Supplementary Material for 'Perfect resonant absorption of guided water-waves by Autler-Townes Splitting'

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I. THEORETICAL ANALYSIS

We consider water wave propagation in a main guide of width d supporting two contiguous channels of width $2a$ and heights (b_1, b_2). Using the assumptions of an inviscid, incompressible fluid, and irrotational motion, the linearized equation for the velocity potential $\phi(x, y, z)$ is

$$\Delta\phi = 0, \qquad \frac{\partial\phi}{\partial z}(x, y, 0) = \frac{\omega^2}{g}\,\phi(x, y, 0), \quad \nabla\phi\cdot\mathbf{n}_{|\Gamma} = 0,$$

with $g = 9.81$ m.s^{-2} the gravity and ω the frequency (in the harmonic regime with time dependence $e^{-i\omega t}$). z is the vertical coordinate with $z = 0$ the undisturbed free surface and $z = -h$ the sea bottom (x the axis of the guide); Γ denotes the boundaries of the vertical walls of the guide and side channels and the horizontal (rigid) sea bottom. The free surface elevation $\eta(x, y)$ is defined as

$$\eta(x, y) = \frac{i\omega}{g}\phi(x, y, 0).$$

With h the constant water depth, the above equations simplify to the two-dimensional Helmholtz equation

$$(\nabla^2 + k^2)\eta = 0, \qquad \nabla\eta\cdot\mathbf{n}_{|\gamma} = 0, \tag{1}$$

where γ denotes the rigid boundaries of the guide and side channels reduced to segments in the (x, y) plane and where k is the wavenumber satisfying the dispersion relation $\omega^2 = gk\tanh(kh)$. In the main waveguide of width d, we shall only need that the solution can be approximated by $\eta(x, y) \simeq \eta(x)$ governed by $\eta''(x) + k^2\eta(x) = 0$. Similarly, in the two channels, we have $\eta(x, y) \approx \eta_n(y)$, $n = 1, 2$, governed by $\eta_n''(y) + k^2\eta_n(y) = 0$. With $\eta_s = \eta_1 + \eta_2$, $\eta_a = \eta_1 - \eta_2$, being the symmetric and antisymmetric fields in the side channels and we define $\overline{\eta} = \frac{1}{2}(\eta^+ + \eta^-)$ and $[\eta] = (\eta^+ - \eta^-)$ to be the average and the jump of η across the junction at $x = 0$ ($\eta^\pm = \eta(0^\pm)$).

$$\left\{\begin{array}{ll} [\eta] = \mathcal{B}\,d\overline{\eta'} - \dfrac{\hat{\mathcal{B}}_a}{2}\,a\eta_a'(0), & d[\eta'] + a\eta_s'(0) = 0, \\ \eta_s(0) = 2\overline{\eta} + \dfrac{\mathcal{B}_s}{2}\,a\eta_s'(0), & \eta_a(0) = -\hat{\mathcal{B}}_a\,d\overline{\eta'} + \dfrac{\mathcal{B}_a}{2}\,a\eta_a'(0). \end{array}\right. \tag{2}$$

In [?], the above jump conditions have been demonstrated and the analysis has been further developed for identical side channels. Here we extend the result to side channels of different heights $b_1 = b(1-\varepsilon)$ and $b_2 = b(1+\varepsilon)$. The solution for an incident wave e^{ikx} reads

$$\left\{\begin{array}{lll} \eta(x) = e^{ikx} + Re^{-ikx}, & & x \in (-\infty, 0), \\ \eta_n(y) = A_n\cos k(y - b_n), & n = 1, 2, & x = 0, \\ \eta(x) = Te^{ikx}, & & x \in (0, +\infty). \end{array}\right.$$

