k + d_k = 1\ \forall\, k \geq 3$. Notice that we have not imposed any explicit constraint on the support of the proposal densities and this implies that the new proposed state might be an intersecting polygon or a polygon with some vertex out of D. The same could happen after the deletion of a vertex. Setting the acceptance probability to zero when this happens solves the problem.

With this move strategy, the transition kernel for going from $v \in C$ to $A \in \mathcal{F}$ is the sum of the probability of moving to A and of staying at v when $v \in A$ (from now on we drop the superscript (k) from the proposals and the acceptance probabilities for convenience):

$$\begin{aligned} P(v, A) &= \int_A P(v, dv') \\ &= \sum_{i=1}^{2} \int_A Q_i(v, dv')\, \alpha_i(v, v') \\ &+ I(v \in A)\, [\sum_{i=1}^{2} \int_A Q_i(v, dv')\, (1 - \alpha_i(v, v')) + (1 - \sum_{i=1}^{2} Q_i(v, C))]. \end{aligned} \tag{8}$$

Note that the last term in the second member can be non null only when configurations out of the space C (such as intersecting polygons) can be proposed.

Once the acceptance probabilities $\alpha_i(\cdot,\cdot), i=1,2$, have been specified so that the kernel has $\pi(v)$ as an invariant distribution and it is irreducible and aperiodic, we obtain a Markov chain with $\pi(v)$ as equilibrium distribution for π-almost all starting points $v \in C$. Such a chain must be positive recurrent; if it is also Harris recurrent the convergence to equilibrium occurs for all starting points (see Tierney [14] for a proof of these assertions). A sufficient condition for the invariance is the reversibility condition for all the pairs of disjoint sets $A, B \in \mathcal{F}$:

$$\int_A \left[\int_B P(v,dv')\right] \pi(v)\,\mu(dv) = \int_B \left[\int_A P(v',dv)\right] \pi(v')\,\mu(dv) \tag{9}$$

which is easily proven once we substitute (8) into (9) and note that the integrals on $A \cap B$ are equal in both members. Substituting then to $P(v,dv')$ its whole expression, we see that in order to prove (9) we need only to show its validity for each move type separately ($i=1,2$):

$$\int_A \left[\int_B Q_i(v,dv')\,\alpha_i(v,v')\right] \pi(v)\,\mu(dv) = \int_B \left[\int_A Q_i(v',dv)\,\alpha_i(v',v)\right] \pi(v')\,\mu(dv'). \tag{10}$$

Considering move type 1 and a pair of disjoint $\mathcal{F}$-measurable sets $A, B \subset C_k$, we get the following expression, which is analogous to that obtained by Geyer and Moller [6] when considering reversibility during the simulation of point processes:

$$\int_A \left[\int_B Q_1(v,dv')\,\alpha_i(v,v')\right] \pi(v)\mu(dv)$$

$$= \int_{D^k} \left[\frac{m_k}{k} \sum_{i=1}^k \int_D q_1(v_i';v_i,v)\; I[v(v_i' \to v_i) \in B]\; \alpha(v, v(v_i' \to v_i))\; dv_i'\right] I(v \in A)\; \pi(v) \frac{dv_1 \dots dv_k}{2k}$$

$$= \int_{D^{k+1}} m_k\; q_1(v_i';v_i,v)\; \alpha(v, v(v_i' \to v_i))\; \pi(v) I[v(v_i' \to v_i) \in B]\; I(v \in A)\; \frac{dv_i'\, dv_1 \dots dv_k}{2k}$$

for any $i=1,\dots,k$, since $q_1()$ and $\alpha_1()$ do not depend on i. If we now write the same for the reverse move we can straightforwardly see that for any $k \geq 3$ and $i=1,\dots,k$ reversibility holds if

$$\alpha_1^{(k)}(v, v(v_i' \to v_i)) = \begin{cases} \min\left\{1, \frac{\pi(v(v_i' \to v_i))\; q_1(v_i; v_i', v(v_i' \to v_i))}{\pi(v)\; q_1(v_i'; v_i, v)}\right\} & \text{if} \quad v, v(v_i' \to v_i) \in C_k \\ 0 & \text{otherwise} \end{cases}$$

