Eigenfrequencies and eigenmodes for a thin rectangular elastic plate with free edges

R. Porter

School of Mathematics, University of Bristol, Bristol, BS8 1TW, UK.

April 6, 2017

1 Introduction

This technical report describes work carried out by the author in March 2017 in preparation for conference talks to be given at the British Applied Mathematics Colloquium at the University of Surrey in April 2017 and on a Isaac Newton Institute program on sea ice at Cambridge in the Autumn of 2017.

The document outlines the mathematical details needed to compute eigensolutions describing the free oscillations of an unloaded thin rectangular elastic plate with four free edges. We essentially reproduce the solution of Ritz (1909) by using the variational method invented precisely to solve this problem. Ritz was able to describe mathematically the frequencies and mode shapes recorded in experiments by Chladni (1787). For a comprehensive review of the general eigenproblem involving thin rectangular elastic plates with all 21 possible combinations of boundary condition, see Leissa (1969), Leissa (1973). The excellent review article by Gander & Wanner (2012) is more recent and provides not only a wonderful description of the history of the problem of the free elastic plate but also a solution and code to accompany it. Gander & Wanner (2012) follow the analysis of Ritz and assume square plates with diagonal symmetries. In their code, Gander & Wanner (2012) take advantage of a symbolic algebra package to perform certain integrals even though they can be done exactly.

Most people interested in solutions to this problem will be able to use the references available to determine solutions themselves. Indeed, this is what I have done, although determining the details apparently not available elsewhere has been more painful than I imagined. This document simply aims to bypass the pain of that derivation and provide necessary computational details (and a brief outline of the derivation) for anyone interested in evaluating solutions. Thus, here, general aspect rectangular plates are assumed and we compute all integrals by hand to provide an explicit numerical recipe for computing the eigenmodes and eigenfrequencies for a rectangular elastic plate with free edges. A selection of results are given to illustrate the output from this approach and to compare with existing solutions and report other benchmark solutions. Fortran 77 code which uses a NAG library call for computing eigenvalues/vectors of real symmetric matrices is also provided as an accompanying online link.

It is quite possible the details provided here can found in other references although they were not found in the preparation of this work. Enjoy.

2 Equations governing the free motions of a two-dimensional elastic plate

An unloaded thin elastic plate occupies the rectangular region $(x, y) \in \mathcal{D} = \{(x, y)| -a < x < a, -b < y < b\}$. The time harmonic vibrations perpendicular to the plane it occupies are described by the function $\Re\{W(x, y)\mathrm{e}^{-\mathrm{i}\omega t}\}$ and, according to Kirchhoff-Love plate theory, $W(x, y)$ satisfies

$$(\nabla^2\nabla^2 - \lambda)W = 0 \tag{2.1}$$

where $\nabla^2 = \partial_{xx} + \partial_{yy}$ is the two-dimensional Laplacian and $\lambda = \rho_s d\omega^2/D$ in terms of ρ_s, the areal density of the plate, d, its thickness, and D, the flexural rigidity defined as $\frac{1}{12}Ed^3/(1-\nu^2)$ in terms of the Young's modulus E and Poisson's ratio ν.

The four straight edges of the plate are free, requiring $W(x, y)$ to satisfy

$$\left.\begin{aligned} (\mathcal{B}W) &\equiv W_{nn} + \nu W_{ss} = 0, \\ (\mathcal{S}W) &\equiv W_{nnn} + (2-\nu)W_{nss} = 0, \end{aligned}\right\} \qquad \text{on } |x| = a,\ |y| < b \text{ and on } |y| = b,\ |x| < a. \tag{2.2}$$

where n and s are used, respectively, to denote derivatives normal and tangential to the edge. The four corners are free of twisting moments implying that

$$W_{ns} = 0 \tag{2.3}$$

as the corner is approached along the edge. Non-straight free edges require additional terms related to curvature in the boundary conditions stated above.

The aim is to find the eigenvalues λ and the corresponding eigenmodes $W(x, y)$ for the problem stated above. For rectangular geometries these cannot be determined explicitly.

