Approximation to acoustic wave scattering by a small aperture in a thin screen across a duct: three ways

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Abstract

Approximating solutions of complicated boundary-value problems arising from applications in some asymptotic regime serves two obviously useful purposes. First, it often provides a simple accessible formula which can be used to inform the behaviour of a physical process subject to some limit of an important physical parameter. Second, it can be used to test the robustness of more general computational methods.

In this report, we consider a prototype problem in wave diffraction theory involving the reflection and transmission of acoustic waves by small gaps in thin screens. Three different methods are described for deriving an approximate formula for the reflection coefficient.

1 The problem

We consider the reflection and transmission of two-dimensional plane acoustic waves propagating along a waveguide of width $2d$ with sound-hard walls along $y = \pm d$ when they interact with an infinitely thin sound-hard baffle at $x = 0$ occupying $a < |y| < d$. The gap $|y| < a$ in the barrier allows energy to be transmitted from the incoming waves from $x = \infty$.

Because of the symmetry in the solution about $y = 0$, a boundary-value problem can be formulated in the strip $0 < y < d$, $-\infty < x < \infty$ and its solution (see, for example, Porter & Evans (1996)) can be reduced to solving the integral equation

$$\int_0^1 u(T)K(Y,T)dT = 1, \qquad 0 < Y < 1 \tag{1}$$

from which the reflection coefficient is given as

$$R = 1/(1 + iA/2kd) \tag{2}$$

(and the transmission coefficient is $T = 1 - R$) where

$$A = \int_0^1 u(T)dT \tag{3}$$

whilst

$$K(Y,T) = \sum_{r=1}^{\infty} \frac{\cos(r\pi Y\epsilon)\cos(r\pi T\epsilon)}{\alpha_r d}, \qquad \alpha_r d = \sqrt{(r\pi)^2 - (kd)^2}. \tag{4}$$

In the above, capitalised independent variables indicate a scaling based on the gap in the barrier; i.e. $y = aY$ and $t = aT$.

We are interested in the solution when $\epsilon = a/d \ll 1$ and have assumed $kd < \pi$ so that only one mode is able to propagate in the waveguide.

2 Method 1: Approximate the kernel

We can write

$$K(Y,T) = -\frac{1}{2\pi}\ln(2|\cos(\pi Y\epsilon) - \cos(\pi T\epsilon)|) + \sum_{r=1}^{\infty}\left(\frac{1}{\alpha_r d} - \frac{1}{r\pi}\right)\cos(r\pi Y\epsilon)\cos(r\pi T\epsilon). \tag{5}$$

If we assume ϵ is small and retain leading order terms only we have

$$K(Y,T) \sim -\frac{1}{2\pi}\ln|Y^2 - T^2| + C \tag{6}$$

where

$$C = -\frac{1}{\pi}\ln(\pi\epsilon) + S(kd) \tag{7}$$

and

$$S(kd) = \sum_{r=1}^{\infty}\left(\frac{1}{\alpha_r d} - \frac{1}{r\pi}\right). \tag{8}$$

Using this in the integral equation and extending $u(T)$ as an even function into $T < 0$ gives

$$\int_{-1}^{1} u(T)\ln|Y - T|dT = 2\pi(CA - 1), \qquad -1 < Y < 1. \tag{9}$$

The solution of an integral equation with a log-difference kernel and a constant on the right hand side is well-known and we find

$$u(T) = \frac{2(CA-1)}{-\ln(2)\sqrt{1-T^2}}. \tag{10}$$

Using this in (3) gives

$$A = \frac{\pi(CA-1)}{-\ln(2)} \tag{11}$$

and it follows that

$$A = \frac{1}{C + (1/\pi)\ln(2)} = \frac{1}{-\frac{1}{\pi}\log(\pi\epsilon/2) + S(kd)}. \tag{12}$$

The accuracy of this approximation is illustrated in the paper of Evans & Porter (2017).

3 Method 2: Variational approximation

The starting point here stems from an approach which can be used to numerically approximate solutions to the problem for *any* gap size to arbitrary precision and use this as the basis to an approximation for small gap size.