If we now consider reversibility for move type 2 when $A \in C_k$ and $B \in C_{k+1}$, we get to the following equivalent condition in the same way we did for move type 1:

$$\frac{b_k}{k} \int_{D^{k+1}} q_2(\xi; v_i, v)\; \alpha_2(v, v(\xi, v_i))\; \pi(v)\; I(v(\xi,v_i) \in B)\; I(v \in A)\; d\xi\, dv_1 \dots dv_k$$

$$= \frac{d_{k+1}}{k+1} \int_{D^{k+1}} \alpha(v', v'(-v_j'))\; I(v'(-v_j') \in A)\; I(v' \in B)\; dv_1' \dots dv_{k+1}'$$

for any $i=1,\dots,k$ and $j=1,\dots,k+1$. Renaming the differential at the second member as $dv_1 \dots dv_k\, d\xi$ and aliasing $v'(-v_i')$ to v and v' to $v(v_i,\xi)$ for any chosen $i \in \{1,\dots,k\}$, we see that the above equivalence holds if we set

$$\alpha_2^{(k)}(v, v(\xi, v_i)) = \begin{cases} \min\left\{1, \frac{\pi(v(\xi,v_i))\; d_{k+1}\; k}{\pi(v)\; q_2(\xi;v_i,v)\; b_k\; (k+1)}\right\} & \text{if} \quad v \in C_k,\; v(\xi,v_i) \in C_{k+1} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\alpha_2^{(k+1)}(v(\xi, v_i), v) = \frac{\pi(v)\, q_2(\xi; v_i, v)\, b_k\, (k+1)}{\pi(v(\xi, v_i))\, d_{k+1}\, k}\, \alpha_2^{(k)}(v, v(\xi, v_i)).$$

The final step for the realization of the reversible jump algorithm is the specification of the proposal densities $q_1(\cdot; \cdot, v)$ and $q_2(\cdot; \cdot, v)$. These densities were chosen to allow with high probability moves that are consistent with the local configuration of the polygon; for example, if we have to add a vertex between two vertices very near to each other, we concentrate the probability mass on a small set of possible positions, thus avoiding a too high probability of the occurrence of illegal configurations (i.e. intersecting polygons). We define in the following the two proposal densities for move types 1 and 2.

1. Move type 1. Let $v \in C_k$ and $i \in \{1, \ldots, k\}$; by $d(\cdot, \cdot)$ the Euclidean distance in $\mathcal{R}^2$:

$$q_1(v_i'; v_i, v) = N_2(v_i, \sigma_1^2(v_i, v)\, I_2) \tag{11}$$

   where I_2 is the 2×2 identity matrix, $N_2()$ is the bivariate normal density, and $\sigma_1(v_i, v) = d(v_i^-, v_i^+)/c(v)$.

2. Move type 2:

$$q_2(\xi; v_i, v) = N_2(\eta(v_i, v), \sigma_2^2(v_i, v)\, I_2) \tag{12}$$

   where $\eta(v_i, v) = (v_i + v_i^+)/2$ and $\sigma_2(v_i, v) = d(v_i, v_i^+)/c(v)$.

The function $c(v)$ appearing above must be chosen to fulfil the condition of coherence to the local configuration when a move is attempted. A simple criterion is to select $c(v)$ so that a new vertex is added between two vertices with a fixed probability (say 0.95) in the square with center $\eta(v_i, v)$ $[\eta_x(v_i, v) \pm d(v_i, v_i^+)/2] \times [\eta_y(v_i, v) \pm d(v_i, v_i^+)/2]$, in which case $c(v)$ is a constant.

With these densities, $\alpha_1(v, v')$ simplifies to $\min(1, \pi(v')/\pi(v))$, while, if we rewrite the position for the new vertex ξ as $\xi = \eta(v_i, v) + \sigma_2(v_i, v)\, u$, where $u \sim N_2(0, I_2)$, $\alpha_2(v, v(\xi, v_i))$ can be rewritten as

$$\min\left\{1, \frac{\pi(v(\xi, v_i))\, d_{k+1}\, k}{\pi(v)\, \phi(u)\, b_k\, (k+1)} \sigma_2^2(v_i, v)\right\}$$

where $\phi(\cdot)$ is the standard normal bivariate density. This formula corresponds to the formula given by Green [7] for the acceptance probability of moves involving a change of space, where, since $v(\xi, v_i)$ is a function of v and u, $\sigma_2^2(v_i, v)$ is the jacobian

$$\left| \frac{\partial v(\xi, v_i)}{\partial(v, u)} \right|.$$

The application of Green's formula is immediate if we consider any single action we can take during the simulation with kernel (8) as a move type and look for reversibility within each of them. In this case, when C_k is the current set, we can choose among $3k$ move types: deletion or new position of any one of k vertices plus insertion of a new vertex between any one of k consecutive pairs.