2.1 The one-dimensional problem

For the one-dimensional analogue of this problem, namely the identification of free bending modes of the Euler-Bernoulli beam equation, solutions are explicit. Thus, the eigenmodes, $w_n(t)$ say, and eigenvalues, k_n^4 say, satisfying the ordinary differential equation

$$w_n''''(t) - k_n^4 w_n(t) = 0, \qquad |t| < 1 \tag{2.4}$$

and free-edge boundary conditions $w_n'''(\pm 1) = w_n''(\pm 1) = 0$ are given by

$$w_0(t) = \frac{1}{2}, \qquad (k_0 = 0) \tag{2.5}$$

$$w_1(t) = \frac{1}{2}t, \qquad (k_1 = 0) \tag{2.6}$$

$$w_{2n}(t) = \frac{1}{4}\left(\frac{\cosh k_{2n}t}{\cosh k_{2n}} + \frac{\cos k_{2n}t}{\cos k_{2n}}\right), \qquad (\tanh k_{2n} + \tan k_{2n} = 0) \tag{2.7}$$

for $n \geq 1$ and

$$w_{2n+1}(t) = \frac{1}{4}\left(\frac{\sinh k_{2n+1}t}{\sinh k_{2n+1}} + \frac{\sin k_{2n+1}t}{\sin k_{2n+1}}\right), \qquad (\tanh k_{2n+1} - \tan k_{2n+1} = 0) \tag{2.8}$$

again, $n \geq 1$. The brackets contain the relations satisfied by k_n, an increasing sequence of positive values beyond k_1.

The eigenmodes defined above are orthogonal, so that

$$\int_{-1}^{1} w_m(t)w_n(t)\,dt = C_n^2\delta_{mn}, \qquad m,n \geq 0. \tag{2.9}$$

where $C_0 = \sqrt{\frac{1}{2}}$, $C_1 = \sqrt{\frac{1}{6}}$ and $C_n = \sqrt{\frac{1}{8}}$ for $n \geq 2$.

2.2 A variational principle

Consider the functional

$$\mathcal{L}(W) = \frac{1}{2}\int_{-b}^{b}\int_{-a}^{a}\{(\nabla^2 W)^2 - 2(1-\nu)(W_{xx}W_{yy} - W_{xy}^2) - \lambda W^2\}\,dxdy. \tag{2.10}$$

With some work it can be shown that this satisfies

$$\delta\mathcal{L} = \int_{-b}^{b}\int_{-a}^{a}\delta W(\nabla^2\nabla^2 - \lambda)W\,dxdy + \oint_C ((\mathcal{S}W)\delta W - (\mathcal{B}W)(\delta W)_n)\,ds + 2(1-\nu)[W_{xy}\delta W]_{corners}. \tag{2.11}$$

The closed loop integral is to be interpreted as the union of four distinct integrals along the four edges of the rectangle. Thus, $\mathcal{L}$ is stationary at the solution of the boundary-value problem in §2.1 subject to arbitrary variations in W and the normal component of its gradient. We use the variational principle to approximate eigensolutions using Ritz's method.

2.3 Approximation by Ritz's method

We write

$$W(x,y) \approx \sum_{m=0}^{N}\sum_{n=0}^{N}\alpha_{m,n}\frac{w_m(x/a)w_n(y/b)}{C_mC_n} \tag{2.12}$$

where the test functions are the eigenmodes of the one-dimensional beam equation with normalising factors in the denominator. These do not satisfy the exact free edge conditions but do allow arbitrary variations in the function and in its first derivative.

Application of the Ritz method (i.e. substituting (2.12) into (2.10) and making $\mathcal{L}$ stationary with respect to $\alpha_{m,n}$) results in the unknown coefficients (and the eigenvalue λ) satisfying the system of equations

$$\sum_{m=0}^{N}\sum_{n=0}^{N}\alpha_{m,n}\left(\frac{K_{m,n,p,q}}{C_nC_mC_pC_q} - \lambda a^4 I_{m,n,p,q}\right) = 0 \tag{2.13}$$

for $p,q = 0,1,\ldots,N$ where

$$K_{m,n,p,q} = \int_{-1}^{1}\int_{-1}^{1}\left(w_m''(t)w_n(u) + \mu^2 w_m(t)w_n''(u)\right)\left(w_p''(t)w_q(u) + \mu^2 w_p(t)w_q''(u)\right)\,dtdu$$
$$-(1-\nu)\mu^2\int_{-1}^{1}\int_{-1}^{1}\left(w_m''(t)w_n(u)w_p(t)w_q''(u) + w_m(t)w_n''(u)w_p''(t)w_q(u) - 2w_m'(t)w_n'(u)w_p'(t)w_q'(u)\right)dtdu \tag{2.14}$$

with $\mu = a/b$ while I is the Identity matrix with entries $I_{m,n,p,q} = \delta_{mp}\delta_{nq}$.

