SEMICLASSICAL WAVE PROPAGATION FOR LARGE TIMES

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ABSTRACT. We study solutions of the time dependent Schrödinger equation on Riemannian manifolds with oscillatory initial conditions given by Lagrangian states. Semiclassical approximations describe these solutions for $\hbar \to 0$, but their accuracy for $t \to \infty$ is in general only understood up to the Ehrenfest time $T \sim \ln 1/\hbar$, and the most difficult case is the one where the underlying classical system is chaotic. We show that on surfaces of constant negative curvature semiclassical approximations remain accurate for times at least up to $1/\sqrt{\hbar}$ in the case that the Lagrangian state is associated with an unstable manifold of the geodesic flow.

1. INTRODUCTION

One of the main open problems in asymptotic analysis of linear wave equations is to understand the accuracy of semiclassical approximations for large times.

Let (M, g) be a Riemannian manifold and let Δ be the Laplace Beltrami operator on $L^2(M)$. In this paper we want to study the long time behaviour of solutions of the Schrödinger equation

$$\mathrm{i}\hbar\partial_t u = -\frac{\hbar^2}{2}\Delta u \tag{1.1}$$

with oscillatory initial conditions of the form

$$u_0(x) = a(x)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi(x)} \ , \tag{1.2}$$

where a is smooth, φ is a smooth real valued function and $\hbar > 0$ is a small parameter. For small $\hbar$ the solutions of (1.1) can be approximated by a geometric optics like constructions which involves the geodesic flow $G^t : T^*M \to T^*M$. By classical results such approximations work well if one restricts the time to a fixed interval $t \in [0, T]$. But the joint limit $\hbar \to 0$, $t \to \infty$ is much less well understood, in particular in the case we will be interested in, namely if the geodesic flow is hyperbolic, the accuracy of the approximations are currently only under control if

$$T \leq \frac{C}{\lambda} \ln \frac{1}{\hbar} \ , \tag{1.3}$$

where λ is a Lyapunov exponent associated with the geodesic flow and C is a constant. This time scale is called the Ehrenfest time. Rigorous results on propagation of coherent states up to this time have been recently derived in a series of papers, [CR97, HJ99, HJ00],

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and analogous results on Egorov's theorem in [BGP99, BR02]. Such results are interesting and useful because the dynamical properties of the geodesic flow become apparent only for large times, and one can use these results to relate the qualitative behaviour of wave propagation for large times and high frequencies to ergodic properties of the geodesic flow. For instance if the geodesic flow is Anosov, it is rapidly mixing and this implies that propagated waves of the form (1.2) become equidistributed for large times, under certain conditions on φ, see [Sch05]. This is in line with the conjecture that for classically chaotic systems propagated waves should for large times behave universally in the semiclassical limit like random superpositions of elementary plane waves, [BB79]. Estimates on time evolution on the scale of the Ehrenfest time have as well recently been used to obtain strong estimates on the distribution of eigenfunctions, namely on the entropy of limit measures obtained from sequences of eigenfunctions, [Ana07, AN06].

But it would be very desirable to understand the behaviour beyond the Ehrenfest time. Propagation of waves is a very abundant physical phenomenon and can be observed and measured easily in many different situations. The Ehrenfest time is rather short, and one would like to be able to use semiclassical approximations for much larger times. In addition to practical applications a better understanding of long time propagation could as well help to approach many open problems about the semiclassical behaviour of eigenfunctions and eigenvalues, e.g., questions like the rate of quantum ergodicity or quantum unique ergodicity.

The accuracy of semiclassical approximations in time evolution has been carefully studied numerically in [TH91, TH93] for the stadium billiard, and they found no breakdown at the Ehrenfest time. In addition they argued that semiclassical approximations should stay accurate up to a time scale of order $1/\hbar$, if the classical system has no singularities. The main problem one faces in the study of semiclassical approximation for chaotic systems is exponential proliferation. The approximations turn out to be a sum of oscillating terms whose number grows exponentially with time and so they are not absolutely convergent, and the error terms one obtains are of the same nature.

The aim in this work is to show that indeed semiclassical approximations can be valid for times far beyond the Ehrenfest time. We do this by developing techniques to control the size of the error term despite the exponential proliferation. This is only the first step to understand semiclassical approximations for large time in more detail, because the main term in the approximation is as well a sum of an exponentially growing number of terms, whose behaviour is not easy to understand.

The system we will study is the Schrödinger equation on a surface M of constant negative curvature. Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk, equipped with the usual metric defined by the line element $\mathrm{d}s^2 = 4\frac{\mathrm{d}x^2+\mathrm{d}y^2}{(1-x^2-y^2)^2}$ and

$$\Delta = \frac{(1-x^2-y^2)^2}{4}\left(\partial_y^2 + \partial_x^2\right) , \tag{1.4}$$

the Laplace Beltrami operator on $\mathbb{D}$. M can be represented as the quotient of $\mathbb{D}$ by a group of isometries Γ,

$$M = \mathbb{D}/\Gamma \ , \tag{1.5}$$

and we will assume that Γ is a Fuchsian group, i.e., acts discontinously on $\mathbb{D}$, this is equivalent to requiring that Γ is a discrete subgroup of $PSU(1,1)$. The most interesting case is the one where Γ is a Fuchsian group of first kind, i.e., M is of finite volume, or even compact. Functions on M can be identified with functions on $\mathbb{D}$ which are invariant under the action of Γ. We can use summation over Γ to build functions on M from functions on $\mathbb{D}$: given a function $u : \mathbb{D} \to \mathbb{C}$ we set

$$u_\Gamma := \sum_{\gamma\in\Gamma} u \circ \gamma^{-1} \ , \tag{1.6}$$

which is a function on M, provided the sum converges. Since Δ commutes with the action of isometries on $\mathbb{D}$, the time evolution operator $\mathcal{U}(t) = \mathrm{e}^{\mathrm{i}\hbar t\Delta/2}$, which is the solution to the Schrödinger equation (1.1) with initial condition $\mathcal{U}(0) = I$, commutes with the action of Γ and we have

$$\mathcal{U}(t)u_\Gamma = \sum_{\gamma\in\Gamma} (\mathcal{U}(t)u) \circ \gamma^{-1} \ . \tag{1.7}$$

Our strategy will be to construct first semiclassical approximations on $\mathbb{D}$ and then use this relation to transfer them to M.

Let us recall the definition of plane waves associated with horocycles. Let $b \in \partial\mathbb{D}$ be a point on the boundary of $\mathbb{D}$, a horocycle associated with b is (euclidean) circle in $\mathbb{D}$ tangent to $\partial\mathbb{D}$ at b, given a point $z \in \mathbb{D}$ we denote by $\xi(z, b)$ the unique horocycle associated with b which passes through z. Furthermore let $\eta(z, b)$ be the geodesic emanating from b and passing through z, see Figure 1 for illustration. Let us write $\xi(z, b) \geq \xi(z', b)$ if $\xi(z', b)$ lies inside $\xi(z, b)$, then given a $b \in \partial\mathbb{D}$ we can define

$$\varphi_b(z) := \begin{cases} d(\xi(z, b), \xi(0, b)) & \text{if} \quad \xi(z, b) \geq \xi(0, b) \\ -d(\xi(z, b), \xi(0, b)) & \text{otherwise} \end{cases} \ , \tag{1.8}$$

where $d(\xi(z, b), \xi(z', b))$ denotes the hyperbolic distance between the two horocycles $\xi(z, b)$ and $\xi(z', b)$. These function are used by Helgason to define a set of plane waves on $\mathbb{D}$ and develop harmonic analysis, [Hel81].

The initial states on $\mathbb{D}$ we will consider are of the form

$$u = a\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_b} \tag{1.9}$$

with a smooth amplitude a. Such functions are known as Lagrangian states, where the Lagrangian submanifold of $T^*\mathbb{D}$ associated with them is the graph of $d\varphi_b$,

$$\Lambda_b := \{(z, d\varphi_b(z))\,;\, z \in \mathbb{D}\} \subset T^*\mathbb{D} \ , \tag{1.10}$$

this manifold is an unstable manifold of the geodesic flow. Let us denote by $G^t : T^*\mathbb{D} \to T^*\mathbb{D}$ the geodesic flow over $\mathbb{D}$ and by $\pi_b : \Lambda_b \to \mathbb{D}$ the restriction of the canonical projection

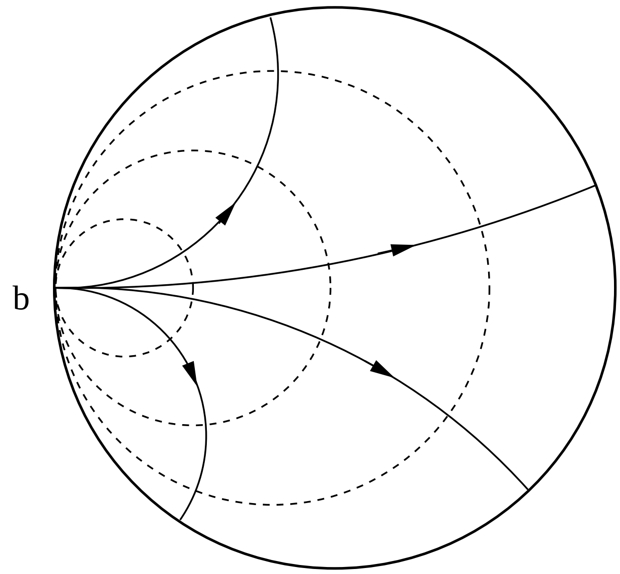

FIGURE 1. The unit disk $\mathbb{D}$ with examples of horocycles and geodesics associated with $b \in \partial\mathbb{D}$. The dashed circles tangent to $\partial\mathbb{D}$ at b are horocycles and the geodesics emanating from b are solid lines. The horocycles are the wavefronts associated with the phase-function φ_b, (1.8), and semiclassical wave propagation is described by transport along the geodesic spray emanating from b (which are the projections of trajectories on the unstable manifold associated with b).

$\pi : T^*\mathbb{D} \to \mathbb{D}$ to Λ_b, then we can define an induced flow on $\mathbb{D}$ by

$$\Phi_b^t = \pi_b G^t \pi_b^{-1} \ , \tag{1.11}$$

which we can then use to define a one parameter family of operators

$$S_b(t) : L^2(\mathbb{D}) \to L^2(\mathbb{D}) \tag{1.12}$$

$$a \mapsto \mathrm{e}^{-\frac{t}{2}} a \circ \Phi_b^{-t} \ . \tag{1.13}$$

We will show in Lemma 2.1 in the the next section that these operators actually form a unitary group. They are defined purely in terms of the geodesic flow, i.e., the classical dynamical system associated with the Schrödinger equation, and they will give the leading semiclassical approximation to the quantum propagation of an initial state of the form $a\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_b}$.

Let us see how $S_b(t)$ is related to the classical picture of the geometric optics approximation to wave propagation at short wavelength, see, e.g., [Tay96] for background. To an initial function $a\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}$ one associates the wavefronts which are the level sets of the phase function φ, the propagated state at time t is then of the same form $a_t\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_t}$ (provided there are no caustics), where the wavefronts of the new phase function φ_t are obtained by transporting the initial wavefronts along the geodesics perpendicular to them a time t. This translates via the method of characteristics into a first order equation for φ_t, the Hamilton Jacobi or eikonal equation, and in addition the new amplitude a_t is obtained by transporting the initial one along the same set of geodesics and multiplying it with a factor related to the expansion rate of the geodesics. Now in our case the wavefronts of φ_b are the horocycles associated with b and these are mapped onto themselves by transport along perpendicular geodesics, so φ_b stays invariant (up to a simple time dependent constant), and only a is transported, which is exactly described by the action of $S_b(t)$.

So our first order semiclassical approximation to $\mathcal{U}(t)\big(a\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_b}\big)$ will be

$$\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{t}{2}}\big(S_b(t)a\big)\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_b}\ , \tag{1.14}$$

and to show that this is a good approximation even when we project it onto M by summing over Γ we have to place some conditions on the amplitude a.

Definition 1.1. *Set $\langle d\rangle := (1+d(0,z)^2)^{1/2}-1$ and let $\alpha=\alpha(\hbar)\geq 0$ and $\beta=\beta(\hbar)\geq 0$ be functions of $\hbar$. Then we define the norm*

$$\|a\|_{\alpha,\beta} := \|\mathrm{e}^{\beta\langle d\rangle}\mathrm{e}^{\alpha\sqrt{-\Delta}}a\|_{L^2(\mathbb{D})} \tag{1.15}$$

and set $H_{\alpha,\beta}(\mathbb{D}) := \{a:(0,1]\times\mathbb{D}\to\mathbb{C}\ :\ \|a(\hbar)\|_{\alpha,\beta}<\infty\}$.

We will usually omit the $\hbar$-dependence from the notation. If $\alpha>0$ then the functions in $H_{\alpha,\beta}(\mathbb{D})$ are analytic, and the factor $\mathrm{e}^{\beta\langle d\rangle}$ makes them exponentially decaying for $\langle d\rangle(z)\to\infty$, i.e., by simple Sobolev imbedding we have for $a\in H_{\alpha,\beta}(\mathbb{D})$

$$|a(z)|\leq C_\alpha \mathrm{e}^{-\beta\langle d\rangle(z)}\ , \tag{1.16}$$

see Lemma 3.11. This exponential decay ensures that the sum over Γ converges, more precisely:

Lemma 1.2. *For $\alpha>0,\beta>1/2$ there is a constant $C_{\alpha,\beta}$ such that*

$$\|a_\Gamma\|_{L^2(M)}\leq C_{\alpha,\beta}\|a\|_{\alpha,\beta}\ . \tag{1.17}$$

A proof of this lemma with the explicit dependence of $C_{\alpha,\beta}$ on α,β will be given in Section 3, see Proposition 3.10. For our applications we are mainly interested in the exponential decay for $\langle d\rangle(z)\to\infty$ of the functions in $H_{\alpha,\beta}(\mathbb{D})$, the analyticity will be necessary to obtain dispersive estimates in Section 4 which show that these exponential decay properties are preserved under the action of certain operators.

We can now state a special version of our main result.

Theorem 1.3. *Let $M=\mathbb{D}/\Gamma$, where Γ is a Fuchsian group, then for all constant $\alpha>0,\beta>1$ there exist constants $C>0$, $\delta>0$ such that for all $a\in H_{\alpha,\beta}(\mathbb{D})$ and $b\in\partial\mathbb{D}$,*

$$\big\|u^{(0)}(t)_\Gamma-\mathcal{U}(t)[a\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_b}]_\Gamma\big\|_{L^2(M)}\leq C\|a\|_{\alpha,\beta}\hbar \tag{1.18}$$

for

$$0\leq t\leq \delta\frac{1}{\sqrt{\hbar}}\ , \tag{1.19}$$

where

$$u^{(0)}(t) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{t}{2}}\big(S_b(t)a\big)\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_b}\ . \tag{1.20}$$

So the semiclassical approximation is accurate at least up to times of order $1/\sqrt{\hbar}$. We will develop below as well higher order approximations which improve the error term in (1.18), but are valid on the same time range. We will as well make the dependence on α explicit which will allow for $\hbar$ dependent α and a. But before we do so let us outline the main ideas behind the proof of Theorem 1.3.

Let $\Delta_b(t)$ be defined by

$$\Delta_b(t) := S_b(t)^*\Delta S_b(t)\ , \tag{1.21}$$

this operator is self-adjoint so we can define the unitary operator $V_b(t)$ as the solution of

$$\mathrm{i}\partial_t V_b(t) = -\frac{\hbar}{2}\Delta_b(t)V_b(t) \tag{1.22}$$

with initial condition $V_b(0) = I$. Then we will show in Section 2 that

$$\mathcal{U}(t)\big(a\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_b}\big) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{t}{2}}\big(S_b(t)V_b(t)a\big)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_b}\ . \tag{1.23}$$

This relation is the main tool of our analysis, the propagation of a state on $\mathbb{D}$ is expressed by the action of the two operators $S_b(t)$ and $V_b(t)$ on the amplitude a. The first one, $S_b(t)$, induces propagation of the state along geodesics associated with b. The second operator $V_b(t)$ describes dispersion, which takes place on a scale of order $\hbar t$, and this is responsible for the error term in (1.18).

Using the unitarity of $V_b(t)$ and (1.23) we can rewrite the leading semiclassical approximation (1.20) as

$$\begin{aligned} u^{(0)}(t) &= \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{t}{2}}\big(S_b(t)V_b(t)V_b^*(t)a\big)\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_b} \\ &= \mathcal{U}(t)\big(V_b^*(t)a\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_b}\big) \end{aligned} \tag{1.24}$$

and so $u^{(0)}(t) - \mathcal{U}(t)\big(a\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_b}\big) = \mathcal{U}(t)\big([V_b^*(t)a - a]\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_b}\big)$. Since $\mathcal{U}(t)$ commutes with the action of Γ and is unitary we then find

$$\|u^{(0)}(t)_\Gamma - \mathcal{U}(t)[a\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_b}]_\Gamma\|_{L^2(M)} = \|\big([V_b^*(t)a - a]\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_b}\big)_\Gamma\|_{L^2(M)} \tag{1.25}$$

and now the right hand side contains only the dispersive part $V_b(t)$. Then from integrating (1.22) we get $V_b(t) = 1 + \mathrm{i}\hbar/2\int_0^t \Delta_b(t')V_b(t')\,\mathrm{d}t'$ and so

$$V_b^*(t)a - a = \frac{\mathrm{i}\hbar}{2}\int_0^t V_b^*(t)\Delta_b(t')V_b(t')a\,\mathrm{d}t' \tag{1.26}$$

which is of order $\hbar t$. So using the unitarity of $\mathcal{U}(t)$ we got rid of the propagating part $S_b(t)$ which would have lead to exponential proliferation in the sum over Γ and are left with the dispersive part which is easier to control. What we need now to conclude the proof is to show that $V_b^*(t)\Delta_b(t')V_b(t')a$ decays sufficiently fast for $\langle d\rangle(z) \to \infty$ so that its sum over Γ is bounded. To this end we have introduced the spaces $H_{\alpha,\beta}$, we will show in Section 4 that for times up to (1.19) we can control the action of $V_b(t)$ on these spaces well enough to ensure the necessary convergence of the sum. But as we will discuss further in Section 4 it is likely that our estimates are not optimal and could be extended to time scales up to $O(1/\hbar)$. This would then imply correspondingly larger times in Theorem 1.3 above and Theorem 1.4 below.

