Spatial mixtures of Poisson distributions and application to disease mapping

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Hierarchical models with spatial priors are commonly used for disease mapping (Clayton and Bernardinelli, 1992). The basic sampling distribution for our data is

$$y_i \sim \text{Poisson}(\lambda_i E_i) \quad \text{independently for } i = 1, 2, \dots, n$$

where y_i denotes the observed count of disease incidences or deaths in region i, E_i is the expected count based on population size, adjusted for age and sex, etc., and λ_i is a region-specific relative risk variable, the object of our inference.
In contrast to conventional schemes, in which $\{\lambda_i\}$ are modelled directly using a spatially-correlated random field (Besag, York and Mollié, 1991), here we consider mixture models of the form

$$\lambda_i = \lambda_{z_i}$$

where $\{\lambda_j, j = 1, 2, \dots, k\}$ characterise k different components, and $\{z_i, i = 1, 2, \dots, n\}$ are *allocation variables* taking values in $\{1, 2, \dots, k\}$.
We place an independence prior on $\{\lambda_j, j = 1, 2, \dots, k\}$:

$$\lambda_j \sim \Gamma(\alpha, \beta) \quad \text{independently for } j = 1, 2, \dots, k.$$

Usually, we take the hyperparameters α and β as fixed, but we occasionally also consider the hierarchical model in which $\beta \sim \Gamma(b_1, b_2)$.
The number of components k is allowed to vary, with prior distribution $p(k|\kappa)$, typically either truncated Poisson in form, or uniform on some range $\{1, 2, \dots, k_{\max}\}$.
Finally, we consider the prior for the allocation variables $\{z_i\}$. This will be a spatially-correlated process, and our aim is to investigate several alternative formulations. Each of these is built on an undirected graph, which plays the role of the prior conditional independence graph of z. Two regions i and i' are said to be neighbours, written $i \sim i'$, $i \in \partial i'$ or $i' \in \partial i$, if they are adjacent with respect to this graph. Typically, regions are neighbours if and only if they are spatially contiguous.
In this talk, we concentrate on an allocation model inspired by image processing applications, the Potts model. Note that, contrary to the hierarchical mixture model defined in Richardson and Green (1997), this formulation does not use weights, but alternative formulations which define the allocations indirectly through spatially related weights are also being explored; see for example, the talk at this conference by Carmen Fernández.
In the Potts model formulation, the z_i are modelled directly:

$$p(z) = \frac{e^{\psi U(z)}}{c_k(\psi)}$$

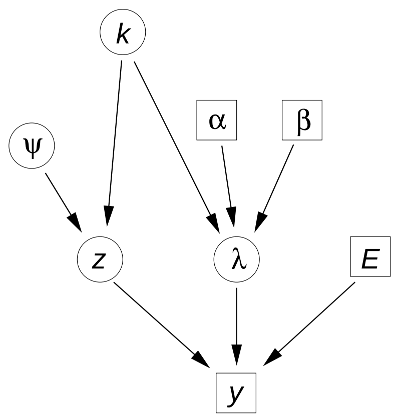

Figure 1: DAG for the Potts spatial mixture model.

where

$$U(z) = \sum_{i \sim i'} I[z_i = z_{i'}] \quad \text{and} \quad c_k(\psi) = \sum_{z \in \{1,2,\ldots,k\}^n} e^{\psi U(z)}$$

are the *number of like-coloured neighbour pairs*, and a normalising constant. The interaction parameter ψ is given a discrete uniform prior on a grid of values $\{\psi_0, \psi_1, \ldots, \psi_m\}$ (mainly for computational convenience).

All the above specifications are expressed somewhat loosely, in the interests of economy of notation. Each of the model ingredients is of course really a conditional distribution for the stated variable, conditional on both its immediate parameters, and hyperparameters higher up in the directed acyclic graph (DAG) describing the model; see Figure 1. Thus the joint distribution of all variables corresponding to the Potts model formulation is

$$p(k|\kappa)p(\psi)p(z|k,\psi)p(\lambda|k,\alpha,\beta)p(y|\lambda,z).$$

Some results The performance of the algorithm and the model will be investigated both on simulated data sets and on data concerning cancer mortality in France for rare sites (Rezvani *et al.*, 1997) for the period 1986–1993, at the level of the 95 French départements; for some preliminary results, see Figure 2.

References

Besag, J., York, J. and Mollié, A. (1991) Bayesian image restoration with applications in spatial statistics (with discussion). *Annals of the Institute of Mathematical Statistics*, **43**, 1–59.

Clayton, D. and Bernardinelli, L. (1992) Bayesian methods for mapping disease risk. In: *Geographical and environment epidemiology : methods for small area studies.* eds Elliott, P., Cuzick, J., English, D. and Stern, R. Oxford University Press, Oxford, 205–220.

Richardson, S. and Green, P. J. (1997) On Bayesian analysis of mixtures with an unknown number of components (with discussion) *Journal of the Royal Statistical Society*, B, **59**, 731–792.

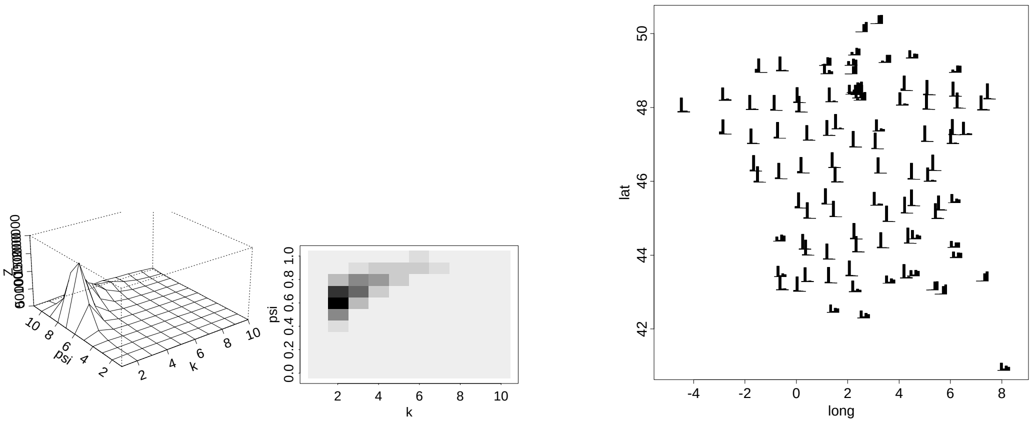

Figure 2: Posterior distributions for larynx cancer data: left, (k, ψ); right, λ_i plotted by spatial location of region i

Rezvani, A., Mollié, A., Doyon, F. and Sancho-Garnier, H. (1997) *Atlas de la mortalité par cancer en France, période 1986-1993.* Editions INSERM.

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