The scattering coefficients (R, T) are deduced from (2) and they read

$$T = \frac{\Re(z_s z_a^*) - z^2}{z_s^* z_a^* + z^2}, \qquad R = -\frac{i\Im(z_s z_a^*) + 2z}{z_s^* z_a^* + z^2}, \tag{3}$$

where, introducing $\mathsf{x}_n = ka\tan kb_n$, $n = 1, 2$,

$$z_s = 1 - \frac{i\gamma_s}{2kd}, \quad z_a = 1 + \frac{ikd\gamma_a}{2}, \quad z = -\frac{\hat{\mathcal{B}}_a}{2D}(\mathsf{x}_1 - \mathsf{x}_2),$$

$$\gamma_a = \mathcal{B} + \frac{\hat{\mathcal{B}}_a^2}{2D}(\mathsf{x}_1 + \mathsf{x}_2 - 2\mathcal{B}_s\mathsf{x}_1\mathsf{x}_2),$$

$$\gamma_s = -\frac{2}{D}(\mathsf{x}_1 + \mathsf{x}_2 - 2\mathcal{B}_a\mathsf{x}_1\mathsf{x}_2),$$

with $D = 1 - \frac{1}{2}(\mathcal{B}_a + \mathcal{B}_s)(\mathsf{x}_1 + \mathsf{x}_2) + \mathcal{B}_a\mathcal{B}_s\mathsf{x}_1\mathsf{x}_2$. Note that the parameters used in the main document are linked to those defined in [?] by the relations

$$\delta_a = \frac{\pi}{4}\mathcal{B}_a, \quad \delta_s = \frac{\pi}{4}\mathcal{B}_s, \quad \delta_0 = \frac{\pi}{8}(\mathcal{B}_s + \mathcal{B}_a - \mathcal{B}), \quad \delta = \frac{\pi}{4}\hat{\mathcal{B}}_a.$$

which allow for more compact expressions. In the main document, Eqs. (8-9) correspond to Taylor expansions of (R, T) in (3) near the resonance at $kb = \pi/2$ (and (4-5) to the case of identical channels with $b_1 = b_2$ as in [?]).

An important result given by this analysis is that two separate transmission zeros are obtained and, depending on the sign of

$$\Delta = \Delta_0 + \left(\frac{\pi b}{2a}\right)^2\varepsilon^2, \quad \Delta_0 = (\delta_0 - \delta_a)^2 - \delta^2, \tag{4}$$

they correspond to real ($\Delta > 0$) or complex conjugate ($\Delta < 0$) wavenumbers k. We also stress that in (4) Δ_0 depends only on the geometry of the junction as the dimensionless parameters $(\delta_a, \delta_s, \delta_0, \delta)$ do, and the dependence of Δ on $\varepsilon = (b_2 - b_1)/(b_1 + b_2)$ is explicit.

II. DERIVATION OF THE RELATION BETWEEN A_a AND A_s

The analysis of the experimental results in figure 6 (main text) suggests that PA is achieved for a particular balance between the symmetric $A_s = (A_1 + A_2)$ and antisymmetric $A_s = (A_1 - A_2)$ complex amplitudes in the channels, of the form $A_s = i\tan(ka/2)A_a$. Below we show that this relationship is consistent with a calculation in which the two channels are replaced by two point sources separated by a distance a/2 (at the center bottom of each channels) and imposing corresponding amplitudes A_1 and A_2. To do so, we introduce the Green's function $g(x, y)$ satisfying

$$(\nabla^2 + k^2)g(x, y) = 0, \qquad x \in (-\infty, +\infty), \quad y \in (0, d),$$

along with the boundary conditions $\frac{\partial g}{\partial y}(x, 0) = 0$, and $\frac{\partial g}{\partial y}(x, d) = \delta(x)$. We then have

$$\lim_{|x|\to+\infty} g(x, y) = \frac{ie^{ik|x|}}{2kd},$$

see *e.g.* [**?**]. We now consider the field $\varphi(x, y)$ generated in the waveguide by two point sources on the upper wall at $y = d$, one located at $x = -a/2$ of amplitude A_1 and one located at $x = a/2$ of amplitude A_2. From above we have

$$\lim_{|x|\to+\infty} \varphi(x, y) = \frac{i}{2kd}\left(A_1 e^{ik|x+a/2|} + A_2 e^{ik|x-a/2|}\right).$$

Highly non-symmetric emission by these two sources is obtained, for what we are interested in, when the radiated field vanishes at $x \to -\infty$, which is obtained when $A_1 = -A_2 e^{ika}$. We notice that this is equivalent to superposing a symmetric source with amplitude $A_s = (A_1 + A_2)$ and a anti-symmetric source with amplitude $A_a = (A_1 - A_2)$ satisfying

$$A_s = i\tan(ka/2)A_a,$$

as observed experimentally for which we have $\tan(ka/2) = 0.48$ at the PA.