We will describe now higher order semiclassical approximations and more refined estimates. To this end let

$$P_k := \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} \Delta_b(t_1)\Delta_b(t_2)\cdots\Delta_b(t_k)\,\mathrm{d}t_k\cdots\mathrm{d}t_1 \tag{1.27}$$

for $k \geq 1$ and $P_0 = 1$, and for $a \in H_{\alpha,\beta}(\mathbb{D})$ set

$$a^{(K)} = \sum_{k=0}^{K} \left(\frac{\mathrm{i}\hbar}{2}\right)^k P_k a \tag{1.28}$$

and

$$u^{(K)}(t) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{t}{2}} S_b(t) a^{(K)}(t) \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi} \ . \tag{1.29}$$

Then our main result is

Theorem 1.4. *Let $M = \mathbb{D}/\Gamma$, where Γ is a Fuchsian group, then for all α, β, with $\beta > 1/2$ and $\alpha^3 \geq \beta^2\hbar$ there exist constants $C > 0$, $\delta > 0$ such that for all $a \in H_{\alpha,\beta}(\mathbb{D})$ and $b \in \partial\mathbb{D}$, and any $K, N \in \mathbb{N}_0$,*

$$\big\|\Delta^N \big[\mathcal{U}(t)[a\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_b}]_\Gamma - u^{(K)}(t)_\Gamma\big]\big\|_{L^2(M)} \leq C^{N+K+1} \frac{N^{2N} K!}{(\alpha\hbar)^{2N+4}} \left(\frac{\hbar t}{\alpha^2}\right)^K \|a\|_{\alpha,\beta} \ . \tag{1.30}$$

for

$$0 \leq t \leq \delta \frac{\alpha^{3/2}}{\hbar^{1/2}} \ , \tag{1.31}$$

where $u^{(K)}(t)$ is given by (1.29).

By choosing K optimally one can obtain an exponentially small remainder term.

Corollary 1.5. *Assume the same conditions as in Theorem 1.4 are satisfied, then for any $\varepsilon > 0$ we have for $K \in \big[C\alpha^2/\hbar t - \varepsilon, C\alpha^2/\hbar t - \varepsilon/2\big]$*

$$\big\|\Delta^N \big[\mathcal{U}(t)[a\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_b}]_\Gamma - u^{(K)}(t)_\Gamma\big]\big\|_{L^2(M)} \leq \frac{1}{\varepsilon^{1/2}} C^{N+1} \frac{N^{2N}}{(\alpha\hbar)^{2N+4}} \mathrm{e}^{-\frac{1-\varepsilon}{C}\frac{\alpha^2}{\hbar t}} \|a\|_{\alpha,\beta} \ , \tag{1.32}$$

where C is the same constant as in (1.30) *and t has to satisfy*

$$0 \leq t \leq \delta \frac{\alpha^{3/2}}{\hbar^{1/2}} \ . \tag{1.33}$$

The reason for the introduction of Δ^N is that it allows us to use Sobolev imbedding to pass to point-wise estimates.

Corollary 1.6. *Assume the same conditions as in Theorem 1.4 are satisfied, then*

$$\big|\Delta^N\big[\mathcal{U}(t)[a\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_b}]_\Gamma - u^{(K)}(t)_\Gamma\big]\big| \leq C^{N+K+1}\frac{1}{(\alpha\hbar)^{2N+8}}(\hbar t)^K(2N)!K!\,\|a\|_{\alpha,\beta}\ , \tag{1.34}$$

for

$$0\leq t\leq \delta\frac{\alpha^{3/2}}{\hbar^{1/2}}\ . \tag{1.35}$$

Furthermore if $K\in\big[C\alpha^2/\hbar t-\varepsilon, C\alpha^2/\hbar t-\varepsilon/2\big]$ *for* $\varepsilon>0$ *then*

$$\big|\Delta^N\big[\mathcal{U}(t)[a\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_b}]_\Gamma - \psi^{(K)}(t)_\Gamma\big]\big| \leq \frac{1}{\varepsilon^{1/2}}C^{N+1}\frac{N^{2N}}{(\alpha\hbar)^{2N+8}}\mathrm{e}^{-\frac{1-\varepsilon}{C}\frac{\alpha^2}{\hbar t}}\,\|a\|_{\alpha,\beta}, \tag{1.36}$$

for

$$0\leq t\leq \delta\frac{\alpha^{3/2}}{\hbar^{1/2}}\ . \tag{1.37}$$

So the semiclassical approximations are even point-wise close to the true evolved states. Let us make a couple of remarks about these results.

Remark 1.7. The α dependence: If α and β are constant, then we have a time range up to $t\ll 1/\sqrt{\hbar}$. But we can let α depend as well on $\hbar$, this allows to use amplitude functions a which depend on $\hbar$ and become, e.g., localised for $\hbar\to 0$. An example would be

$$\frac{1}{\hbar^{\delta/2}}\mathrm{e}^{-\frac{1}{\hbar^\delta}\langle d\rangle}\ , \tag{1.38}$$

for $\delta\geq 0$. For $\hbar\leq 1$ this function is in $H_{\alpha,\beta}$ with $\beta=1$ and $\alpha=\hbar^\delta$ and so the semiclassical approximations work at least up to $t\ll 1/\hbar^{(1-3\delta)/2}$. This means that we have to have $\delta<1/3$ to be able to reach large times. One can improve this by refining the semiclassical approximations and write $V_b(t)$ when applied to a function localised at $z_0\in\mathbb{D}$ as a product of a metaplectic operator times another unitary operator. This allows to treat coherent states for which $\alpha=\sqrt{\hbar}$, and we hope to discuss this in more detail in the future.

Remark 1.8. One can as well allow larger spaces than $H_{\alpha,\beta}$, in particular Gevrey type spaces defined by the norm $\|a\|^{(\delta)}_{\alpha,\beta}:=\|\mathrm{e}^{\beta\langle d\rangle}\mathrm{e}^{\alpha\sqrt{-\Delta}^{1+\delta}}a\|_{L^2}$ could be useful, because they contain functions of compact support. For these spaces with constant α and β we would expect that with the mollification introduced in Section 4 to be able to control semiclassical approximations up to $t\ll\hbar^{(1-\delta)/2}$.

Remark 1.9. Our semiclassical approximations are of the form

$$\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\frac{t}{2}}\sum_\Gamma (S_b(t)a)\circ\gamma^{-1}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_b\circ\gamma^{-1}} \tag{1.39}$$

for some $a\in H_{\alpha,\beta}$ and the action of $S_b(t)$ increases the effective support of a at an exponential rate in t, so if Γ is a Fuchsian group of the first kind then one can show that, even if the sum is absolutely convergent for $t = 0$, we still have

$$\sum_{\Gamma} \left|(S_b(t)a) \circ \gamma^{-1}\right| \gg C_a \mathrm{e}^{t/2} \ . \tag{1.40}$$

So the phase factors $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_b \circ \gamma^{-1}}$ are absolutly crucial to ensure uniformly bounded L^2-norms for large t.

Remark 1.10. As we mentioned already in the discussion after Theorem 1.3 our methods can possibly be improved to extend the time range from $1/\sqrt{\hbar}$ to $1/\hbar$. In order to do so we would need some stronger estimates on the action of the operator $V_b(t)$.

Remark 1.11. In order to keep the presentation as simple as possible we have restricted ourselves here to two-dimensional manifolds of constant negative curvature, but it should be possible to generalise the results. The generalisation to higher dimensional manifolds of constant curvature should be straightforward and we expect the same results to hold, in particular the time scales our methods give do not depend on the dimension. A natural general time scale in semiclassical problems is the Heisenberg time $T_H \sim 1/\hbar^{\dim M - 1}$ which is related to the mean spacing of the eigenvalues, it is the time scale on which the system starts to resolve individual eigenvalues. We see that the optimal time range we can hope to reach with our methods coincides in two-dimensions with the Heisenberg time but is shorter in higher dimensions. It is not clear if this is an artefact of the method, or some change of behaviour can happen at that time.

Since our constructions are mainly of a geometric nature paired with some general analytic estimates on the action of pseudodifferential operators, one should be able to generalise them to Riemannian manifolds of non-constant negative curvature. The phase functions φ_b are Busemann functions and the operators $S_b(t)$ and $V_b(t)$ together with the decomposition $\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\varphi_b}\mathcal{U}(t)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_b} = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{t}{2}} S_b(t)V_b(t)$ can be constructed in exactly the same way. But some of the ensuing estimates become more complicated since the operator $\Delta_b(t) = S_b^*(t)\Delta S_b(t)$ can have coefficients which become highly oscillatory, although with a very small amplitude.

Remark 1.12. Similar results should hold for other hyperbolic problems, e.g., the standard wave equation and the Dirac equation, with oscillatory initial conditions. The methods developed here can probably be generalised to such cases.

The plan of the paper is as follows. In Section 2 we discuss time evolution on the universal cover and prove the decomposition (1.23). In Section 3 we study the action of differential and pseudodifferential operators on the spaces $H_{\alpha,\beta}$ and show how they can be used together with Sobolev imbeddings to get precise estimates on functions a_Γ on the quotient in terms of a. We then proceed in Section 4 to discuss the crucial properties of the action of V_b on $H_{\alpha,\beta}$, and in Section 5 we finally use the material collected in the previous sections to prove our main Theorems and some related results. Some auxiliary material on pseudodifferential operators on $\mathbb{D}$ has been collected in the Appendix.

Note on notation: We will denote by C a generic constant which can change from line to line. We write as well sometimes $a \ll b$ if there is a constant $C > 0$ such that $a \leq Cb$.

2. Time evolution on the universal cover

2.1. **Coordinates adapted to φ_b.** It will be useful to choose special coordinates adapted to the phase-function φ_b. Since any rotation around the origin is an isometry on $\mathbb{D}$, there is an isometry γ_b such that $\varphi_b \circ \gamma_b = \varphi_{-\pi}$, where $b = -\pi$ is the point on $\partial\mathbb{D}$ at $z = -1$. Composing γ_b with the standard mapping $\mathbb{D} \to \mathbb{H}$ from the unit disk model to the upper half plane $\mathbb{H} = \{x + \mathrm{i}y \in \mathbb{C}\,; y > 0\}$, the geodesics emanating from b are mapped to straight lines parallel to the y-axis and the corresponding horocycles are horizontal lines. The phase function φ_b takes in these coordinates the simple form

$$\varphi_b(x, y) = -\ln y \tag{2.1}$$

and in order to keep the notation light we will from now on fix the point $b \in \partial\mathbb{D}$ and drop the reference to it from the notation.

We recall as well the expressions for the metric $\mathrm{d}s^2 = \frac{\mathrm{d}x^2+\mathrm{d}y^2}{y^2}$, the Laplacian

$$\Delta = y^2(\partial_x^2 + \partial_y^2) \tag{2.2}$$

and the volume element

$$\mathrm{d}\nu = \frac{1}{y^2}\mathrm{d}y\mathrm{d}x \tag{2.3}$$

in these coordinates.

2.2. **Time evolution.** The geodesics emanating from b are given in the adapted coordinates by $\eta_x = \{x + \mathrm{i}y\,; y \in \mathbb{R}^2\}$ and the flow on $\mathbb{H}$ induced by shifting with constant speed along these geodesics can be easily seen to be

$$\Phi^t(x, y) = x + \mathrm{i}\mathrm{e}^{-t}y\ . \tag{2.4}$$

Therefore the action of the operator $S(t)$ defined in (1.12) is given in these coordinates by

$$\big(S(t)a\big)(x, y) = \mathrm{e}^{-t/2}a(x, \mathrm{e}^t y)\ . \tag{2.5}$$

Lemma 2.1. *The operator $S(t) : L^2(\mathbb{H}) \to L^2(\mathbb{H})$ is unitary and*

$$\big(S^*(t)a\big)(x, y) = \mathrm{e}^{t/2}a(x, \mathrm{e}^{-t}y)\ . \tag{2.6}$$

Furthermore

$$\mathrm{i}\partial_t S(t)a = \mathrm{i}\left[y\partial_y - \frac{1}{2}\right]S(t)a\ . \tag{2.7}$$

i.e., the generator of $S(t)$ is $Y = \mathrm{i}\big[y\partial_y - \frac{1}{2}\big]$.

Proof. The unitarity follows using a simple change of coordinates

$$\begin{aligned}\langle S(t)a, S(t)b\rangle &= \int_{\mathbb{H}} \mathrm{e}^{-t/2}a^*(x, \mathrm{e}^t y)\mathrm{e}^{-t/2}b(x, \mathrm{e}^t y)\frac{1}{y^2}\mathrm{d}y\mathrm{d}x \\ &= \int_{\mathbb{H}} a^*(x, y)b(x, y)\frac{1}{y^2}\mathrm{d}y\mathrm{d}x \\ &= \langle a, b\rangle\ ,\end{aligned} \tag{2.8}$$

and (2.7) is a straightforward computation. □

Using (2.5) we find that the generator $\Delta(t) := S^*(t)\Delta S(t)$ of the unitary operator $V(t)$, see (1.22), has as well a simple explicit expression in the adapted coordinates

$$\Delta(t) = y^2\big(\partial_y^2 + \mathrm{e}^{-2t}\partial_x^2\big)\ . \tag{2.9}$$

Proposition 2.2. *Let* $\mathcal{U}(t) = \mathrm{e}^{\mathrm{i}\frac{\hbar t}{2}\Delta}$ *and* $\varphi(x,y) = -\ln y$ *then we have the identity*

$$\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\varphi}\mathcal{U}(t)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi} = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{t}{2}}S(t)V(t) \tag{2.10}$$

as operators on $L^2(\mathbb{H})$.

Proof. Since $\mathcal{U}(t)$ is a solution of $\mathrm{i}\hbar\partial_t\mathcal{U}(t) = -\frac{\hbar^2}{2}\Delta\mathcal{U}(t)$, $\mathcal{U}_\varphi(t) := \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\varphi}\mathcal{U}(t)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}$ satisfies

$$\mathrm{i}\hbar\partial_t\mathcal{U}_\varphi(t) = -\frac{\hbar^2}{2}\Delta_\varphi\mathcal{U}_\varphi(t) \tag{2.11}$$

with the initial condition $\mathcal{U}_\varphi(0) = I$ and where $\Delta_\varphi = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\varphi}\Delta\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}$. Now a short calculation gives

$$-\frac{\hbar^2}{2}\Delta_\varphi = \frac{1}{2} + \hbar Y - \frac{\hbar^2}{2}\Delta \tag{2.12}$$

where Y is the generator of $S(t)$ from Lemma 2.1. On the other hand we have

$$\begin{aligned}\mathrm{i}\hbar\partial_t\Big[\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{t}{2}}S(t)V(t)\Big] &= \left[\frac{1}{2} + \hbar Y\right]\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{t}{2}}S(t)V(t) - \frac{\hbar^2}{2}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{t}{2}}S(t)\Delta(t)V(t)\\ &= \left[\frac{1}{2} + \hbar Y - \frac{\hbar^2}{2}S(t)\Delta(t)S^*(t)\right]\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{t}{2}}S(t)V(t)\end{aligned} \tag{2.13}$$

and since $S(t)\Delta(t)S^*(t) = \Delta$ we find that $\mathcal{U}_\varphi(t)$ and $\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{t}{2}}S(t)V(t)$ satisfy the same first order differential equation with the same initial condition, so they coincide. □

Thus we have separated the action of $\mathcal{U}(t)$ on oscillatory states $a\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}$ into two parts. The part described by $S_b(t)$ is the propagation which is induced by the classical dynamics, note that $S_b(t)$ does not depend on $\hbar$. The second part, coming from $V_b(t)$, is responsible for dispersion which takes place on a scale of order $\hbar t$ as we will see in Section 4.

3. The spaces $H_{\alpha,\beta}$ and estimates on a quotient

In this section we will discuss how to use the spaces $H_{\alpha,\beta}$ to obtain precise estimates when passing from $\mathbb{D}$ to a quotient $M = \mathbb{D}/\Gamma$.