The equilibrium distribution of kernel (8) is $\pi(\cdot)$. In fact, we have shown above that $\pi(\cdot)$ is an invariant distribution for $P(v, \cdot)$, which is obviously an irreducible and aperiodic kernel, as long as the probabilities m_k, b_k and d_k are bounded away from zero by some positive constant for all $k \geq 3$.

Moreover, because Metropolis-Hastings kernels are Harris recurrent (see Tierney [14]), $\pi(\cdot)$ is the equilibrium distribution for all starting points.

The advantage of this method is that we do not need to explicitly impose any constraint of closeness or non intersection of polygons within the probability distribution, because the admissibility of the configuration is controlled during the execution of the reversible jump algorithm, simply refusing the illegal configurations. But there are two disadvantages: checking the admissibility of the proposed configurations is time consuming; the variances $\sigma_1^2(\cdot)$ and $\sigma_2^2(\cdot)$ must be carefully specified because if they were too high too many illegal configuration would be proposed, while if they were too low, the algorithm would be too"timid", slowing down in both cases the convergence to equilibrium of the chain.

4 Experiments

As an application of the method we have tried to restore two 64×64 artificial pictures with a feature in common: a central body, whose dimensions are not too small compared to the dimension of the entire image, with some tiny extensions. A method based on ordinary smoothing techniques would probably shorten or even cause the extensions to part from the central body, while our method, exploiting the additional information that the image portrays only one object and the connectivity among the polygon vertices, has succeeded in recovering these features, starting from a very poor initial configuration (a square located more or less in the centre of the object).

The parameter values were set to $\mu_1 = 1$, $\mu_0 = 0$ and $\sigma^2 = 0.5$ in model (1) and to $\beta = 0.5$, $\gamma = 1.6$ and $\alpha = 3$ in the prior model (2), because simulating realizations from the prior distribution we saw that these values produce polygons which have features quite similar to our true polygons: visible angles, a simple shape (which means not too many vertices) and some tiny extensions. An example of such a polygon is shown in figure 1, where we have drawn the last realization of a chain of length 50000. The estimated marginal mode of the number of vertices, found by performing the ergodic average of the number of vertices, with a *burn-in* period of 20000 iterations, is 31. The good convergence to equilibrium is shown by the sequence in figure 2. Finally, we set $m_k = b_k = d_k = 1/3$ $\forall k \geq 3$ and the constant c to 20, according to the criterion stated in the previous section.

In figure 3 we show two different test images perturbed according to model (1) with the parameters specified above. The restorations according to (7), together with the true object and the starting configuration, appear in figure 4. Seventy thousand iterations were performed and the procedure took about three minutes of CPU time on a Sun workstation.

The starting configuration was found automatically simply producing a smoothed version of the noisy images with a running mean performed by a 5×5 window and thresholding the result to two greylevels. Considering the resulting image as a bivariate probability distribution on D, we have calculated its two marginal means and variances (say m_x, m_y, σ_x and σ_y) and then have taken as initial configuration the square $[m_x \pm 2\,\sigma_x] \times [m_y \pm 2\,\sigma_y]$. The same procedure without smoothing and thresholding did not produce any meaningful result.