Using Galerkin's method to approximate the solution of the integral equation numerically (e.g. Porter & Evans (1996), Evans & Porter (2017)) gives

$$A \approx a_0 \tag{13}$$

where a_0 is the first unknown coefficient in the solution to

$$\sum_{n=0}^{N} a_n K_{mn} = \delta_{m0}, \qquad m = 0, 1, \ldots, N \tag{14}$$

and

$$K_{mn} = \sum_{r=1}^{\infty} \frac{J_{2n}(r\pi\epsilon)J_{2m}(r\pi\epsilon)}{\alpha_r d}. \tag{15}$$

Here N is a numerical parameter which represents the number of terms used in an expansion of the unknown function $u(T)$ and which, when increased, gives higher accuracy to solutions.

As $\epsilon \to 0$, $K_{mn} \to 0$ for all m, n apart from $m = n = 0$ and so

$$A \approx 1/K_{00} \tag{16}$$

where

$$K_{00} = \sum_{r=1}^{\infty} \frac{J_0^2(r\pi\epsilon)}{\alpha_r d}. \tag{17}$$

We write this as

$$K_{00} = \sum_{r=1}^{\infty} \left(\frac{1}{\alpha_r d} - \frac{1}{r\pi} \right) J_0^2(r\pi\epsilon) + E \sim S(kd) + E \tag{18}$$

to leading order in ϵ where

$$E = \frac{1}{\pi} \sum_{r=1}^{\infty} \frac{J_0^2(r\pi\epsilon)}{r}. \tag{19}$$

We need to be careful now and write this as

$$E = \frac{1}{\pi} \sum_{r=1}^{Q} \left(\frac{J_0^2(r\pi\epsilon)}{r\pi\epsilon} - \frac{1}{r\pi\epsilon} \right) \pi\epsilon + \frac{1}{\pi} \sum_{r=1}^{Q} \frac{1}{r} + \frac{1}{\pi} \sum_{r=Q+1}^{\infty} \frac{J_0^2(r\pi\epsilon)}{r\pi\epsilon} \pi\epsilon \tag{20}$$

and convert the two series to integrals on assumption $\epsilon \to 0$ so that

$$E = \frac{1}{\pi} \int_{\pi\epsilon}^{(Q+1)\pi\epsilon} \left(\frac{J_0^2(x)}{x} - \frac{1}{x} \right) dx + \int_{(Q+1)\pi\epsilon}^{\infty} \frac{J_0^2(x)}{x} dx + \frac{1}{\pi} \sum_{r=1}^{Q} \frac{1}{r}. \tag{21}$$

Note, the integrands need to be bounded for this to work accurately, hence the need for the subtraction of the singularity at the origin. The value of Q is arbitrary by must scale like $1/\epsilon$, implying $Q \to \infty$ and this allows us to write

$$E = \frac{\gamma}{\pi} + \frac{1}{\pi} \int_{\pi\epsilon}^{\infty} \frac{J_0^2(x)}{x} dx \tag{22}$$

where $\gamma = 0.5772\ldots$.

Using wolfram alpha (how to do by hand ?) we find that

$$E = \frac{\gamma}{\pi} + \frac{1}{\pi} \int_{\pi\epsilon}^{\infty} \frac{J_0^2(x)}{x} dx = -\frac{1}{\pi} \log(\pi\epsilon/2) \tag{23}$$

as $\epsilon \to 0$.

It follows that

$$A = \frac{1}{-\frac{1}{\pi} \log(\pi\epsilon/2) + S(kd)} \tag{24}$$

as before.