3.1. **Sobolev embedding and passing to the quotient.** Recall that M is the quotient of $\mathbb{D}$ by the fundamental group Γ, $M = \mathbb{D}/\Gamma$. Given a function u on $\mathbb{D}$ we defined a function $u_\Gamma := \sum_{\gamma\in\Gamma} u\circ\gamma^{-1}$ on M, provided that the sum converges. We will now discuss some conditions on u which ensure convergence of u_Γ. These are based on Sobolev imbeddings combined with the following simple estimate for the L^1 norm:

Lemma 3.1. *Let $u \in L^1(\mathbb{D})$, then $u_\Gamma \in L^1(M)$ and*

$$\|u_\Gamma\|_{L^1(M)} \leq \|u\|_{L^1(\mathbb{D})} \ . \tag{3.1}$$

Proof. Let $F \subset \mathbb{D}$ be a fundamental domain for M, then

$$\begin{aligned} \|u_\Gamma\|_{L^1(M)} &= \int_F \bigg| \sum_{\gamma\in\Gamma} u \circ \gamma^{-1} \bigg| \,\mathrm{d}\nu \\ &\leq \int_F \sum_{\gamma\in\Gamma} |u| \circ \gamma^{-1} \,\mathrm{d}\nu \\ &= \sum_{\gamma\in\Gamma} \int_{\gamma(F)} |u| \,\mathrm{d}\nu \\ &= \int_{\mathbb{D}} |u| \,\mathrm{d}\nu = \|u\|_{L^1(\mathbb{D})} \end{aligned} \tag{3.2}$$

since $\bigcup_\Gamma \overline{\gamma(F)} = \mathbb{D}$. □

Let us recall two of the standard Sobolev imbedding results. For every $s > 1$ there is a constant $C_s > 0$ such that

$$\|u_\Gamma\|_{L^2(M)} \leq C_s\big(\|\Delta^s u_\Gamma\|_{L^1(M)} + \|u_\Gamma\|_{L^1(M)}\big) \tag{3.3}$$

and for every $s > 1/2$ there is another constant $C_s' > 0$ such that

$$|u_\Gamma| \leq C_s'\big(\|\Delta^s u_\Gamma\|_{L^1(M)} + \|u_\Gamma\|_{L^1(M)}\big) \ . \tag{3.4}$$

Combining Lemma 3.1 with the Sobolev imbedding (3.3) gives

Proposition 3.2. *Assume that $u \in L^1(\mathbb{D})$ and $\Delta^2 u \in L^1(\mathbb{D})$, then $u_\Gamma \in L^2(M)$ and there is a constant $C > 0$ such that*

$$\|u_\Gamma\|_{L^2(M)} \leq C\big(\|\Delta^2 u\|_{L^1(\mathbb{D})} + \|u\|_{L^1(\mathbb{D})}\big) \tag{3.5}$$

To obtain an estimate on $(S(t)u)_\Gamma$ we use the following simple Lemma. Note that we continue to use the notation $S(t)$ instead of $S_b(t)$, since there is no b dependence in the estimates.

Lemma 3.3. *For $u \in L^1(\mathbb{D})$ we have*

$$\|S(t)u\|_{L^1(\mathbb{D})} = \|u\|_{L^1(\mathbb{D})}\mathrm{e}^{t/2} \ . \tag{3.6}$$

Proof. This follows from

$$\|S(t)u\|_{L^1(\mathbb{D})} = \langle S(t)|u|, 1\rangle = \langle |u|, S^*(t)1\rangle = \mathrm{e}^{\frac{t}{2}}\|u\|_{L^1(\mathbb{D})} \tag{3.7}$$

where we have used $S^*(t)1 = \mathrm{e}^{\frac{t}{2}}$. □

Corollary 3.4. *Assume that $u \in L^1(\mathbb{D})$ and $\Delta(t)^2 u \in L^1(\mathbb{D})$ for $t \geq 0$, then there is a constant $C > 0$ such that for $t \geq 0$*

$$\|(S(t)u)_\Gamma\|_{L^2(M)} \leq C\big(\|\Delta(t)^2 u\|_{L^1(\mathbb{D})} + \|u\|_{L^1(\mathbb{D})}\big)\mathrm{e}^{\frac{t}{2}} \tag{3.8}$$

Proof. By Proposition 3.2 we have to estimate $\|S(t)u\|_{L^1(\mathbb{D})}$ and $\|\Delta^2 S(t)u\|_{L^1(\mathbb{D})}$. But by Lemma 3.3 $\|S(t)u\|_{L^1(\mathbb{D})} = \|u\|_{L^1(\mathbb{D})}\mathrm{e}^{t/2}$ and

$$\|\Delta^2 S(t)u\|_{L^1(\mathbb{D})} = \|S(t)\Delta(t)^2 u\|_{L^1(\mathbb{D})} = \mathrm{e}^{t/2}\|\Delta(t)^2 u\|_{L^1(\mathbb{D})}\ . \tag{3.9}$$

□

3.2. **The action of pseudodifferential operators on $H_{\alpha,\beta}$.** The drawback of working with $L^1(\mathbb{D})$ on the universal cover is that the action of $V(t)$ on $L^1(\mathbb{D})$ is difficult to control, this is the reason that we introduced the spaces $H_{\alpha,\beta}(\mathbb{D})$. We now analyse the action of pseudodifferential operators on the spaces $H_{\alpha,\beta}$. The classes of pseudodifferential operators we use are a semiclassical version of the ones developed by Zelditch in [Zel86] based on Helgason's harmonic analysis on $\mathbb{D}$. The small semiclassical parameter will be denoted by $\varepsilon > 0$ and for reference we have collected the definitions and basic properties in Appendix 6.

Proposition 3.5. *Assume $A \in \Psi_\varepsilon^{m,k}(\mathbb{D})$, $m \geq 0$, has an analytic symbol, then there are $\alpha_0, \beta_0, \varepsilon_0 > 0$ and a constant $C_A > 0$ such that for all α, α', β with $\alpha_0 \geq \alpha > \alpha' \geq 0$ and $\beta_0 \geq \beta$, and for all $\varepsilon \in (0, \varepsilon_0]$,*

$$\|Au\|_{\alpha',\beta} \leq C_A \frac{1}{\varepsilon^{k-m}(\alpha-\alpha')^m}\|u\|_{\alpha,\beta} \tag{3.10}$$

for $u \in H_{\alpha,\beta}$.

Proof. We have $\|Au\|_{\alpha',\beta} = \|\mathrm{e}^{\alpha'\sqrt{-\Delta}}\mathrm{e}^{\beta\langle d\rangle}Au\|_{L^2}$ and we write

$$\mathrm{e}^{\alpha'\sqrt{-\Delta}}\mathrm{e}^{\beta\langle d\rangle}A = B\mathrm{e}^{\alpha\sqrt{-\Delta}}\mathrm{e}^{\beta\langle d\rangle} \tag{3.11}$$

with

$$B = \mathrm{e}^{\alpha'\sqrt{-\Delta}}\mathrm{e}^{\beta\langle d\rangle}A\mathrm{e}^{-\beta\langle d\rangle}\mathrm{e}^{-\alpha\sqrt{-\Delta}}\ , \tag{3.12}$$

so that $\|Au\|_{\alpha',\beta} = \|B\mathrm{e}^{\alpha\sqrt{-\Delta}}\mathrm{e}^{\beta\langle d\rangle}u\|_{L^2}$. Therefore we have to estimate the L^2 norm of B. We first observe that by Theorem 7.2

$$B_0 := \mathrm{e}^{\alpha'\sqrt{-\Delta}}\mathrm{e}^{\beta\langle d\rangle}A\mathrm{e}^{-\beta\langle d\rangle}\mathrm{e}^{-\alpha'\sqrt{-\Delta}} \in \Psi_\varepsilon^{m,k}(\mathbb{D}) \tag{3.13}$$

and since $B_0 = B\mathrm{e}^{(\alpha-\alpha')\sqrt{-\Delta}}$ we can write

$$B^*B = \mathrm{e}^{(\alpha-\alpha')\sqrt{-\Delta}}B_0^*B_0\mathrm{e}^{(\alpha-\alpha')\sqrt{-\Delta}} \leq C\frac{1}{\varepsilon^{2k}}(-\varepsilon^2\Delta+1)^m\mathrm{e}^{2(\alpha-\alpha')\sqrt{-\Delta}} \tag{3.14}$$

because $B_0^*B_0 \leq C\frac{1}{\varepsilon^{2k}}(-\varepsilon^2\Delta+1)^m$, since $\varepsilon^2\Delta$ is elliptic. Now the operator $(-\varepsilon^2\Delta + 1)^{m/2}\mathrm{e}^{(\alpha-\alpha')\sqrt{-\Delta}}$ has symbol $(\lambda^2+1)^{m/2}\mathrm{e}^{-(\alpha-\alpha')\lambda/\varepsilon}$ which can be bounded using the following auxiliary Lemma whose proof we leave to the reader:

Lemma 3.6. *Let $m, \delta \geq 0$ and set $f_{m,\delta}(\lambda) := (\lambda^2+1)^{m/2}\mathrm{e}^{-\delta\lambda}$, then for every $\delta_0 > 0$ there is a constant $C > 0$ such that*

$$|f_{m,\delta}(\lambda)| \leq C\frac{m!}{\delta^m} \tag{3.15}$$

for $\delta \leq \delta_0$ and $\lambda > 0$.

So using this Lemma and the Calderon Vallaincourt Theorem we see that

$$(-\varepsilon^2\Delta+1)^{m/2}\mathrm{e}^{(\alpha'-\alpha)\sqrt{-\Delta}} \leq C_m \frac{\varepsilon^m}{(\alpha-\alpha')^m} \ , \tag{3.16}$$

for $\varepsilon/(\alpha-\alpha')$ small enough, and this gives

$$\|B\|_{L^2\to L^2} \leq C\frac{1}{\varepsilon^{k-m}(\alpha-\alpha')^m} \ . \tag{3.17}$$

□

In case $A = B^N$ or that we have products of N different operators we would like to determine how the norms depend on N.

Corollary 3.7. *Let $B_n \in \Psi_\varepsilon^{m,m}$, $n = 1, 2, \cdots N$, and assume that $\|B_n u\|_{\alpha',\beta} \leq C\|u\|_{\alpha,\beta}/(\alpha-\alpha')^m$ with the same C for all n, then for $A = \prod_{n=1}^N B_n$ we have*

$$\|Au\|_{\alpha',\beta} \leq C^N \frac{N^{mN}}{(\alpha-\alpha')^{mN}}\|u\|_{\alpha,\beta} \tag{3.18}$$

Proof. Set $\alpha_n = \alpha' + (\alpha-\alpha')n/N$ for $n = 0, 1, \cdots N$, then

$$\begin{aligned}
\|Au\|_{\alpha',\beta} &= \left\| \prod_{n=1}^N B_n u \right\|_{\alpha_0,\beta} \\
&\leq \frac{C}{(\alpha_1-\alpha_0)^m}\left\| \prod_{n=2}^N B_n u \right\|_{\alpha_1,\beta} \\
&\leq \frac{C}{(\alpha_1-\alpha_0)^m}\frac{C}{(\alpha_2-\alpha_1)^m}\left\| \prod_{n=3}^N B_n u \right\|_{\alpha_2,\beta} \\
&\ \ \vdots \\
&\leq \prod_{n=1}^N \frac{C}{(\alpha_n-\alpha_{n-1})^m}\|u\|_{\alpha_N,\beta}
\end{aligned} \tag{3.19}$$

and since $\alpha_N = \alpha$, $\alpha_n - \alpha_{n-1} = (\alpha-\alpha')/N$ we find

$$\|Au\|_{\alpha',\beta} \leq C^N \frac{N^{mN}}{(\alpha-\alpha')^{mN}}\|u\|_{\alpha,\beta} \ . \tag{3.20}$$

□

We can use Proposition 3.5 as well to estimate the L^1 norm of Au in terms of $\|u\|_{\alpha,\beta}$. In order to do so we need an auxiliary Lemma.

Lemma 3.8. *For $\beta > 1/2$ we have*

$$\|\mathrm{e}^{-\beta\langle d\rangle}\|_{L^2(\mathbb{D})} \leq C\frac{1}{2\beta-1} \tag{3.21}$$

for some $C > 0$.

Proof. In geodesic polar coordinates (r,θ) centred at the origin of $\mathbb{D}$ the Riemannian volume element is $\mathrm{d}\nu = 2\sinh r \mathrm{d}r\mathrm{d}\theta$, and so we find

$$\|\mathrm{e}^{-\beta\langle d\rangle}\|_{L^2}^2 = 4\pi\int_0^\infty \mathrm{e}^{-2\beta\langle r\rangle}\sinh r\mathrm{d}r \leq C\frac{1}{2\beta-1} \tag{3.22}$$

for some $C>0$. □

Corollary 3.9. *Let $A\in\Psi_\varepsilon^{m,m}$ and assume $\alpha>0$ and $\beta>1/2$, then there is a $C>0$ such that*

$$\|Aa\|_{L^1(\mathbb{D})} \leq C\frac{1}{\alpha^m}\|a\|_{\alpha,\beta}\ . \tag{3.23}$$

Furthermore if $A=\prod_{n=1}^N B_n$ with $B_n\in\Psi_\varepsilon^{m,m}$ (uniformly, i.e., with the same constants in (3.10)*) then there is a constant C independent of N such that*

$$\|Aa\|_{L^1(\mathbb{D})} \leq C^N\frac{N^{mN}}{\alpha^{mN}}\|a\|_{\alpha,\beta}\ . \tag{3.24}$$

Proof. We write $Aa = \mathrm{e}^{-\beta\langle d\rangle}\mathrm{e}^{\beta\langle d\rangle}Aa$ and apply the Cauchy Schwarz inequality

$$\|\mathrm{e}^{-\beta\langle d\rangle}\mathrm{e}^{\beta\langle d\rangle}Aa\|_{L^1} \leq \|\mathrm{e}^{-\beta\langle d\rangle}\|_{L^2}\|\mathrm{e}^{\beta\langle d\rangle}Aa\|_{L^2}\ . \tag{3.25}$$

By Lemma 3.8 the first factor on the right hand side is finite since $\beta>1/2$, and we notice that the second is

$$\|\mathrm{e}^{\beta\langle d\rangle}Aa\|_{L^2} = \|Aa\|_{0,\beta}\ , \tag{3.26}$$

and so the results follow from Proposition 3.5 and Corollary 3.7 with $\alpha'=0$. □

If we combine this with the estimates in Proposition 3.2 and Corollary 3.4 this implies the

Proposition 3.10. *Assume that $\alpha>0, \beta>1/2$, then there is a constant $C>0$ such that for any $a\in H_{\alpha,\beta}(\mathbb{D})$*

$$\|a_\Gamma\|_{L^2(M)} \leq C\bigg(\frac{1}{\alpha^4}+1\bigg)\|a\|_{\alpha,\beta} \tag{3.27}$$

and

$$\|(S(t)a)_\Gamma\|_{L^2(M)} \leq C\bigg(\frac{1}{\alpha^4}+1\bigg)\|a\|_{\alpha,\beta}\mathrm{e}^{\frac{t}{2}} \tag{3.28}$$

Furthermore if $\alpha\hbar\leq c$ for some $c>0$, then there is a $C>0$ such that for all $N\in\mathbb{N}_0$

$$\big\|\Delta^N\big[a\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}\big]_\Gamma\big\|_{L^2(M)} \leq C^{N+1}\frac{N^{2N}}{(\alpha\hbar)^{2N+4}}\|a\|_{\alpha,\beta} \tag{3.29}$$

and

$$\big\|\Delta^N\big[(S(t)a)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}\big]_\Gamma\big\|_{L^2(M)} \leq C^{N+1}\frac{N^{2N}}{(\alpha\hbar)^{2N+4}}\|a\|_{\alpha,\beta}\mathrm{e}^{t/2}\ . \tag{3.30}$$

Proof. The first two estimates, (3.27) and (3.28), follow directly by combining Proposition 3.2 and Corollary 3.4 with Corollary 3.9. To prove (3.30) we first use that Δ commutes with the action of Γ and Proposition 3.2

$$\begin{aligned}\big\|\Delta^N\big[(S(t)a)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}\big]_\Gamma\big\|_{L^2(M)} &= \big\|\big[\Delta^N(S(t)a)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}\big]_\Gamma\big\|_{L^2(M)}\\ &\leq C\big(\|\Delta^{N+2}(S(t)a)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}\|_{L^1(\mathbb{D})}+\|\Delta^N(S(t)a)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}\|_{L^1(\mathbb{D})}\big)\end{aligned} \tag{3.31}$$

Now $\Delta^N(S(t)a)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}=\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}S(t)\Delta_\varphi^N(t)a$ with $\Delta_\varphi(t)=S^*(t)\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\varphi}\Delta\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}S(t)$ and since by (2.12)

$$\hbar^2\Delta_\varphi=-1-\hbar Y+\hbar^2\Delta\ , \tag{3.32}$$

where Y is the generator of $S(t)$, we have

$$\hbar^2\Delta_\varphi(t)=\hbar^2S^*(t)\Delta_\varphi S(t)=-1-\hbar Y+\hbar^2\Delta(t)\in\Psi_\varepsilon^{2,2} \tag{3.33}$$

uniformly for $t\geq 0$. Then with Lemma 3.3 we find

$$\|\Delta^N(S(t)a)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}\|_{L^1(\mathbb{D})}=\|S(t)\Delta_\varphi^N(t)a\|_{L^1(\mathbb{D})}=\|\Delta_\varphi^N(t)a\|_{L^1(\mathbb{D})}\mathrm{e}^{t/2}\ , \tag{3.34}$$

and applying Corollary 3.9 gives then

$$\|\Delta_\varphi^N(t)a\|_{L^1(\mathbb{D})}\leq C^N\frac{N^{2N}}{(\alpha\hbar)^{2N}}\|a\|_{\alpha,\beta}\ . \tag{3.35}$$

This, together with the same estimate for $N+2$ proves then (3.30), and (3.29) follows from (3.30) by setting $t=0$. □

We will need as well some point-wise estimates on a for $a\in H_{\alpha,\beta}$, these follow again from Sobolev imbedding.