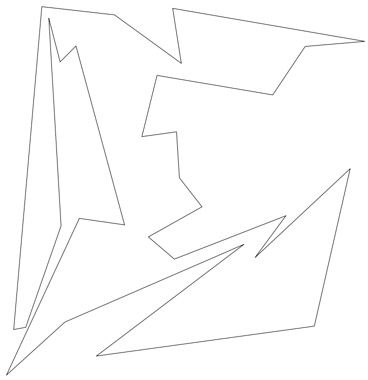

Figure 1: The last realisation from an MCMC simulation from the prior

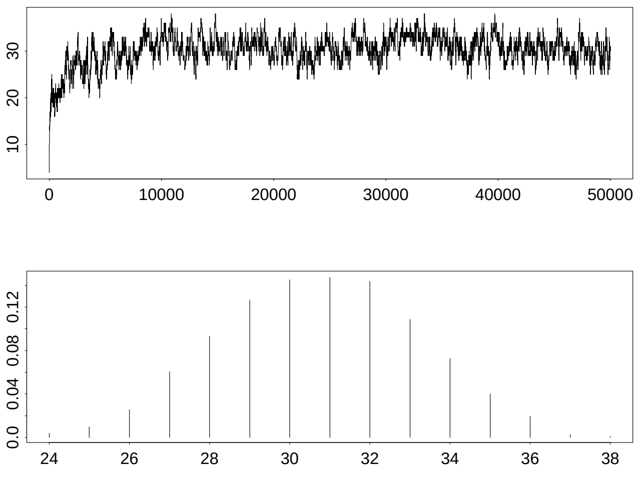

Figure 2: Trajectory of the number of vertices and their marginal distribution

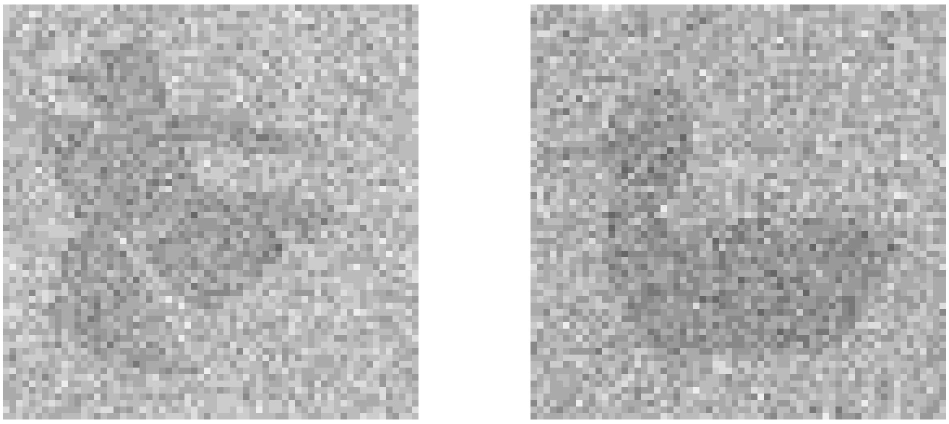

Figure 3: Test images

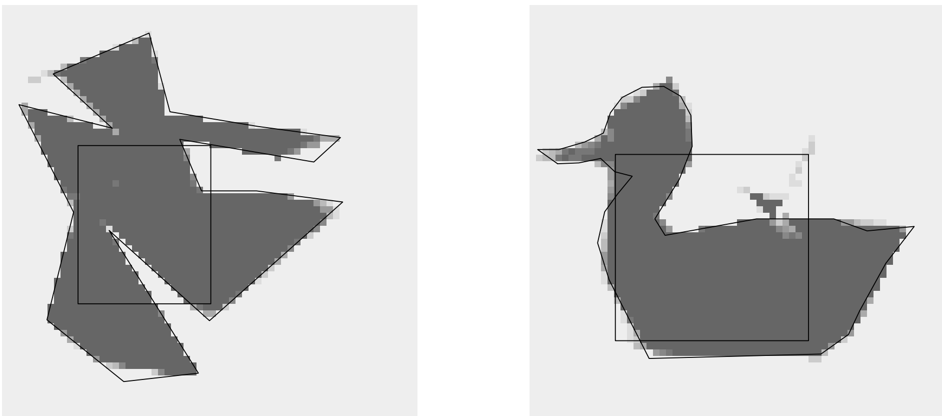

Figure 4: Restorations of figure 3

5 Discussion

In this paper we have illustrated a method for the restoration of a noisy binary image portraying a single object whose outline is entirely defined by a polygon (with an indefinite number of sides). The computing machine provided by the reversible jump MCMC algorithm was used to perform the restoration, exploiting the capability of the algorithm of switching between spaces of different dimension, that is, of drawing inference about the correct number of sides of the polygon to be fitted to the object.

The problem setting was as simple as possible: the values of the two greylevels present in the original image were assumed known, the move type probabilities were kept constant throughout the execution of the algorithm, regardless of the current state of the chain, and there was no problem of estimation of the parameters of the prior distribution. Indeed, it is not possible to find a maximum pseudo-likelihood estimate of the parameters in 2 without a preliminary guess about the true underlying polygon.

Thus many other routes should be tried. First, the prior model should be extended to more general situations, substituting μ_0 and μ_1 with unknown function to be estimated, like e.g. in Qian *et al.* [11]. This should be done in such a way that the gain in speed of convergence given by the capability of the reversible jump algorithm of discarding bad models during the simulation is not counterbalanced by the richer parametrization. Second, this setting should be extended to allow multiple objects in the image, which is by no means trivial. Third, we may improve the realization of the reversible jump algorithm to accelerate the convergence to equilibrium, e.g. by choosing more sensibly the vertex visiting scheme or the probabilities of the move types. Fourth, a full Bayesian approach could allow us to include in the restoration process the estimation of prior parameters, provided we are able to define some hyperpriors for the parameters which describe meaningfully our prior expectations about the shape of the polygon.

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