4 Method 3: Matched Asymptotic Expansion

Finally we approach the problem using a matched asymptotic expansion method. In $x > 0$, $0 < y < d$ (unscaled coordinates) we have

$$\phi(x, y) \approx 2 \cos kx + mG(x, y) \tag{25}$$

where

$$G(x, y) = \frac{i}{2kd} \mathrm{e}^{ikx} + \sum_{n=1}^{\infty} \frac{\mathrm{e}^{-\alpha_n x}}{\alpha_n d} \cos(n\pi y/d) \tag{26}$$

and $\alpha_n d = \sqrt{(n\pi)^2 - (kd)^2}$ and it follows from the far field that that

$$R = 1 + im/2kd \tag{27}$$

In $x < 0$,

$$\phi(x, y) \approx mG(-x, y) \tag{28}$$

so that $T = -im/2kd$ and $R + T = 1$. Here, m is an unknown source strength. Letting $r = \sqrt{x^2 + y^2}$ we find that as $r \to 0$,

$$G(x, y) \sim -\frac{1}{\pi} \ln r + (i/2kd) + S(kd) - \frac{1}{\pi} \ln(\pi/d). \tag{29}$$

Thus, the inner expansion of the outer solution in $x > 0$ is

$$\phi(x, y) \sim 2 - \frac{m}{\pi} \ln r + (im/2kd) + mS(kd) - \frac{m}{\pi} \ln(\pi/d) \tag{30}$$

and of the outer solution in $x < 0$ is

$$\phi(x, y) \sim \frac{m}{\pi} \ln r - (im/2kd) - mS(kd) + \frac{m}{\pi} \ln(\pi/d). \tag{31}$$

In the inner region, variables are scaled by $a \ll 1$ so that $(x, y) = a(X, Y)$ and $\phi(x, y) \equiv \Phi(X, Y)$. On this scaling $(\nabla^2 + k^2)\phi = 0$ becomes $(\nabla^2_{XY} + (ka)^2)\Phi = 0$ which is approximated by $\nabla^2_{XY}\Phi = 0$. We use the complex variable $z = X + iY$ and define a Schwarz-Christoffel mapping via

$$z = f(w) = -\frac{1}{4} \int_w^1 \frac{(s+1)}{s^{3/2}} ds = \frac{1}{2} \left(w^{1/2} - w^{-1/2} \right). \tag{32}$$

Thus, f maps the positive real w-axis to the line $y = 0$ and the negative real w-axis to $x = 0$, $1 \le y < \infty$. The point $w = -1$ corresponds to $z = i$, the end of the barrier.

The flow through the barrier is represented by a source at the origin in the w-plane given by the potential

$$\Psi = -\frac{M}{2\pi} \ln |w| + D \tag{33}$$

where M and D are constants to be determined.

As $|w| \to \infty$, $|w| = 4r^2/a^2$ and so the outer expansion of the inner solution into $x > 0$ is

$$\phi(x, y) \sim -\frac{M}{\pi} \ln(2r/a) + D. \tag{34}$$

Matching with (30) means that $M = m$ and

$$D - (m/\pi)\ln(2/a) = 2 + (im/2kd) + mS(kd) - \frac{m}{\pi}\ln(\pi/d). \tag{35}$$

As $|w| \to 0$, $|w| \sim a^2/4r^2$ and so the outer expansion into $x < 0$ of the inner solution is

$$\phi(x,y) \sim \frac{M}{\pi}\ln(2r/a) + D \tag{36}$$

as $r \to \infty$. Matching with (31) above gives $M = m$, as we already know, and

$$D + (m/\pi)\ln(2/a) = -(im/2kd) - mS(kd) + \frac{m}{\pi}\ln(\pi/d). \tag{37}$$

Eliminating D and m between (36) and (37) gives $D = 1$ and

$$m = \frac{-1}{\dfrac{i}{2kd} + S(kd) - \dfrac{1}{\pi}\ln(\pi\epsilon/2)}. \tag{38}$$

Using this in (27) and rearranging we find that

$$R = 1/(1 + iA/2kd) \tag{39}$$

with

$$A = \frac{1}{S(kd) - \dfrac{1}{\pi}\ln(\pi\epsilon/2)} \tag{40}$$

as before.

References

[1] R. Porter & D.V. Evans, 1996. Wave scattering by periodic arrays of breakwaters. *Wave Motion*, **23**, 97-120.

[2] D.V. Evans & R. Porter, 2017. Total transmission of waves through narrow gaps in channels *Q. J. Mech. Appl. Math.* **70**(1): 87-101.