Lemma 3.11. *There is a $C>0$ such that for all $\alpha>0$ we have*

$$|a(z)|\leq C\bigg(\frac{1}{\alpha^4}+1\bigg)\|a\|_{\alpha,\beta}\mathrm{e}^{-\beta\langle d\rangle(z)} \tag{3.36}$$

for $a\in H_{\alpha,\beta}$

Proof. By Sobolev imbedding, (3.4), we have

$$\|u\|_{L^\infty(\mathbb{D})}\leq C\big(\|\Delta^2u\|_{L^2(\mathbb{D})}+\|u\|_{L^2(\mathbb{D})}\big) \tag{3.37}$$

and applying this to $u=a\mathrm{e}^{\beta\langle d\rangle}$ gives

$$|a(z)|\leq C\big(\|\Delta^2\mathrm{e}^{\beta\langle d\rangle}a\|_{L^2(\mathbb{D})}+\|\mathrm{e}^{\beta\langle d\rangle}a\|_{L^2(\mathbb{D})}\big)\mathrm{e}^{-\beta\langle d\rangle}\ . \tag{3.38}$$

But $\|\mathrm{e}^{\beta\langle d\rangle}a\|_{L^2(\mathbb{D})}=\|a\|_{0,\beta}\leq\|a\|_{\alpha,\beta}$ and

$$\begin{aligned}\|\Delta^2\mathrm{e}^{\beta\langle d\rangle}a\|_{L^2(\mathbb{D})} &= \|\Delta^2\mathrm{e}^{-\alpha\sqrt{-\Delta}}\mathrm{e}^{\alpha\sqrt{-\Delta}}\mathrm{e}^{\beta\langle d\rangle}a\|_{L^2(\mathbb{D})}\\ &\leq\frac{C}{\alpha^4}\|\mathrm{e}^{\alpha\sqrt{-\Delta}}\mathrm{e}^{\beta\langle d\rangle}a\|_{L^2(\mathbb{D})}\\ &\leq\frac{C}{\alpha^4}\|a\|_{\alpha,\beta}\ .\end{aligned} \tag{3.39}$$

since $\Delta^2 \mathrm{e}^{-\alpha\sqrt{-\Delta}} \leq \frac{C}{\alpha^4}$ by Lemma 3.8 and in the last step we used the equivalence of different expressions for the norm $\|a\|_{\alpha,\beta}$ in Proposition 7.1. □

This Lemma is the main tool in the proof of

Proposition 3.12. *There is a $C > 0$ such that for all $\varphi : \mathbb{D} \to \mathbb{R}$ and $a \in H_{\alpha,\beta}$ with $\alpha > 0$ and $\beta > 1$ we have*

$$\big\| \big(a\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}\big)_\Gamma \big\|_{L^2(M)} \leq \frac{C\beta^4}{\beta - 1}\left(\frac{1}{\alpha^4} + 1\right)\|a\|_{\alpha,\beta} \ . \tag{3.40}$$

Proof. We have $\big|\big(a\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}\big)_\Gamma\big| \leq |a|_\Gamma$ and by Lemma 3.11

$$|a|_\Gamma \leq C(1/\alpha^4 + 1)\|a\|_{\alpha,\beta}\big(\mathrm{e}^{-\beta\langle d\rangle}\big)_\Gamma \ . \tag{3.41}$$

But by Proposition 3.2

$$\|(\mathrm{e}^{-\beta\langle d\rangle})_\Gamma\|_{L^2(M)} \leq C\big(\|\Delta^2\mathrm{e}^{-\beta\langle d\rangle}\|_{L^1(\mathbb{D})} + \|\mathrm{e}^{-\beta\langle d\rangle}\|_{L^1(\mathbb{D})}\big) \tag{3.42}$$

and now using polar coordinates as in the proof of Lemma 3.8 gives $\|\mathrm{e}^{-\beta\langle d\rangle}\|_{L^1(\mathbb{D})} \leq C/(\beta - 1)$ and $\|\Delta^2\mathrm{e}^{-\beta\langle d\rangle}\|_{L^1(\mathbb{D})} \leq C\beta^4/(\beta-1)$ since the derivatives of $\langle d\rangle$ are bounded. □

This Proposition is quite similar to (3.29) in Proposition 3.10 for $N = 0$, but we have no powers of $1/\hbar$ on the right hand side, instead we had to increase the lower bound on the value of β from 1/2 to 1.

4. The dispersive part

We will study in this section how to control that action of $V(t)$ on $H_{\alpha,\beta}(\mathbb{D})$.

4.1. **Estimates on the rate of dispersion.** We have to discuss now some a priori estimates on the action of unitary groups generated by second order operators on functions from the spaces $H_{\alpha,\beta}$. These belong to the family of energy estimates which are a standard tool. But we will think of the particular estimates we need rather as estimates on the rate of dispersion, and to explain this let us first describe what we need these estimates for.

In Proposition 2.2 we have shown how to write the action of the time evolution operator $\mathcal{U}(t)$ on oscillatory functions $a\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}$ in terms of the action of two operators $S(t)$ and $V(t)$ on the amplitude a. Here the operator $S(t)$ described transport along geodesics, whereas $V(t)$ is the dispersive part. Using this partition we are able, as sketched after Theorem 1.3, to get rid of $S(t)$ in the remainder estimates and reduce them to expressions involving only $V(t)$. The problem is now that we have to estimate the sum over Γ of $V(t)a$. Using Sobolev imbedding we could reduce this to L^1-estimates of $V(t)$, but these seem to be very difficult, so we decided to pose the problem in the following form; assume that a satisfies

$$|a(z)| \leq C\mathrm{e}^{-\beta\langle d\rangle(z)} \tag{4.1}$$

for $\beta > 1/2$ (which ensures by Proposition 3.10 that a_Γ is convergent), under which conditions (on t and a) do we have then

$$|V(t)a(z)| \leq C'\mathrm{e}^{-\beta\langle d\rangle(z)} \ ? \tag{4.2}$$

(So that $(V(t)a)_\Gamma$ is still convergent). To answer this question it is natural to look at the action of $V(t)$ on weighted L^2-Sobolev spaces, with a weight $\mathrm{e}^{\beta\langle d\rangle(z)}$, which in turn leads to the study of the operator $\tilde{V}(t) := \mathrm{e}^{-\beta\langle d\rangle(z)}V(t)\mathrm{e}^{\beta\langle d\rangle(z)}$ on $L^2(\mathbb{D})$ which satisfies the equation

$$\mathrm{i}\partial_t\tilde{V}(t) = -\frac{\hbar}{2}\tilde{\Delta}(t)\tilde{V}(t) \tag{4.3}$$

where

$$\tilde{\Delta}(t) = \mathrm{e}^{-\beta\langle d\rangle}\Delta(t)\mathrm{e}^{\beta\langle d\rangle} \ . \tag{4.4}$$

Now $\tilde{V}(t)$ is no longer unitary and $\tilde{\Delta}(t)$ not selfadjoint.

In order to understand the consequences of this let us look at a simple model problem. Let Δ be the Laplacian on $\mathbb{R}^d$ and let us conjugate it with $\mathrm{e}^{\beta\langle k,x\rangle}$, where $k\in\mathbb{R}^d$ is fixed with $|k|=1$, then

$$\tilde{\Delta}_\beta = \mathrm{e}^{-\beta\langle k,x\rangle}\Delta\mathrm{e}^{\beta\langle k,x\rangle} = \Delta+\beta^2+2\beta\langle k,\nabla\rangle \tag{4.5}$$

is not selfadjoint due to the term $2\beta\langle k,\nabla\rangle$. The equation $\mathrm{i}\partial_t a = -\frac{\hbar}{2}\tilde{\Delta}_\beta a$ can easily be solved using Fourier transformation which gives $\hat{a}(t,\xi) = \mathrm{e}^{\mathrm{i}\frac{\hbar t}{2}\xi^2+\mathrm{i}\frac{\beta^2}{\hbar}}\mathrm{e}^{\hbar t\beta\langle k,\xi\rangle}\widehat{a_0}(\xi)$, for the Fourier-transformed a, where a_0 denotes the initial condition. So we have an exponentially growing factor

$$|\widehat{a(t)}(\xi)| \sim |\widehat{a_0}(\xi)|\mathrm{e}^{\beta\hbar t\langle k,\xi\rangle} \ , \tag{4.6}$$

and in order to balance this exponential growth we require that for our initial function a_0 we have $|\widehat{a_0}(\xi)|\leq \mathrm{e}^{-\alpha|\xi|}$, then $a(t)$ is well behaved for

$$\beta\hbar t<\alpha \ . \tag{4.7}$$

But this requirement on the Fourier transformation of a_0 is equivalent to requiring analyticity and leads directly to the definition of the norms $\|a_0\|_{\alpha,\beta}$.

These heuristic arguments lead us to the following

Conjecture 1. *For $\alpha,\beta>0$ and $\alpha'<\alpha$ there exist $C,\delta>0$ such that*

$$\|V(t)a\|_{\alpha',\beta}\leq C\|a\|_{\alpha,\beta} \tag{4.8}$$

$$\|V^*(t)a\|_{\alpha',\beta}\leq C\|a\|_{\alpha,\beta} \tag{4.9}$$

for $a\in H_{\alpha,\beta}$ and

$$t\leq\delta\frac{\alpha-\alpha'}{\beta}\frac{1}{\hbar} \ . \tag{4.10}$$

Since a proof of this conjecture remained elusive, we have to work around it by mollifying the generator of $V(t)$, this will be described in the rest of this section. For the mollified operator we obtain a result similar to Conjecture 1 but the time scale we eventually reach is of order $1/\sqrt{\hbar}$. Conjecture 1 would allow us to extend the time scales in Theorem 1.4 from $1/\sqrt{\hbar}$ to $1/\hbar$.

As support for the conjecture let us show that it is rather easy to prove for $\mathcal{U}(t)$.

Theorem 4.1. *There exist $C, c > 0$ such that for $\alpha, \beta > 0$ and $\alpha' < \alpha$ we have*

$$\|\mathcal{U}(t)a\|_{\alpha',\beta} \leq C\|a\|_{\alpha,\beta} \tag{4.11}$$

for $a \in H_{\alpha,\beta}$ and

$$|t| \leq c\frac{\alpha-\alpha'}{\beta}\frac{1}{\hbar} \ . \tag{4.12}$$

Proof. By using that $\mathcal{U}(t)$ is unitary and commutes with Δ we have

$$\|\mathcal{U}(t)a\|_{\alpha',\beta} = \|\mathrm{e}^{\beta\langle d\rangle}\mathrm{e}^{\alpha'\sqrt{-\Delta}}\mathcal{U}(t)a\|_{L^2} = \|\mathcal{U}^*(t)\mathrm{e}^{\beta\langle d\rangle}\mathcal{U}(t)\mathrm{e}^{\alpha'\sqrt{-\Delta}}a\|_{L^2} \tag{4.13}$$

so we have to estimate the operator $\mathcal{U}^*(t)\mathrm{e}^{\beta\langle d\rangle}\mathcal{U}(t)$. Let us set $\psi = \langle d\rangle$, we have $\mathcal{U}^*(t)\mathrm{e}^{\beta\langle d\rangle}\mathcal{U}(t) = \mathrm{e}^{\beta\mathcal{U}^*(t)\psi\mathcal{U}(t)}$ and therefore we have to consider $\psi(t) := \mathcal{U}^*(t)\psi\mathcal{U}(t)$. Using the Schrödinger equation for $\mathcal{U}(t)$ we find

$$\partial_t\psi(t) = \frac{\mathrm{i}\hbar}{2}[\Delta, \psi(t)] \tag{4.14}$$

and integrating this equation gives

$$\begin{aligned} \psi(t) - \psi &= \frac{\hbar}{2}\int_0^t \mathrm{i}[\Delta, \psi(t')]\ \mathrm{d}t' \\ &= \frac{\hbar}{2}\int_0^t \mathcal{U}^*(t')\mathrm{i}[\Delta, \psi]\mathcal{U}(t')\ \mathrm{d}t' \ . \end{aligned} \tag{4.15}$$

Now $\mathrm{i}[\Delta, \psi]$ is a symmetric first order operator, and since Δ is elliptic there exists a constant $C > 0$ such that $\mathrm{i}[\Delta, \psi] \leq 2C(1 + \sqrt{-\Delta})$ and so

$$\psi(t) - \psi \leq \hbar tC(1 + \sqrt{-\Delta}) \ . \tag{4.16}$$

This yields

$$\mathcal{U}^*(t)\mathrm{e}^{\beta\langle d\rangle}\mathcal{U}(t) \leq \mathrm{e}^{\beta\hbar tC}\mathrm{e}^{\beta\langle d\rangle + \hbar t\beta C\sqrt{-\Delta}} \tag{4.17}$$

and by Lemma 7.1

$$\begin{aligned} \|\mathcal{U}(t)a\|_{\alpha',\beta} &\leq \mathrm{e}^{\beta\hbar tC}\|\mathrm{e}^{\beta\langle d\rangle + \hbar t\beta C\sqrt{-\Delta}}\mathrm{e}^{\alpha'\sqrt{-\Delta}}a\|_{L^2} \\ &\leq C'\mathrm{e}^{\beta\hbar tC}\|\mathrm{e}^{\beta\langle d\rangle}\mathrm{e}^{(\alpha'+\hbar t\beta C)\sqrt{-\Delta}}a\|_{L^2} \end{aligned} \tag{4.18}$$

for some $C' > 0$. So we get the condition $\hbar t\beta C \leq \alpha - \alpha'$ or

$$t \leq \frac{\alpha - \alpha'}{C\beta}\frac{1}{\hbar} \ . \tag{4.19}$$

$\square$

4.2. **Mollifying.** Let us choose a $\chi \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp} \chi \in [-2,2]$ and $\chi(x) = 1$ for $x \in [-1,1]$, furthermore let $g(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2}$ and set $g_\varepsilon := g(x/\varepsilon)/\varepsilon$. We define

$$\chi_\varepsilon = g_\varepsilon * \chi \tag{4.20}$$

then χ_ε is analytic and exponentially small in $1/\varepsilon$ for x outside any neighbourhood of $[-2,2]$. Our mollifying operator will then be

$$J_\varepsilon := \chi_\varepsilon(\varepsilon\sqrt{-\Delta}) \ . \tag{4.21}$$

J_ε is a smoothed analytic version of $\chi(\varepsilon\sqrt{-\Delta})$ and so for $\varepsilon \to 0$ we have $J_\varepsilon \to 1$. In the next Lemma we quantify how fast this limit is reached on $H_{\alpha,\beta}$. Notice that the symbol of J_ε is $\chi_\varepsilon(\lambda)$, see Appendix A.

Lemma 4.2. *We have for any $a \in H_{\alpha,\beta}$ and $\alpha' < \alpha$*

$$\|(1-J_\varepsilon)a\|_{\alpha',\beta} \leq C\mathrm{e}^{-(\alpha-\alpha')/\varepsilon}\|a\|_{\alpha,\beta} \ . \tag{4.22}$$

Proof. We have

$$\|(1-J_\varepsilon)a\|_{\alpha',\beta} = \|\mathrm{e}^{\beta\psi}\mathrm{e}^{\alpha'\sqrt{-\Delta}}(1-J_\varepsilon)a\|_{L^2} = \|A\mathrm{e}^{\alpha\sqrt{-\Delta}}\mathrm{e}^{\beta\psi}a\|_{L^2} \tag{4.23}$$

with

$$A = \mathrm{e}^{\beta\psi}\mathrm{e}^{\alpha'\sqrt{-\Delta}}(1-J_\varepsilon)\mathrm{e}^{-\alpha\sqrt{-\Delta}}\mathrm{e}^{-\beta\psi} \ , \tag{4.24}$$

and so we have to estimate the L^2 norm of A. To begin with we note that

$$\mathrm{e}^{\alpha'\sqrt{-\Delta}}(1-J_\varepsilon)\mathrm{e}^{-\alpha\sqrt{-\Delta}} = (1-J_\varepsilon)\mathrm{e}^{-(\alpha-\alpha')\sqrt{-\Delta}} \tag{4.25}$$

is an analytic ε-pseudodifferential operator with symbol

$$B(\lambda) = \mathrm{e}^{-(\alpha-\alpha')\lambda/\varepsilon}(1-\chi_\varepsilon(\lambda)) \tag{4.26}$$

which is analytic and satisfies

$$\partial_\lambda^k B(\lambda) \leq CR^k k!\mathrm{e}^{-\frac{\alpha-\alpha'}{\varepsilon}} \ , \tag{4.27}$$

for some constants $C > 0, R > 0$. On the other hand in local normal coordinates the standard full symbol of B is a function $b(z,\xi)$ which satisfies similar estimates (with different C, R, see [Shu92] for a calculus on non-compact manifolds based on local normal coordinates), i.e., the integral kernel of B can be locally written as

$$\frac{1}{(2\pi\varepsilon)^2}\int \mathrm{e}^{\frac{\mathrm{i}}{\varepsilon}\xi(z-z')}b(z,\xi)\ \mathrm{d}\xi \tag{4.28}$$

and so $A = \mathrm{e}^{\beta\psi}B\mathrm{e}^{-\beta\psi}$ has integral kernel

$$k_A(z,z') = \frac{1}{(2\pi\varepsilon)^2}\int \mathrm{e}^{\frac{\mathrm{i}}{\varepsilon}[\xi(z-z')+\mathrm{i}\varepsilon\beta(\psi(z)-\psi(z'))]}b(z,\xi)\ \mathrm{d}\xi \ . \tag{4.29}$$

Now we use the Kuranishi trick and expand $\psi(z)-\psi(z')=R(z,z')(z-z')$ using Taylors Theorem, and so the phase function becomes $\xi(z-z')+\mathrm{i}\varepsilon\beta(\psi(z)-\psi(z'))=(\xi+\mathrm{i}\varepsilon\beta R(z,z'))(z-z')$ and then the coordinate change $\xi\to\xi-\mathrm{i}\varepsilon\beta R(z,z')$ gives

$$k_A(z,z')=\frac{1}{(2\pi\varepsilon)^2}\int \mathrm{e}^{\frac{\mathrm{i}}{\varepsilon}\xi(z-z')}a(z,z',\xi)\,\mathrm{d}\xi\ . \tag{4.30}$$

with the amplitude $a(z,z',\xi)=b(z,\xi-\mathrm{i}\varepsilon\beta R(z,z'))$. But $R(z,z')$ is bounded and b is analytic, so the amplitude $a(z,z',\xi)$ satisfies for ε small enough the same estimate (4.27) and so by the Calderon Vallaincourt Theorem the L^2-norm of A is bounded by $C\mathrm{e}^{-\frac{\alpha-\alpha'}{\varepsilon}}$ and therefore

$$\|A\mathrm{e}^{\beta\psi}\mathrm{e}^{\alpha\sqrt{-\Delta}}a\|_{L^2}\leq C\mathrm{e}^{-\frac{\alpha-\alpha'}{\varepsilon}}\|\mathrm{e}^{\beta\psi}\mathrm{e}^{\alpha\sqrt{-\Delta}}a\|_{L^2}=C\mathrm{e}^{-\frac{\alpha-\alpha'}{\varepsilon}}\|a\|_{\alpha,\beta}\ . \tag{4.31}$$

□

4.3. **A dispersive estimate.** Let J_ε be the mollifier introduced in (4.21), and set

$$\Delta_\varepsilon(t):=J_\varepsilon\Delta(t)J_\varepsilon \tag{4.32}$$

and let $V_\varepsilon(t)$ be the unitary operator generated by $\Delta_\varepsilon(t)$, i.e., the solution to

$$\mathrm{i}\partial_t V_\varepsilon(t)=-\frac{\hbar}{2}\Delta_\varepsilon(t)V_\varepsilon(t)\quad\text{with}\quad V_\varepsilon(t=0)=I\ . \tag{4.33}$$

Since $\Delta(t)\in\Psi_\varepsilon^{2,2}$ uniformly for $t\geq 0$ and $J_\varepsilon\in\Psi_\varepsilon^{-\infty,0}$ we have

Lemma 4.3. *We have*

$$\Delta_\varepsilon(t)=\frac{1}{\varepsilon^2}H_\varepsilon(t)\ , \tag{4.34}$$

where $H_\varepsilon(t)\in\Psi_\varepsilon^{-\infty,0}$ is uniformly bounded for all $t\geq 0$.

In this subsection we want to prove the following dispersive estimate

Theorem 4.4. *There exist constants $C,c>0$ such that for any $\varepsilon,\alpha,\alpha'>0,\beta\geq 0$ with $\alpha\geq\beta\varepsilon$ and $\alpha'<\alpha$ we have*

$$\|V_\varepsilon(t)a\|_{\alpha',\beta}\leq C\|a\|_{\alpha,\beta} \tag{4.35}$$

$$\|V_\varepsilon^*(t)a\|_{\alpha',\beta}\leq C\|a\|_{\alpha,\beta} \tag{4.36}$$

for $a\in H_{\alpha,\beta}$ and $t\geq 0$ satisfying

$$t\leq c\frac{\alpha-\alpha'}{\alpha}\frac{\varepsilon}{\hbar}\ . \tag{4.37}$$

If the condition $\alpha\geq\beta\varepsilon$ is not fulfilled, then the theorem remains true if one replaces (4.37) by

$$t\leq c\min\left(\frac{\alpha}{\beta}\frac{1}{\varepsilon},1\right)\frac{\alpha-\alpha'}{\alpha}\frac{\varepsilon}{\hbar}\ , \tag{4.38}$$

as follows from the proof. Since we use this Theorem mostly for the case that $\varepsilon\sim\sqrt{\hbar}$, $\beta=const.$ and $\alpha\gg\sqrt{\hbar}$, we don't need this case. The proof gives as well a larger time range if we have $\alpha' \to 0$, for $\alpha' = 0$ we can actually get $t \ll 1/\hbar$ but this transition of the time scales takes place on a $\ln \alpha'$ scale, so we need very small α' to see it.

To prepare the proof we need several Lemmas

Lemma 4.5. *There exist a $C \geq 0$ such that*

$$\mathrm{e}^{-C\hbar t/\varepsilon}\sqrt{-\Delta} \leq V_\varepsilon^*(t)\sqrt{-\Delta}V_\varepsilon(t) \leq \mathrm{e}^{C\hbar t/\varepsilon}\sqrt{-\Delta}\ . \tag{4.39}$$

Proof. Let us introduce the operator $P(t) := (1+\hbar t/\varepsilon)V_\varepsilon^*(t)\sqrt{-\Delta}V_\varepsilon(t)$, then

$$\frac{\mathrm{d}P(t)}{\mathrm{d}t} = (1+\hbar t/\varepsilon)V_\varepsilon^*(t)\left[\mathrm{i}\hbar[\Delta_\varepsilon(t),\sqrt{-\Delta}] + \frac{\hbar/\varepsilon}{1+\hbar t/\varepsilon}\sqrt{-\Delta}\right]V_\varepsilon(t)\ , \tag{4.40}$$

and we rewrite the term in brackets as

$$\begin{aligned} &\mathrm{i}\hbar[\Delta_\varepsilon(t),\sqrt{-\Delta}] + \frac{\hbar/\varepsilon}{1+\hbar t/\varepsilon}\sqrt{-\Delta} \\ &\quad = (-\Delta)^{1/4}\left[\mathrm{i}\hbar(-\Delta)^{-1/4}[\Delta_\varepsilon(t),\sqrt{-\Delta}](-\Delta)^{-1/4} + \frac{\hbar/\varepsilon}{1+\hbar t/\varepsilon}\right](-\Delta)^{1/4}\ . \end{aligned} \tag{4.41}$$

With Lemma 4.3 we have $\Delta_\varepsilon(t) \in \Psi_\varepsilon^{0,2}$ and since $\sqrt{-\Delta} \in \Psi_\varepsilon^{1,1}$ the pseudodifferential calculus gives $[\Delta_\varepsilon(t),\sqrt{-\Delta}] \in \Psi_\varepsilon^{0,2}$ and $(-\Delta)^{-1/4}[\Delta_\varepsilon(t),\sqrt{-\Delta}](-\Delta)^{-1/4} \in \Psi_\varepsilon^{-1,1}$. Therefore

$$\varepsilon\mathrm{i}(-\Delta)^{-1/4}[\Delta_\varepsilon(t),\sqrt{-\Delta}](-\Delta)^{-1/4} \in \Psi_\varepsilon^{-1,0} \tag{4.42}$$

and hence is bounded, so there is a constant $C-1>0$ such that

$$\begin{aligned} \mathrm{i}\hbar[\Delta_\varepsilon(t),\sqrt{-\Delta}] + \frac{\hbar/\varepsilon}{1+\hbar t/\varepsilon}\sqrt{-\Delta} &\leq \frac{\hbar}{\varepsilon}\left[C-1+\frac{1}{1+\hbar t/\varepsilon}\right]\sqrt{-\Delta} \\ &\leq C\frac{\hbar}{\varepsilon}\sqrt{-\Delta} \end{aligned} \tag{4.43}$$

and

$$\begin{aligned} \mathrm{i}\hbar[\Delta_\varepsilon(t),\sqrt{-\Delta}] + \frac{\hbar/\varepsilon}{1+\hbar t/\varepsilon}\sqrt{-\Delta} &\geq \frac{\hbar}{\varepsilon}\left[-C+\frac{1}{1+\hbar t/\varepsilon}\right]\sqrt{-\Delta} \\ &\geq -C\frac{\hbar}{\varepsilon}\sqrt{-\Delta}\ . \end{aligned} \tag{4.44}$$

So by using the estimate (4.43) in (4.40) we get

$$\frac{\mathrm{d}P(t)}{\mathrm{d}t} \leq C\frac{\hbar}{\varepsilon}P(t) \tag{4.45}$$

which implies $P(t) \leq \mathrm{e}^{C\hbar t/\varepsilon}P(0)$, i.e.,

$$V_\varepsilon^*(t)\sqrt{-\Delta}V_\varepsilon(t) \leq \frac{\mathrm{e}^{C\hbar t/\varepsilon}}{1+\hbar t/\varepsilon}\sqrt{-\Delta} \leq \mathrm{e}^{C\hbar t/\varepsilon}\sqrt{-\Delta}\ . \tag{4.46}$$

On the other hand side, if we use (4.44) in (4.40) we have

$$\frac{\mathrm{d}P(t)}{\mathrm{d}t} \geq -(C-1)\frac{\hbar}{\varepsilon}P(t) \tag{4.47}$$

which implies $P(t) \geq \mathrm{e}^{-(C-1)\hbar t/\varepsilon} P(0)$, i.e.,

$$V_\varepsilon^*(t)\sqrt{-\Delta}V_\varepsilon(t) \geq \frac{\mathrm{e}^{\hbar t/\varepsilon}}{1+\hbar t/\varepsilon}\mathrm{e}^{-C\hbar t/\varepsilon}\sqrt{-\Delta} \geq \mathrm{e}^{-C\hbar t/\varepsilon}\sqrt{-\Delta}\ , \tag{4.48}$$

since $\frac{\mathrm{e}^{\hbar t/\varepsilon}}{1+\hbar t/\varepsilon} \geq 1$. $\square$

Lemma 4.6. *Let $\psi : \mathbb{D} \to \mathbb{R}$ be a smooth function with $|\sqrt{-\Delta}\psi| \leq C$ for some $C > 0$, then*

$$-C\hbar t\sqrt{-\Delta} \leq V_\varepsilon^*(t)\psi V_\varepsilon(t) - \psi \leq C\hbar t\sqrt{-\Delta} \tag{4.49}$$

Proof. We have $\partial_t V_\varepsilon^*(t)\psi V_\varepsilon(t) = \frac{\mathrm{i}\hbar}{2}V_\varepsilon^*(t)[\Delta_\varepsilon(t),\psi]V_\varepsilon(t)$ and integrating this equation gives

$$V_\varepsilon^*(t)\psi V_\varepsilon(t) - \psi = \frac{\mathrm{i}\hbar}{2}\int_0^t V_\varepsilon^*(t')[\Delta_\varepsilon(t'),\psi]V_\varepsilon(t')\ \mathrm{d}t'\ . \tag{4.50}$$

But there is a constant $C > 0$ such that

$$-C\sqrt{-\Delta} \leq \mathrm{i}[\Delta_\varepsilon(t'),\psi] \leq C\sqrt{-\Delta} \tag{4.51}$$

and so by Lemma 4.5 we find

$$-\hbar tC\sqrt{-\Delta} \leq \frac{\mathrm{i}\hbar}{2}\int_0^t V_\varepsilon^*(t')[\Delta_\varepsilon(t'),\psi]V_\varepsilon(t')\ \mathrm{d}t' \leq \hbar tC\sqrt{-\Delta} \tag{4.52}$$

for $t \leq \hbar/\varepsilon$. $\square$

We can now prove Theorem 4.4

Proof. By Proposition 7.1 the norm $\|a\|_{\alpha,\beta}$ is equivalent to $\|\mathrm{e}^{\beta\psi+\alpha\sqrt{-\Delta}}a\|_{L^2}$, and we will work with that norm. So we have to estimate $\|\mathrm{e}^{\beta\psi+\alpha'\sqrt{-\Delta}}V_\varepsilon(t)a\|_{L^2}$ and using unitarity of $V_\varepsilon(t)$ we have

$$\|\mathrm{e}^{\beta\psi+\alpha'\sqrt{-\Delta}}V_\varepsilon(t)a\|_{L^2} = \|V_\varepsilon^*(t)\mathrm{e}^{\beta\psi+\alpha'\sqrt{-\Delta}}V_\varepsilon(t)a\|_{L^2}\ . \tag{4.53}$$

Now

$$V_\varepsilon^*(t)\mathrm{e}^{\beta\psi+\alpha'\sqrt{-\Delta}}V_\varepsilon(t) = \mathrm{e}^{\beta V_\varepsilon^*(t)\psi V_\varepsilon(t)+\alpha' V_\varepsilon^*(t)\sqrt{-\Delta}V_\varepsilon(t)} \tag{4.54}$$

and Lemma 4.5 and Lemma 4.6 give

$$\beta V_\varepsilon^*(t)\psi V_\varepsilon(t) + \alpha' V_\varepsilon^*(t)\sqrt{-\Delta}V_\varepsilon(t) \leq \beta\psi + (C\beta\hbar t + \alpha'\mathrm{e}^{C\hbar t/\varepsilon})\sqrt{-\Delta} \tag{4.55}$$

and so we have

$$\|\mathrm{e}^{\beta\psi+\alpha'\sqrt{-\Delta}}V_\varepsilon(t)a\|_{L^2} \leq \|\mathrm{e}^{\beta\psi+\alpha\sqrt{-\Delta}}a\|_{L^2} \tag{4.56}$$

if

$$C\beta\hbar t + \alpha'\mathrm{e}^{C\hbar t/\varepsilon} \leq \alpha\ . \tag{4.57}$$

We will now analyse this inequality, in order to simplify the notation let us introduce $\delta := C\hbar t$ and $\gamma = \alpha'/\alpha$, then (4.57) can be rewritten as $\beta\delta/\alpha + \gamma \mathrm{e}^{\delta\varepsilon} \leq 1$, and this is certainly satisfied if we have

$$\beta\delta/\alpha \leq \frac{1-\gamma}{2} \ , \quad \text{and} \quad \gamma \mathrm{e}^{\delta\varepsilon} \leq \frac{1+\gamma}{2} \ . \tag{4.58}$$

The first of these inequalities easily reduces to

$$\hbar t \leq \frac{1}{2C}\frac{\alpha-\alpha'}{\beta} \ . \tag{4.59}$$

By convexity of the log, the second inequality is satisfied if we have

$$\delta \leq \varepsilon \ln \frac{1+\gamma}{2\gamma} \tag{4.60}$$

but $\ln\frac{1+\gamma}{2\gamma} \geq (1-\gamma)/2$ for $\gamma \leq 1$ and so we obtain the condition $\delta \leq \varepsilon(1-\gamma)/2$, which is

$$\hbar t \leq \frac{\varepsilon}{2C}\frac{\alpha-\alpha'}{\alpha} \ . \tag{4.61}$$

So we have shown that if (4.59) and (4.61) are satisfied, that then also (4.57) holds. But (4.61) is (4.37) and for $\alpha \geq \beta\varepsilon$ (4.37) implies (4.59). This proves the upper bound in (4.35).

On the other hand we obtain from Lemma 4.5 and Lemma 4.6 as well that

$$\beta V_\varepsilon^*(t)\psi V_\varepsilon(t) + \alpha V_\varepsilon^*(t)\sqrt{-\Delta}V_\varepsilon(t) \geq \beta\psi + (-C\beta\hbar t + \alpha \mathrm{e}^{-C\hbar t/\varepsilon})\sqrt{-\Delta} \tag{4.62}$$

and so we find

$$\|\mathrm{e}^{\beta\psi+\alpha\sqrt{-\Delta}}V_\varepsilon(t)a\|_{L^2} \geq \|\mathrm{e}^{\beta\psi+\alpha'\sqrt{-\Delta}}a\|_{L^2} \ , \tag{4.63}$$

provided

$$-C\beta\hbar t + \alpha \mathrm{e}^{C\hbar t/\varepsilon} \geq \alpha' \ . \tag{4.64}$$

We analyse this inequality along the same lines as (4.57), with the same abbreviations it can be rewritten as

$$-\frac{\delta\beta}{\alpha'} + \frac{1}{\gamma}\mathrm{e}^{-\delta/\varepsilon} \geq 1 \tag{4.65}$$

which follows from the two separate inequalities

$$\frac{\delta\beta}{\alpha'} \leq -1 + \frac{1}{2}\left(1+\frac{1}{\gamma}\right) \ , \quad \text{and} \quad \frac{1}{\gamma}\mathrm{e}^{-\delta/\varepsilon} \geq \frac{1}{2}\left(1+\frac{1}{\gamma}\right) \ . \tag{4.66}$$

The first one reduces again to (4.59), the second one is equivalent to $\mathrm{e}^{\delta/\varepsilon} \leq 2/(1+\gamma)$ and by convexity this holds if $\delta \leq \varepsilon \ln 2/(1+\gamma)$. But like above we have $\ln 2/(1+\gamma) \geq (1-\gamma)/2$ and so the second inequality follows as well from (4.61). So under these conditions we have the lower bound $\|V_\varepsilon(t)a\|_{\alpha,\beta} \geq \|a\|_{\alpha',\beta}$. From this we obtain

$$\|a\|_{\alpha,\beta} = \|V_\varepsilon(t)V_\varepsilon^*(t)a\|_{\alpha,\beta} \geq \|V_\varepsilon^*(t)a\|_{\alpha',\beta} \tag{4.67}$$

which is (4.36) . $\square$

5. Proof of the main theorems

In this section we combine the semiclassical approximations on the upper half plane developed in Section 2 with the dispersive estimates from Section 4 and the estimates on $H_{\alpha,\beta}$ from Section 3 to provide the proof of the main theorems.

The first step is to show that we can replace the operator $V(t)$ with its mollified version $V_\varepsilon(t)$. To this end we show first that the generators are close on $H_{\alpha,\beta}$.

Lemma 5.1. *We have*

$$\|(\Delta(t)-\Delta_\varepsilon(t))a\|_{\alpha',\beta}\leq C\frac{1}{(\alpha-\alpha')^2}\mathrm{e}^{-\frac{(\alpha-\alpha')}{2}\frac{1}{\varepsilon}}\|a\|_{\alpha,\beta} \tag{5.1}$$

Proof. We write $\Delta(t)-\Delta_\varepsilon(t)=\Delta(t)(1-J_\varepsilon)+(1-J_\varepsilon)\Delta(t)J_\varepsilon$ and then applying Lemma 3.5 and Lemma 4.2 gives

$$\begin{aligned}\|\Delta(t)-\Delta_\varepsilon(t)a\|_{\alpha',\beta}&\leq\|\Delta(t)(1-J_\varepsilon)a\|_{\alpha',\beta}+\|(1-J_\varepsilon)\Delta(t)J_\varepsilon a\|_{\alpha',\beta}\\&\leq C\frac{1}{(\alpha_0-\alpha')^2}\|(1-J_\varepsilon)a\|_{\alpha_0,\beta}+C\mathrm{e}^{-\frac{\alpha_0-\alpha'}{\varepsilon}}\|\Delta(t)J_\varepsilon a\|_{\alpha_0,\beta}\\&\leq\left[C\frac{1}{(\alpha_0-\alpha')^2}\mathrm{e}^{-\frac{\alpha-\alpha_0}{\varepsilon}}+C\frac{1}{(\alpha-\alpha_0)^2}\mathrm{e}^{-\frac{\alpha_0-\alpha'}{\varepsilon}}\right]\|a\|_{\alpha,\beta}\end{aligned} \tag{5.2}$$

and with the choice $\alpha_0=(\alpha+\alpha')/2$ the claim follows. ☐

Now we can proceed to show that $V(t)$ and $V_\varepsilon(t)$ are close.

Lemma 5.2. *There exist constants $C,c>0$ such that for $\beta>1/2$, $\alpha\geq\beta\varepsilon$, $a\in H_{\alpha,\beta}$ and $N\in\mathbb{N}_0$*

$$\|\Delta^N\big([V^*(t)V_\varepsilon(t)-1]a\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}\big)_\Gamma\|_{L^2(M)}\leq\|a\|_{\alpha,\beta}\frac{C^{N+1}N^{2N}}{\alpha^3(\hbar\alpha)^{2N+3}}\mathrm{e}^{-\frac{1}{4}(\alpha/\varepsilon-2t)}\ , \tag{5.3}$$

for $t\leq c\frac{\varepsilon}{\hbar}$.

Notice that the right hand side of (5.3) is small if $t\ll\alpha/\varepsilon$, whereas we have as well the condition $t\ll\varepsilon/\hbar$, so we see that the largest time range for which $V_\varepsilon(t)$ is close to $V(t)$ on $H_{\alpha,\beta}$ is obtained if we choose

$$\varepsilon\sim\sqrt{\alpha\hbar}\ . \tag{5.4}$$

Then $V_\varepsilon(t)$ is close to $V(t)$ on $H_{\alpha,\beta}$ if

$$t\ll\sqrt{\alpha/\hbar}\ . \tag{5.5}$$

Proof. We have

$$\mathrm{i}\partial_t\big(V^*(t)V_\varepsilon(t)\big)=-\frac{\hbar}{2}V^*(t)[\Delta(t)-\Delta_\varepsilon(t)]V_\varepsilon(t) \tag{5.6}$$

and integrating this equation gives

$$V^*(t)V_\varepsilon(t)-1=\mathrm{i}\frac{\hbar}{2}\int_0^t V^*(t')[\Delta(t')-\Delta_\varepsilon(t')]V_\varepsilon(t')\ \mathrm{d}t'\ . \tag{5.7}$$

If we set

$$b(t') := [\Delta(t') - \Delta_\varepsilon(t')]V_\varepsilon(t')a \ , \tag{5.8}$$

and use

$$[V^*(t')b]\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi} = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{t'}{2}}\mathcal{U}^*(t')[(S(t')b)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}] \ , \tag{5.9}$$

which follows from Proposition 2.2, we find

$$[V^*(t)V_\varepsilon(t)a - a]\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi} = \mathrm{i}\frac{\hbar}{2}\int_0^t \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{t'}{2}}\mathcal{U}^*(t')[(S(t')b)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}] \, \mathrm{d}t' \tag{5.10}$$

which gives

$$\|\Delta^N\big([V^*(t)V_\varepsilon(t)a - a]\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}\big)_\Gamma\|_{L^2(M)} \leq \frac{\hbar}{2}\int_0^t \|\Delta^N[(S(t')b(t'))\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}]_\Gamma\|_{L^2(M)} \, \mathrm{d}t' \tag{5.11}$$

since $\mathcal{U}(t)$ commutes with action of Γ and is unitary. But by Proposition 3.10

$$\|\Delta^N[(S(t')b(t'))\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}]_\Gamma\|_{L^2(M)} \leq C^{N+1}\frac{N^{2N}}{(\hbar\alpha_0)^{2N+4}}\|b(t')\|_{\alpha_0,\beta}\mathrm{e}^{t'/2} \ , \tag{5.12}$$

an furthermore by Lemma 5.1 and the dispersive estimate in Theorem 4.4

$$\begin{aligned} \|b(t')\|_{\alpha_0,\beta} &\leq C\frac{1}{(\alpha_1-\alpha_0)^2}\mathrm{e}^{-(\alpha_1-\alpha_0)/\varepsilon}\|V_\varepsilon^*(t')a\|_{\alpha_1,\beta} \\ &\leq C\frac{1}{(\alpha_1-\alpha_0)^2}\mathrm{e}^{-(\alpha_1-\alpha_0)/\varepsilon}\|a\|_{\alpha_2,\beta} \end{aligned} \tag{5.13}$$

for $t \leq c\frac{(\alpha_2-\alpha_1)}{\alpha_1}\varepsilon/\hbar$ and $\alpha_2 > \alpha_1 > \alpha_0$ (and of course C changes from line to line). Combining these estimates gives

$$\|\Delta^N[(S(t')b(t'))\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}]_\Gamma\|_{L^2(M)} \leq C^{N+1}\frac{N^{2N}}{(\hbar\alpha_0)^{2N+4}}\frac{\mathrm{e}^{-(\alpha_1-\alpha_0)/\varepsilon}}{(\alpha_1-\alpha_0)^2}\|a\|_{\alpha_2,\beta}\mathrm{e}^{t'/2} \tag{5.14}$$

and if we choose now $\alpha_2 = \alpha$, $\alpha_1 = 3\alpha/4$ and $\alpha_0 = \alpha/4$ this is

$$\|\Delta^N[(S(t')b(t'))\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}]_\Gamma\|_{L^2(M)} \leq C^{N+1}\frac{N^{2N}}{\alpha^2(\hbar\alpha)^{2N+4}}\|a\|_{\alpha,\beta}\mathrm{e}^{-(\alpha/\varepsilon-2t')/4}\|a\|_{\alpha,\beta} \tag{5.15}$$

for $t \leq c\varepsilon/\hbar$ and so finally

$$\|\Delta^N\big([V^*(t)V_\varepsilon(t)a - a]\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}\big)_\Gamma\|_{L^2(M)} \leq \frac{C^{N+1}\hbar N^{2N}}{\alpha^2(\hbar\alpha)^{2N+4}}\|a\|_{\alpha,\beta}\mathrm{e}^{-(\alpha/\varepsilon-2t)/4}\|a\|_{\alpha,\beta} \ . \tag{5.16}$$

□

The operator $V_\varepsilon(t)$ can be approximated recursively by a Volterra series as follows,

Lemma 5.3. *Let $a \in C^\infty(\mathbb{D})$, then for any $K \in \mathbb{N}$*

$$V_\varepsilon(t)a = \sum_{k<K}\left(\frac{\mathrm{i}\hbar}{2}\right)^k P_k^{(\varepsilon)}(t)a + \left(\mathrm{i}\frac{\hbar}{2}\right)^K R_K^{(\varepsilon)}(t)a \tag{5.17}$$

where

$$P_0^{(\varepsilon)} = 1 \ , \tag{5.18}$$

$$P_k^{(\varepsilon)}(t) = \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} \Delta_\varepsilon(t_1)\Delta_\varepsilon(t_2)\cdots\Delta_\varepsilon(t_k)\,\mathrm{d}t_k\cdots\mathrm{d}t_1 \tag{5.19}$$

for $k \geq 1$ *and*

$$R_K^{(\varepsilon)}(t) = \int_0^t \int_0^{t_1} \cdots \int_0^{t_{K-1}} \Delta_\varepsilon(t_1)\Delta_\varepsilon(t_2)\cdots\Delta_\varepsilon(t_K)V_\varepsilon(t_K)\,\mathrm{d}t_1\cdots\mathrm{d}t_K \ . \tag{5.20}$$

Proof. This is a standart argument. We integrate equation (4.33)

$$V_\varepsilon(t)a = a + \frac{\mathrm{i}\hbar}{2}\int_0^t \Delta_\varepsilon(t_1)V_\varepsilon(t_1)a\mathrm{d}t_1 \tag{5.21}$$

and iterating this equation gives the Lemma. □

We now estimate the terms in this expansion and the remainder.

Lemma 5.4. *There exists a constant* $C > 0$ *such that for* α, α', β *with* $\alpha > \alpha'$ *and* $\alpha \geq \beta\varepsilon$ *we have for all* $a \in H_{\alpha,\beta}$

$$\|P_k^{(\varepsilon)}a\|_{\alpha',\beta} \leq C^k \frac{t^k}{(\alpha-\alpha')^{2k}}\|a\|_{\alpha,\beta} \tag{5.22}$$

and for every $\delta > 0$ *there is a constant* C_δ *such that*

$$\bigg\| \sum_{k<K} (\mathrm{i}\hbar/2)^k P_k^{(\varepsilon)} a \bigg\|_{\alpha',\beta} \leq C_\delta \|a\|_{\alpha,\beta} \tag{5.23}$$

if

$$|t| \leq \frac{2(1-\delta)}{C}\frac{\varepsilon}{\hbar} \ . \tag{5.24}$$

Furthermore

$$\|R_K^{(\varepsilon)}a\|_{\alpha',\beta} \leq C^K \frac{t^K}{(\alpha-\alpha')^{2K}}\|a\|_{\alpha,\beta} \tag{5.25}$$

if $t < c\frac{\alpha-\alpha'}{\alpha}\frac{\varepsilon}{\hbar}$.

Proof. We can view $\Delta_\varepsilon(t)$ as an operator in $\Psi_\varepsilon^{m,2}$ for all $m \geq 0$, this allows to balance the powers of ε and $\alpha - \alpha'$ appearing in the estimate (3.10), we will choose $m = 2$, which gives the estimate

$$\|\Delta_\varepsilon(t)a\|_{\alpha',\beta} \leq C\frac{1}{(\alpha-\alpha')^2}\|a\|_{\alpha,\beta} \ . \tag{5.26}$$

Then by Corollary 3.7

$$\|\Delta_\varepsilon(t_1)\Delta_\varepsilon(t_2)\cdots\Delta_\varepsilon(t_k)a\|_{\alpha',\beta} \leq C^k \frac{k^k}{(\alpha-\alpha')^{2k}}\|a\|_{\alpha,\beta} \ , \tag{5.27}$$

and together with $k^k \ll k!$ and

$$\int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} \mathrm{d}t_k \cdots \mathrm{d}t_1 = \frac{t^k}{k!} \tag{5.28}$$

this gives (5.22). From (5.22) we directly obtain

$$\bigg\| \sum_{k<K} (\mathrm{i}\hbar/2)^k P_k^{(\varepsilon)} a \bigg\|_{\alpha',\beta} \leq \sum_{k<K} C^k \frac{t^k \hbar^k}{2^k (\alpha-\alpha')^{2k}} \|a\|_{\alpha,\beta} \tag{5.29}$$

and if $C\frac{t\hbar}{2(\alpha-\alpha')^2} \leq 1-\delta$ the sum is uniformly bounded. Finally the same argument leading to (5.22) gives

$$\|R_K^{(\varepsilon)} a\|_{\alpha',\beta} \leq C^K \frac{t^K}{(\alpha_0-\alpha'')^{2K}} \|V_\varepsilon(t)a\|_{\alpha_0,\beta} \tag{5.30}$$

and by Theorem 4.4 $\|V_\varepsilon(t)a\|_{\alpha_0,\beta} \leq C'\|a\|_{\alpha,\beta}$ if $t \leq c\frac{\alpha-\alpha_0}{\alpha}\varepsilon/\hbar$ and $\alpha \geq \beta\varepsilon$, and so choosing $\alpha_0 = (\alpha+\alpha')/2$ proves (5.25). □

We have now collected most of the material we need to prove Theorem 1.4. We will do this in two steps. We first prove a theorem similar to Theorem 1.4 but with the semiclassical approximation done in terms of the Volterra series defined by the mollified operator $V_\varepsilon(t)$. And then we will show that the Volterra series defined by the mollified operator and the original operator $V(t)$ are close.

For $a \in H_{\alpha,\beta}$ let us set

$$a_\varepsilon^{(K)} = \sum_{k=0}^K \left(\frac{\mathrm{i}\hbar}{2}\right)^k P_k^{(\varepsilon)}(t)a \tag{5.31}$$

and

$$u_\varepsilon^{(K)}(t) := \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{t}{2}}\big(S(t)a_\varepsilon^{(K)}\big)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi} \ . \tag{5.32}$$

Theorem 5.5. *There are constants $C, c > 0$ such that for $a \in H_{\alpha,\beta}$, with $\beta > 1/2$, $\alpha \geq \beta\varepsilon$, and $N \in \mathbb{N}$ we have*

$$\begin{aligned} &\big\|\Delta^N\big[u_\varepsilon^{(K)}(t)_\Gamma - \mathcal{U}(t)(a\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi})_\Gamma\big]\big\|_{L^2(M)} \\ &\qquad \leq C^{N+1}\frac{N^{2N}}{(\hbar\alpha)^{2N+4}}\left[C^K\left(\frac{\hbar|t|}{\alpha^2}\right)^{K+1} + \frac{\hbar}{\alpha^2}\mathrm{e}^{-\frac{1}{8}(\alpha/\varepsilon-4t)}\right]\|a\|_{\alpha,\beta} \end{aligned} \tag{5.33}$$

for $t \leq c\varepsilon/\hbar$.

Proof. We start by using Proposition 2.2 to write

$$\begin{aligned} u_\varepsilon^{(K)} &= \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{t}{2}}\big(S(t)a_\varepsilon^{(K)}\big)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi} \\ &= \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{t}{2}}\big(S(t)V(t)V^*(t)a_\varepsilon^{(K)}\big)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi} \\ &= \mathcal{U}(t)\big([V^*(t)a_\varepsilon^{(K)}]\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}\big) \end{aligned} \tag{5.34}$$

and so

$$\begin{aligned}\|\Delta^N[u_\varepsilon^{(K)}-\mathcal{U}(t)(a\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi})]_\Gamma\|_{L^2(M)}&=\|\Delta^N[\mathcal{U}(t)((V^*(t)a_\varepsilon^{(K)})\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi})]_\Gamma\|_{L^2(M)}\\&=\|\Delta^N[(V^*(t)a_\varepsilon^{(K)}-a)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}]_\Gamma\|_{L^2(M)}\end{aligned}\tag{5.35}$$

since $\mathcal{U}(t)$ commutes with Δ^N and the action of Γ and is unitary. In the next step we want to replace $V^*(t)$ with the mollified version $V_\varepsilon^*(t)$, to this end we write

$$\begin{aligned}V^*(t)a_\varepsilon^{(K)}-V_\varepsilon^*(t)a_\varepsilon^{(K)}&=V^*(t)V_\varepsilon(t)V_\varepsilon^*(t)a_\varepsilon^{(K)}-V_\varepsilon^*(t)a_\varepsilon^{(K)}\\&=(V^*(t)V_\varepsilon(t)-1)V_\varepsilon^*(t)a_\varepsilon^{(K)}\end{aligned}\tag{5.36}$$

and so with Lemma 5.2 we find

$$\begin{aligned}&\|\Delta^N[(V^*(t)a_\varepsilon^{(K)}-V_\varepsilon^*(t)a_\varepsilon^{(K)})\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}]_\Gamma\|_{L^2(M)}\\&\qquad\leq C^{N+1}\frac{N^{2N}}{\alpha^3(\hbar\alpha)^{2N+3}}\mathrm{e}^{-\frac{1}{8}(\alpha/\varepsilon-4t)}\|V_\varepsilon^*(t)a_\varepsilon^{(K)}\|_{\alpha/2,\beta}\end{aligned}\tag{5.37}$$

for $t\leq c\frac{\varepsilon}{\hbar}$. And then Theorem 4.4 and Lemma 5.4 finally give

$$\|V_\varepsilon^*(t)a_\varepsilon^{(K)}\|_{\alpha/2,\beta}\leq C\|a\|_{\alpha,\beta}\tag{5.38}$$

for $t\leq c\max(1,\alpha)\hbar/\varepsilon$. So we have

$$\begin{aligned}\|\Delta^N[(V^*(t)a_\varepsilon^{(K)}-a)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}]_\Gamma\|_{L^2(M)}&=\|\Delta^N[(V_\varepsilon^*(t)a_\varepsilon^{(K)}-a)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}]_\Gamma\|_{L^2(M)}\\&\quad+O\left(\|a\|_{\alpha,\beta}\frac{C^{N+1}N^{2N}}{\alpha^3(\hbar\alpha)^{2N+3}}\mathrm{e}^{-\frac{1}{8}(\alpha/\varepsilon-4t)}\right).\end{aligned}\tag{5.39}$$

Now we can use the Volterra series for $V_\varepsilon(t)$ from Lemma 5.3

$$\begin{aligned}V_\varepsilon^*(t)a_\varepsilon^{(K)}-a&=V_\varepsilon^*(t)(a_\varepsilon^{(K)}-V_\varepsilon(t)a)\\&=\left(\frac{\mathrm{i}\hbar}{2}\right)^K V_\varepsilon^*(t)R_K(t)a\end{aligned}\tag{5.40}$$

and so with (3.29) from Proposition 3.10, the dispersive estimates from Theorem 4.4 and the estimates for $R_K(t)a$ from Lemma 5.4 we obtain

$$\begin{aligned}\|\Delta^N[(V_\varepsilon^*(t)a_\varepsilon^{(K)}-a)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}]_\Gamma\|_{L^2(M)}&=\left(\frac{\hbar}{2}\right)^K\|\Delta^N[(V_\varepsilon^*(t)R_K(t)a)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}]_\Gamma\|_{L^2(M)}\\&\leq C^{N+1}\frac{\hbar^K N^{2N}}{(\hbar\alpha)^{2N+4}}\|V_\varepsilon^*(t)R_K(t)a\|_{\alpha/3,\beta}\\&\leq C^{N+1}\frac{\hbar^K N^{2N}}{(\hbar\alpha)^{2N+4}}\|R_K(t)a\|_{2\alpha/3,\beta}\\&\leq C^{K+N+1}\frac{N^{2N}}{(\hbar\alpha)^{2N+4}}\left(\frac{\hbar|t|}{\alpha^2}\right)^K\|a\|_{\alpha,\beta}\,.\end{aligned}\tag{5.41}$$

for $t\leq c\hbar/\varepsilon$ and $2\alpha/3\geq\beta\varepsilon$. $\square$

Let us discuss for which choice of ε we obtain the maximal time range for which the right hand side of (5.33) is small. In order that the exponential term $\mathrm{e}^{-\frac{1}{8}(\alpha/\varepsilon-4t)}$ we must have $t \ll \alpha/\varepsilon$. This must hold together with $t \ll \varepsilon/\hbar$, and these two upper bounds are equal if

$$\varepsilon = \sqrt{\alpha\hbar}\ . \tag{5.42}$$

With this choice of ε we have $t \ll \sqrt{\alpha/\hbar}$ and then $\hbar t/\alpha^2 \ll \sqrt{\hbar/\alpha}/\alpha$, using these bounds together with $\mathrm{e}^{-\frac{1}{8}(\alpha/\varepsilon-4t)} \ll C^K K!(\alpha/\hbar)^{K/2}$ the estimate (5.33) becomes

$$\|\Delta^N[u_\varepsilon^{(K)} - \mathcal{U}(t)(a\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi})]_\Gamma\|_{L^2(M)} \leq \|a\|_{\alpha,\beta} C^{N+K+1}\frac{N^{2N}K!}{(\hbar\alpha)^{2N+4}}\frac{1}{\alpha^{K+1}}\left(\frac{\hbar}{\alpha}\right)^{\frac{K+1}{2}} \tag{5.43}$$

if

$$t \leq c\sqrt{\alpha/\hbar} \tag{5.44}$$

with a sufficiently small constant $c > 0$

This gives us already a good approximation for $\mathcal{U}(t)(a\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi})_\Gamma$, but it is defined in terms of the mollified operator Δ_ε. In the final step we replace Δ_ε by Δ in the approximations. But before doing so we want to show how to prove Theorem 1.3 using (5.43).

Proof of Theorem 1.3. We set $N = 0$ in (5.43) which gives

$$\|[u_\varepsilon^{(K)} - \mathcal{U}(t)(a\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi})]_\Gamma\|_{L^2(M)} \leq \|a\|_{\alpha,\beta} C^{K+1}\frac{K!}{(\hbar\alpha)^4}\frac{1}{\alpha^{K+1}}\left(\frac{\hbar}{\alpha}\right)^{\frac{K+1}{2}} \tag{5.45}$$

for all $a \in H_{\alpha,\beta}$ and $t \leq c\sqrt{\alpha/\hbar}$. We would like to use this with $K = 0$, but the factor $1/\hbar^4$ on the right hand side prevents us from doing so. Instead we will write $u^{(0)}$ as a sum of terms to which we can apply (5.45) with large K. To this end we use

Lemma 5.6. *For $K \in \mathbb{N}_0$ let us set*

$$\hat{P}^{(K)} := \sum_{k=0}^{K}\left(\frac{\mathrm{i}\hbar}{2}\right)^k P_k^{(\varepsilon)} \tag{5.46}$$

and furthermore

$$\hat{P}_1^{(K)} := \sum_{k=0}^{K-1}\left(\frac{\mathrm{i}\hbar}{2}\right)^k P_{k+1}^{(\varepsilon)} \tag{5.47}$$

if $K \geq 1$ and $\hat{P}_1^{(0)} = 0$ for $K = 0$. Using these operators we then set for $a \in H_{\alpha,\beta}$, $k \geq 1$

$$a_k^{(K)} := \hat{P}_1^{(K-k+1)}\hat{P}_1^{(K-k+2)}\cdots\hat{P}_1^{(K)}a \tag{5.48}$$

and $a_0^{(K)} = a$. Then we have for all $K \in \mathbb{N}_0$

$$a = \sum_{k=0}^{K}\left(\frac{-\mathrm{i}\hbar}{2}\right)^k \hat{P}^{(K-k)}a_k^{(K)}\ . \tag{5.49}$$

Proof. We have for all K

$$\begin{aligned}\hat{P}^{(K)}a &= a + \sum_{k=1}^{K} (\frac{\mathrm{i}\hbar}{2})^k P_k^{(\varepsilon)} a \\ &= a + \frac{\mathrm{i}\hbar}{2}\hat{P}_1^{(K)} a \ ,\end{aligned} \tag{5.50}$$

and this can be rewritten as

$$a = \hat{P}^{(K)}a - \frac{\mathrm{i}\hbar}{2}\hat{P}_1^{(K)} a \ . \tag{5.51}$$

By iterating this relation we arrive at (5.49). But in order to prove that (5.49) is actually correct it is easier to use (5.50) with K replaced by $K-k$ and a by $a_k^{(K)}$ which gives

$$\begin{aligned}\hat{P}^{(K-k)}a_k^{(K)} &= a_k^{(K)} + \frac{\mathrm{i}\hbar}{2}\hat{P}_1^{(K-k)} a_k^{(K)} \\ &= a_k^{(K)} + \frac{\mathrm{i}\hbar}{2} a_{k+1}^{(K)} \ ,\end{aligned} \tag{5.52}$$

by (5.48). Summing this over k then yields

$$\begin{aligned}\sum_{k=0}^{K}\left(\frac{-\mathrm{i}\hbar}{2}\right)^k \hat{P}^{(K-k)}a_k^{(K)} &= \sum_{k=0}^{K}\left(\frac{-\mathrm{i}\hbar}{2}\right)^k \left(a_k^{(K)} + \frac{\mathrm{i}\hbar}{2}a_{k+1}^{(K)}\right) \\ &= \sum_{k=0}^{K}\left(\frac{-\mathrm{i}\hbar}{2}\right)^k a_k^{(K)} - \sum_{k=1}^{K+1}\left(\frac{-\mathrm{i}\hbar}{2}\right)^k a_k^{(K)} \\ &= a_0^{(K)} - \left(\frac{-\mathrm{i}\hbar}{2}\right)^{K+1} a_{K+1}^{(K)} \\ &= a\end{aligned} \tag{5.53}$$

□

Using this Lemma we now set

$$u_{\varepsilon,k}^{(K-k)} = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{t}{2}}\big(S(t)P^{K-k}a_k^{(K)}\big)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi} \tag{5.54}$$

and we notice that this is close to $\mathcal{U}(t)\big(a_k^{(K)}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}\big)$ by Theorem 5.5, and therefore we rewrite $u^{(0)}(t)$ as

$$\begin{aligned}u^{(0)}(t) &= \sum_{k=0}^{K}\left(\frac{-\mathrm{i}\hbar}{2}\right)^k u_{\varepsilon,k}^{(K-k)} \\ &= \sum_{k=0}^{K}\left(\frac{-\mathrm{i}\hbar}{2}\right)^k \big[u_{\varepsilon,k}^{(K-k)} - \mathcal{U}(t)\big(a_k^{(K)}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}\big)\big] + \sum_{k=0}^{K}\left(\frac{-\mathrm{i}\hbar}{2}\right)^k \mathcal{U}(t)\big(a_k^{(K)}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}\big)\end{aligned} \tag{5.55}$$

which finally gives

$$
\begin{aligned}
u^{(0)}(t)-\mathcal{U}(t)\big(a\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}\big) &= \sum_{k=0}^{K}\left(\frac{-\mathrm{i}\hbar}{2}\right)^k\big[u_{\varepsilon,k}^{(K-k)}-\mathcal{U}(t)\big(a_k^{(K)}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}\big)\big] \\
&\quad+\sum_{k=1}^{K}\left(\frac{-\mathrm{i}\hbar}{2}\right)^k\mathcal{U}(t)\big(a_k^{(K)}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}\big)\ .
\end{aligned} \tag{5.56}
$$

Now we can take the L^2-norms of the projections to M and with the unitarity of $\mathcal{U}$, the estimate (5.45) and Proposition 3.12 we obtain

$$
\begin{aligned}
\|u^{(0)}(t)_\Gamma-\mathcal{U}(t)[a\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}]_\Gamma\|_{L^2(M)} &\leq \sum_{k=0}^{K}\left(\frac{\hbar}{2}\right)^k\|\big[u_{\varepsilon,k}^{(K-k)}-\mathcal{U}(t)\big(a_k^{(K)}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}\big)\big]_\Gamma\|_{L^2(M)} \\
&\quad+\sum_{k=1}^{K}\left(\frac{\hbar}{2}\right)^k\|[a_k^{(K)}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}]_\Gamma\|_{L^2(M)} \\
&\leq \sum_{k=0}^{K}\left(\frac{\hbar}{2}\right)^k\|a_k^{(K)}\|_{\alpha,\beta}\frac{(K-k)!}{(\hbar\alpha)^4}\frac{C^{K-k+1}}{\alpha^{K-k+1}}\left(\frac{\hbar}{\alpha}\right)^{\frac{K-k+1}{2}} \\
&\quad+\frac{C\beta^4}{\beta-1}\left(\frac{1}{\alpha^4}+1\right)\sum_{k=1}^{K}\left(\frac{\hbar}{2}\right)^k\|a_k^{(K)}\|_{\alpha,\beta}\ .
\end{aligned} \tag{5.57}
$$

We have by assumption $\alpha=const.$ (independent of $\hbar$) and $\beta>1$ fixed, so then the second sum is for finite K of order $O_K(\|a\|_{\alpha,\beta}\hbar)$. In the first sum the power of $\hbar$ in the k'th term is $\hbar^{(K+k+1)/2-4}$ and so if we choose $K=9$ this sum is as well of order $O_K(\|a\|_{\alpha,\beta}\hbar)$. Therefore we have

$$
\|u^{(0)}(t)_\Gamma-\mathcal{U}(t)[a\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}]_\Gamma\|_{L^2(M)}\ll\|a\|_{\alpha,\beta}\hbar \tag{5.58}
$$

for $t\leq c/\sqrt{\hbar}$. □

What is left now in order to complete the proof of our main result, Theorem 1.4, is to estimate the difference between the semiclassical approximations in terms of the mollified operator Δ_ε and the original Δ. Let us set

$$
a^{(K)}=\sum_{k=0}^{K}\left(\frac{\mathrm{i}\hbar}{2}\right)^k P_k(t)a\ , \tag{5.59}
$$

and

$$
u^{(K)}:=\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{t}{2}}\big(S(t)a^{(K)}\big)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}\ , \tag{5.60}
$$

Then we have

Proposition 5.7. *There is a constant $C > 0$ such that for $\varepsilon = \sqrt{\hbar/\alpha}$, $\beta > 1/2$, $\alpha^3 \geq \beta^2\hbar$, $a \in H_{\alpha,\beta}$, $K \geq 1$ and $N \in \mathbb{N}$*

$$\|\Delta^N[u_\varepsilon^{(K)} - u^{(K)}]_\Gamma\|_{L^2(M)} \leq C^{N+K+1}\frac{N^{2N}K!}{(\alpha\hbar)^{2N+4}}\left(\frac{|t|\hbar}{\alpha^2}\right)^K\|a\|_{\alpha,\beta} \tag{5.61}$$

if

$$t \ll \frac{\alpha^{3/2}}{\hbar^{1/2}} \ . \tag{5.62}$$

The proof of this Proposition relies on two Lemmas. In the first we estimate the difference between the expansions of $V(t)$ and $V_\varepsilon(t)$ on $H_{\alpha,\beta}$.

Lemma 5.8. *There is a constant $C > 0$ such that for all $\alpha > \alpha'$ we have*

$$\left\|\sum_{k\leq K}(\mathrm{i}\hbar/2)^kP_ka - \sum_{k\leq K}(\mathrm{i}\hbar/2)^kP_k^{(\varepsilon)}a\right\|_{\alpha',\beta} \leq \sum_{k=1}^K C^kk!\frac{t^k\hbar^k}{(\alpha-\alpha')^{2k}}\mathrm{e}^{-\frac{\alpha-\alpha'}{2k}\frac{1}{\varepsilon}}\|a\|_{\alpha,\beta} \ . \tag{5.63}$$

Proof. We start by estimating the norm of

$$\Delta_\varepsilon(t_1)\Delta_\varepsilon(t_2)\cdots\Delta_\varepsilon(t_k)a - \Delta(t_1)\Delta(t_2)\cdots\Delta(t_k)a \ , \tag{5.64}$$

to this end we introduce for $0 \leq k_1 \leq k$

$$D_{k_1} := \Delta(t_1)\cdots\Delta(t_{k_1})\Delta_\varepsilon(t_{k_1+1})\cdots\Delta_\varepsilon(t_k) \tag{5.65}$$

and then write

$$\begin{aligned}\Delta_\varepsilon(t_1)\Delta_\varepsilon(t_2)\cdots\Delta_\varepsilon(t_k) - \Delta(t_1)\Delta(t_2)\cdots\Delta(t_k)&\\ &= D_0 - D_k\\ &= D_0 - D_1 + D_1 - D_2 + D_2 - D_3 + \cdots - D_k\\ &= \sum_{j=0}^{k-1} D_j - D_{j+1} \ .\end{aligned} \tag{5.66}$$

Now by combining Lemma 5.1 and Proposition 3.5 we see that

$$\|(D_j - D_{j+1})a\|_{\alpha',\beta} \leq C^k\frac{k^{2k}}{(\alpha-\alpha')^{2k}}\mathrm{e}^{-\frac{\alpha-\alpha'}{2k}\frac{1}{\varepsilon}}\|a\|_{\alpha,\beta} \tag{5.67}$$

and so therefore

$$\|\Delta_\varepsilon(t_1)\Delta_\varepsilon(t_2)\cdots\Delta_\varepsilon(t_k)a - \Delta(t_1)\Delta(t_2)\cdots\Delta(t_k)a\|_{\alpha',\beta} \leq kC^k\frac{k^{2k}\mathrm{e}^{-\frac{\alpha-\alpha'}{2k}\frac{1}{\varepsilon}}}{(\alpha-\alpha')^{2k}}\|a\|_{\alpha,\beta} \ . \tag{5.68}$$

Taking the t-integral into account as in (5.28) this leads to

$$\|P_k(t)a - P_k^{(\varepsilon)}(t)a\|_{\alpha',\beta} \leq C^k\frac{t^kk!}{(\alpha-\alpha')^{2k}}\mathrm{e}^{-\frac{\alpha-\alpha'}{2k}\frac{1}{\varepsilon}}\|a\|_{\alpha,\beta} \ . \tag{5.69}$$

□

Now what remains to do is to estimate the sum in (5.63).

Lemma 5.9. *For every $C_0 > 0$ there is are constants $C, c > 0$ such that for $t \leq c\alpha^2\varepsilon/\hbar$ we have*

$$\sum_{k=1}^{K} C^k k! \frac{t^k \hbar^k}{\alpha^{2k}} \mathrm{e}^{-\frac{\alpha}{4k}\frac{1}{\varepsilon}} \leq C^{K+1} K! \left(\frac{\hbar|t|}{\alpha^2}\right)^K \tag{5.70}$$

Proof. We write

$$\sum_{k=1}^{K} C^k k! \frac{t^k \hbar^k}{\alpha^{2k}} \mathrm{e}^{-\frac{\alpha}{4k}\frac{1}{\varepsilon}} = \left(\frac{Ct\hbar}{\alpha^2}\right)^K \sum_{k=1}^{K} k! \left(\frac{\alpha^2}{Ct\hbar}\right)^{K-k} \mathrm{e}^{-\frac{\alpha}{4k}\frac{1}{\varepsilon}} \tag{5.71}$$

and setting $\lambda = \frac{\alpha}{4\varepsilon}$ and $\delta = \frac{\alpha^2\varepsilon}{Ct\hbar}$ the sum becomes

$$\sum_{k=1}^{K} k! \left(\frac{\alpha^2}{Ct\hbar}\right)^{K-k} \mathrm{e}^{-\frac{\alpha K}{4k}\frac{1}{\varepsilon}} = \sum_{k=1}^{K} k! \delta^{K-k} \lambda^{K-k} \mathrm{e}^{-\frac{1}{k}\lambda} . \tag{5.72}$$

Now by Lemma 3.6 we have $\lambda^{K-k}\mathrm{e}^{-\frac{1}{k}\lambda} \ll k^{K-k}(K-k)!$ and using $k! \ll k^k \mathrm{e}^{-k}$ we have

$$k!\delta^{K-k}\lambda^{K-k}\mathrm{e}^{-\frac{1}{k}\lambda} \ll \delta^{K-k}(K-k)! k^K \mathrm{e}^{-k} , \tag{5.73}$$

so using Lemma 3.6 once more to see that $k^K \mathrm{e}^{-k} \ll k!$ we finally have

$$\sum_{k=1}^{K} k!\delta^{K-k}\lambda^{K-k}\mathrm{e}^{-\frac{1}{k}\lambda} \ll \sum_{k=1}^{K} k!(K-k)! \leq C^K K! \tag{5.74}$$

for $\delta \leq 1$ and some $C > 0$. □

Proof of Proposition 5.7. The first part of the proof is similar to the proof of Theorem 5.5. Let us set

$$b := a_\varepsilon^{(K)} - a^{(K)} \tag{5.75}$$

and write

$$\begin{aligned} u_\varepsilon^{(K)} - u^{(K)} &= [S(t)b]\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi} \\ &= [S(t)V(t)V^*(t)b]\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi} \\ &= \mathcal{U}(t)\big([V^*(t)b]\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}\big) \end{aligned} \tag{5.76}$$

and since $\mathcal{U}(t)$ is unitary and commutes with Δ and the action of Γ we find

$$\|\Delta^N [u_\varepsilon^{(K)} - u^{(K)}]_\Gamma\|_{L^2(M)} = \|\Delta^N \big([V^*(t)b]\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}\big)_\Gamma\|_{L^2(M)} . \tag{5.77}$$

Now we want to replace V^* by V_ε^* as in the proof of Theorem 5.5. To this end we write

$$V^*(t)b = V_\varepsilon^*(t)b + (V^*(t)V_\varepsilon(t) - 1)V_\varepsilon^*(t)b \tag{5.78}$$

and then by Proposition 3.10 and Lemma 5.2 we obtain

$$
\begin{aligned}
\|\Delta^N\big([V^*(t)b]\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}\big)_\Gamma\|_{L^2(M)} &\leq \|\Delta^N\big([V_\varepsilon^*(t)b]\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}\big)_\Gamma\|_{L^2(M)} \\
&\quad + \|\Delta^N\big([(V^*(t)V_\varepsilon(t)-1)V_\varepsilon^*(t)b]\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi}\big)_\Gamma\|_{L^2(M)} \\
&\leq C^{N+1}\frac{N^{2N}}{(\alpha\hbar)^{2N+4}}\|V_\varepsilon^*(t)b\|_{\alpha/3,\beta} \\
&\quad + C^{N+1}\frac{N^{2N}}{\alpha^3(\alpha\hbar)^{2N+3}}\mathrm{e}^{-\frac{1}{4}(\frac{\alpha}{\varepsilon}-2t)}\|V_\varepsilon^*(t)b\|_{\alpha/3,\beta} \\
&= C^{N+1}\frac{N^{2N}}{(\alpha\hbar)^{2N+4}}\left(1+\frac{\hbar}{\alpha^2}\mathrm{e}^{-\frac{1}{4}(\frac{\alpha}{\varepsilon}-2t)}\right)\|V_\varepsilon^*(t)b\|_{\alpha/3,\beta}\ .
\end{aligned} \tag{5.79}
$$

But by the dispersive estimate in Theorem 4.4 we have

$$\|V_\varepsilon^*(t)b\|_{\alpha/3,\beta} \leq C\|b\|_{2\alpha/3,\beta} \tag{5.80}$$

for $t \ll \varepsilon/\hbar$ and $\alpha \gg \beta\varepsilon$. Now we can apply Lemma 5.8 to $b = a_\varepsilon^{(K)} - a^{(K)}$, which gives

$$\|b\|_{2\alpha/3,\beta} \leq \sum_{k=1}^{K} C^k k!\left(\frac{|t|\hbar}{\alpha^2}\right)^k \mathrm{e}^{-\frac{\alpha}{6k}\frac{1}{\varepsilon}}\|a\|_{\alpha,\beta} \tag{5.81}$$

and then Lemma 5.9 allows to estimate the sum which yields

$$\|b\|_{2\alpha/3,\beta} \leq C^K K!\left(\frac{|t|\hbar}{\alpha^2}\right)^K\|a\|_{\alpha,\beta} \tag{5.82}$$

if $t \ll \alpha^2\varepsilon/\hbar$. If we require in addition that $t \ll \alpha/\varepsilon$ then the exponential term $\mathrm{e}^{-\frac{1}{4}(\frac{\alpha}{\varepsilon}-2t)}$ is bounded and the optimal choice for ε is then

$$\varepsilon = \sqrt{\frac{\hbar}{\alpha}}\ . \tag{5.83}$$

With this choice for ε the condition $\alpha \geq \beta\varepsilon$ becomes $\alpha^3 \geq \beta^2\hbar$. Combining the successive estimates gives then finally

$$\|\Delta^N[u_\varepsilon^{(K)} - u^{(K)}]_\Gamma\|_{L^2(M)} \leq C^{N+K+1}\frac{N^{2N}K!}{(\alpha\hbar)^{2N+4}}\left(\frac{|t|\hbar}{\alpha^2}\right)^K\|a\|_{\alpha,\beta} \tag{5.84}$$

for $\varepsilon = \sqrt{\hbar/\alpha}$, $\alpha^3 \geq \beta^2\hbar$ and

$$t \ll \frac{\alpha^{3/2}}{\hbar^{1/2}}\ . \tag{5.85}$$

$\square$

Notice that for K fixed one actually can obtain an error estimate of order $O_K(\mathrm{e}^{\frac{\alpha}{6K}\frac{1}{\varepsilon}})$. Now we can prove our main Theorem.

Proof of Theorem 1.4. If we combine the estimates from Theorem 5.5 and Proposition 5.7 and set $\varepsilon = \sqrt{\hbar/\alpha}$ we obtain

$$\begin{aligned} \|\Delta^N[u^{(K)} - \mathcal{U}(t)(a\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi})]_\Gamma\|_{L^2(M)} &\leq \|\Delta^N[u^{(K)} - u_\varepsilon^{(K)}]_\Gamma\|_{L^2(M)} \\ &\quad + \|\Delta^N[u_\varepsilon^{(K)} - \mathcal{U}(t)(a\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi})]_\Gamma\|_{L^2(M)} \\ &\leq C^{N+K+1}\frac{N^{2N}K!}{(\alpha\hbar)^{2N+4}}\left(\frac{|t|\hbar}{\alpha^2}\right)^K \|a\|_{\alpha,\beta} \end{aligned} \tag{5.86}$$

for

$$t \ll \frac{\alpha^{3/2}}{\hbar^{1/2}} \ . \tag{5.87}$$

Finally the condition $\alpha \geq \beta\varepsilon$ from Theorem 4.4 together with the choice $\varepsilon = \sqrt{\hbar/\alpha}$ gives $\alpha^3 \geq \beta^2\hbar$. □

The proof of Corollary 1.5 is now a standard estimate.

Proof of Corollary 1.5. We have

$$\big\|\Delta^N[\mathcal{U}(t)\big(a\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi_b}\big)_\Gamma - u^{(K)}(t)_\Gamma]\big\|_{L^2(M)} \leq C^{N+K+1}\frac{N^{2N}K!}{(\alpha\hbar)^{2N+4}}\left(\frac{\hbar t}{\alpha^2}\right)^K \|a\|_{\alpha,\beta} \ , \tag{5.88}$$

let us set $\delta = C\frac{\hbar t}{\alpha^2}$, then using Sterlings formula we find

$$\begin{aligned} \left(C\frac{\hbar t}{\alpha^2}\right)^K K! &\ll \mathrm{e}^{-K\ln 1/\delta}K^{1/2}\mathrm{e}^{K(\ln K-1)} \\ &= K^{1/2}\mathrm{e}^{-K}\mathrm{e}^{K\ln(K\delta)} \\ &\ll \frac{1}{\varepsilon^{1/2}}\mathrm{e}^{-K} \end{aligned} \tag{5.89}$$

if $K\delta \leq 1-\varepsilon/2$. But if $K\delta \geq 1-\varepsilon$ then $\mathrm{e}^{-K} \leq \mathrm{e}^{-(1-\varepsilon)/\delta}$ and so

$$\left(C\frac{\hbar t}{\alpha^2}\right)^K K! \ll \frac{1}{\varepsilon^{1/2}}\mathrm{e}^{-(1-\varepsilon)\frac{\alpha^2}{C\hbar t}} \ . \tag{5.90}$$

□

Finally Corollary 1.6 follows from Sobolev imbedding.

Proof of Corollary 1.6. We use the standard relation

$$\|u\|_{L^\infty(M)} \leq C\big(\|\Delta^2 u\|_{L^2(M)} + \|u\|_{L^2(M)}\big) \ . \tag{5.91}$$

Applying this to (1.30) gives (1.34), and to (1.32) gives (1.36). □

Notice that if $\beta > 1$ we could use as well use Proposition 3.12 which would reduce the power of $1/(\alpha\hbar)$ in Corollary 1.5 and 1.6.

6. Pseudodifferential operators on $\mathbb{D}$

We collect here some elements of a semiclassical calculus of pseudodifferential operators on $\mathbb{D}$, which is a simple extension of the calculus developed in [Zel86]. We denote by $C_b^\infty(\mathbb{D})$ the space of uniformly bounded smooth functions on $\mathbb{D}$, i.e., $u \in C_b^\infty(\mathbb{D})$ if for every $n \in \mathbb{N}_0$ there is a constant C_n such that

$$|\Delta^n u| \leq C_n \ . \tag{6.1}$$

Let $\varepsilon \in (0,1]$ be small parameter, we say a family of operators $A_\varepsilon : C_b^\infty(\mathbb{D}) \to C_b^\infty(\mathbb{D})$ has symbol $a(\varepsilon; z, b, \lambda)$ if

$$A_\varepsilon \mathrm{e}^{(\mathrm{i}\lambda/\varepsilon+1/2)\varphi_b} = a(\varepsilon; z, b, \lambda)\mathrm{e}^{(\mathrm{i}\lambda/\varepsilon+1/2)\varphi_b} \tag{6.2}$$

for all $(b, \lambda) \in \partial\mathbb{D} \times \mathbb{R}^+$ [1]. For $u \in C_0^\infty(\mathbb{D})$ we have the Non-Euclidean Fourier-transform

$$\tilde{u}(b, \lambda) = \int_{\mathbb{D}} \mathrm{e}^{(-\mathrm{i}\lambda/\varepsilon+1/2)\varphi_b(z)} u(z)\mathrm{d}\nu(z) \tag{6.3}$$

and the inversion formula

$$u(z) = \frac{1}{2\pi\varepsilon^2} \iint_{\mathbb{R}^+\times\partial\mathbb{D}} \mathrm{e}^{(\mathrm{i}\lambda/\varepsilon+1/2)\varphi_b(z)} \tilde{u}(b, \lambda)\lambda \tanh \frac{2\pi\lambda}{\varepsilon} \, \mathrm{d}\lambda \mathrm{d}b \ . \tag{6.4}$$

Applying the definition of the symbol (6.2) to the inversion formula (6.4) gives an integral formula for the action of the operator A_ε,

$$A_\varepsilon u(z) = \frac{1}{2\pi\varepsilon^2} \iint_{\mathbb{R}^+\times\partial\mathbb{D}} \mathrm{e}^{(\mathrm{i}\lambda/\varepsilon+1/2)\varphi_b(z)} a(\varepsilon; z, b, \lambda)\tilde{u}(b, \lambda)\lambda \tanh \frac{2\pi\lambda}{\varepsilon} \, \mathrm{d}\lambda \mathrm{d}b \ . \tag{6.5}$$

Pseudodifferential operators are defined by requiring conditions on the symbol of an operator. We will view (z, b, λ) as coordinates on the co-tangent bundle $T^*\mathbb{D}$ via the mapping

$$\mathbb{D} \times (\partial\mathbb{D} \times \mathbb{R}^+) \to T^*\mathbb{D} \tag{6.6}$$

$$(z, b, \lambda) \mapsto \lambda d\varphi_b(z) \ . \tag{6.7}$$

Let $\hat{g}$ be the Sasaki metric on $T^*\mathbb{D}$, $\hat{g}_{S^*\mathbb{D}}$ the restriction to the unit cotangent bundle $S^*\mathbb{D}$ and $\Delta_{S^*\mathbb{D}}$ the corresponding Laplace Beltrami operator on $S^*\mathbb{D}$. We say that $a \in S^{m,k}$ if for all $\alpha, \beta \in \mathbb{N}_0$ there are constants $C_{\alpha,\beta}$ such that

$$\left|\partial_\lambda^\alpha \Delta_{S^*\mathbb{D}}^\beta a(\varepsilon)\right| \leq C_{\alpha,\beta} \frac{1}{\varepsilon^m}(1+\lambda)^{k-\alpha} \tag{6.8}$$

The corresponding class of operators are defined by (6.5) will be denoted by $\Psi_\varepsilon^{m,k}(\mathbb{D})$.

These classes of pseudodifferential operators satisfy the usual properties

- Product-formula: For $A \in \Psi_\varepsilon^{m,k}(\mathbb{D})$ and $B \in \Psi_\varepsilon^{m',k'}(\mathbb{D})$ we have $AB \in \Psi_\varepsilon^{m+m',k+k'}(\mathbb{D})$ and $[A, B] \in \Psi_\varepsilon^{m+m'-1,k+k'-1}(\mathbb{D})$

[1] Note that in [Zel86] a slightly different convention was used, the plane waves there are $\mathrm{e}^{(\mathrm{i}\lambda/\varepsilon+1)\varphi_b}$, this is due to the fact that we use the metric $\mathrm{d}s^2 = \frac{(1-|z|^2)^2}{4}|dz|^2$ instead of $\mathrm{d}s^2 = (1-|z|^2)^2|dz|^2$, in order to have curvature -1.

- Boundedness: The Calderon Vallaincourt Theorem holds: The L^2 norm of operators can be estimated by a finite number of derivatives of the symbol, in particular the operators in $\Psi^{0,0}_\varepsilon(\mathbb{D})$ are bounded on $L^2(\mathbb{D})$.

In particular we have $\Delta \in \Psi^{2,2}_\varepsilon(\mathbb{D})$ since its symbol is λ^2/ε^2

7. Equivalent norms

In this appendix we sketch a proof of

Proposition 7.1. *There is a constant $C > 0$ such that for all $a \in H_{\alpha,\beta}$*

$$\frac{1}{C}\|a\|_{\alpha,\beta} \leq \|\mathrm{e}^{\alpha\sqrt{-\Delta}+\beta\langle d\rangle}a\|_{L^2} \leq C\|a\|_{\alpha,\beta} \tag{7.1}$$

$$\frac{1}{C}\|a\|_{\alpha,\beta} \leq \|\mathrm{e}^{\alpha\sqrt{-\Delta}}\mathrm{e}^{\beta\langle d\rangle}a\|_{L^2} \leq C\|a\|_{\alpha,\beta} \tag{7.2}$$

i.e. the norms $\|a\|_{\alpha,\beta}$, $\|\mathrm{e}^{\alpha\sqrt{-\Delta}+\beta\langle d\rangle}a\|_{L^2}$ and $\|\mathrm{e}^{\alpha\sqrt{-\Delta}}\mathrm{e}^{\beta\langle d\rangle}a\|_{L^2}$ are equivalent.

Let us set in the following

$$P_{\alpha,\beta} := \alpha\sqrt{-\Delta} + \beta\langle d\rangle$$

Our main technical tool will be the following complex version of Egorov's Theorem.

Theorem 7.2. *Let $A \in \Psi^{m,k}_\varepsilon(\mathbb{D})$ be analytic, then there is a constant $c > 0$ such that for all $\alpha, \beta \in [-c, c]$*

$$A_{\alpha,\beta} := \mathrm{e}^{-P_{\alpha,\beta}}A\mathrm{e}^{P_{\alpha,\beta}} \in \Psi^{m,k}_\varepsilon(\mathbb{D}) \ . \tag{7.3}$$

This follows basically from work in [Sjö82] by noticing that $\mathrm{e}^{-P_{\alpha,\beta}}$ is a Fourier integral operator with complex phase function.

From this we derive

Lemma 7.3. *Let $\alpha, \beta \in [-c, c]$ and $\alpha', \beta' \in \mathbb{R}$, then*

$$R := \mathrm{e}^{-P_{\alpha,\beta}}P_{\alpha',\beta'}\mathrm{e}^{P_{\alpha,\beta}} - P_{\alpha',\beta'} \in \Psi^{0,0}_\varepsilon(\mathbb{D}) \ . \tag{7.4}$$

Proof. We have

$$\partial_t \mathrm{e}^{-tP_{\alpha,\beta}}P_{\alpha'.\beta'}\mathrm{e}^{tP_{\alpha,\beta}} = \mathrm{e}^{-tP_{\alpha,\beta}}[P_{\alpha'.\beta'}, P_{\alpha,\beta}]\mathrm{e}^{tP_{\alpha,\beta}} \tag{7.5}$$

and $[P_{\alpha'.\beta'}, P_{\alpha,\beta}] \in \Psi^{0,0}_\varepsilon$ is analytic, so by Theorem 7.2

$$A = \mathrm{e}^{-tP_{\alpha,\beta}}[P_{\alpha'.\beta'}, P_{\alpha,\beta}]\mathrm{e}^{tP_{\alpha,\beta}} \in \Psi^{0,0}_\varepsilon \tag{7.6}$$

for $|t| \leq 1$ and therefore by integrating in t we find

$$R = \int_0^1 \partial_t \mathrm{e}^{-tP_{\alpha,\beta}}P_{\alpha'.\beta'}\mathrm{e}^{tP_{\alpha,\beta}}\,\mathrm{d}t = \int_0^1 A\,\mathrm{d}t \in \Psi^{0,0}_\varepsilon \ . \tag{7.7}$$

$\square$

Proof of Proposition 7.1. Let us define $B(t)$ by

$$\mathrm{e}^{tP_{\alpha,\beta}} = B(t)\mathrm{e}^{tP_{0,\beta}}\mathrm{e}^{tP_{\alpha,0}} \tag{7.8}$$

i.e., $B(t) = \mathrm{e}^{tP_{\alpha,\beta}}\mathrm{e}^{-tP_{\alpha,0}}\mathrm{e}^{-tP_{0,\beta}}$, and taking the derivative with respect to t gives

$$\begin{aligned} \partial_t B(t) &= \mathrm{e}^{tP_{\alpha,\beta}}[P_{\alpha,\beta} - P_{\alpha,0} - \mathrm{e}^{-tP_{\alpha,0}}P_{0,\beta}\mathrm{e}^{tP_{\alpha,0}}]\mathrm{e}^{-tP_{\alpha,0}}\mathrm{e}^{-tP_{0,\beta}} \\ &= \mathrm{e}^{tP_{\alpha,\beta}}[P_{\alpha,\beta} - P_{\alpha,0} - \mathrm{e}^{-tP_{\alpha,0}}P_{0,\beta}\mathrm{e}^{tP_{\alpha,0}}]\mathrm{e}^{-tP_{\alpha,\beta}}B(t)\ . \end{aligned} \tag{7.9}$$

Now we have $R := P_{\alpha,\beta} - P_{\alpha,0} - \mathrm{e}^{-tP_{\alpha,0}}P_{0,\beta}\mathrm{e}^{tP_{\alpha,0}} = P_{0,\beta} - \mathrm{e}^{-tP_{\alpha,0}}P_{0,\beta}\mathrm{e}^{tP_{\alpha,0}} \in \Psi_\varepsilon^{0,0}$ by Lemma 7.3 and so by Theorem 7.2 $A := \mathrm{e}^{tP_{\alpha,\beta}}R\mathrm{e}^{-tP_{\alpha,\beta}} \in \Psi_\varepsilon^{0,0}$ and from $\partial_t B(t) = AB(t)$ a comparison argument gives that $B^*(1)B(1)$ is bounded from above and below, which proves the equivalence of the norms. □

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