Now

$$\int_{-1}^{1} w_m''(t)w_p''(t)\,dt = k_m^4\delta_{mp} \tag{2.15}$$

by integrating by parts and using the ODE and BCs for $w_m(t)$ along with the orthogonality condition. Also

$$\int_{-1}^{1} w_m'(t)w_p'(t)\,dt = L_m - J_{m,p} \tag{2.16}$$

where

$$L_m = [w_m'(t)w_p(t)]_{-1}^{1} \qquad \text{and} \qquad J_{m,p} = \int_{-1}^{1} w_m''(t)w_p(t)\,dt. \tag{2.17}$$

It follows that

$$K_{m,n,p,q} = (k_m^4 + \mu^4 k_n^4)C_mC_nC_pC_q\delta_{mp}\delta_{nq} + \nu\mu^2(J_{m,p}J_{q,n} + J_{p,m}J_{n,q}) + 2\mu^2(1-\nu)(L_m - J_{m,p})(L_n - J_{n,q}). \tag{2.18}$$

The four intrinsic symmetry classes of a rectangular plate mean that many of these elements are zero and (2.13) can be decoupled into four separate systems for each symmetry class.

In practical terms, we take advantage of this from the outset and write in place of (2.12)

$$W^{(\mu\nu)}(x,y) = \sum_{m=0}^{N}\sum_{n=0}^{N} \alpha_{2m+\mu,2n+\nu}\frac{w_{2m+\mu}(x/a)w_{2n+\nu}(y/b)}{C_{2m+\mu}C_{2n+\nu}} \tag{2.19}$$

with $\mu = 0,1$, $\nu = 0,1$ to denote symmetry/antisymmetry in x and y respectively. Note: it is an unfortunate accident that μ and ν are simultaneously used as both parameters and indices; however, it should be clear which values they take in relation to the context of their use.

After using this, (2.13) is replaced with four uncoupled equations for each symmetry group:

$$\sum_{m=0}^{N}\sum_{n=0}^{N} \alpha_{2m+\mu,2n+\nu}\left(\frac{K_{2m+\mu,2n+\nu,2p+\mu,2q+\nu}}{C_{2m+\mu}C_{2n+\nu}C_{2p+\mu}C_{2q+\nu}} - \lambda^{(\mu\nu)}a^4 I_{2m+\mu,2n+\nu,2p+\mu,2q+\nu}\right) = 0. \tag{2.20}$$

The terms required to calculate the necessary elements of the matrix $K_{m,n,p,q}$ directly are as follows. First

$$L_{2m} = \tfrac{1}{2}k_{2m}\tanh k_{2m} \tag{2.21}$$

for all m, and

$$J_{2m,2p} = \frac{L_{2m} - L_{2p}}{1 - k_{2p}^4/k_{2m}^4} \tag{2.22}$$

with special cases $J_{0,2p} = 0$ for all p, $J_{2m,0} = L_{2m} - L_0$ for $m \geq 0$ and

$$J_{2m,2m} = \tfrac{1}{4}L_{2m} - \tfrac{1}{2}L_{2m}^2 \tag{2.23}$$

for $m > 0$.

Next, we have

$$L_{2m+1} = \tfrac{1}{2}k_{2m+1}\coth k_{2m+1} \tag{2.24}$$

for $m > 0$ and $L_1 = \frac{1}{2}$ whilst

$$J_{2m+1,2p+1} = \frac{L_{2m+1} - L_{2p+1}}{1 - k_{2p+1}^4/k_{2m+1}^4} \tag{2.25}$$

with $J_{1,2p+1} = 0$, for all $p \geq 0$, $J_{2m+1,1} = L_{2m+1} - L_1$ and when $m = p \neq 0$

$$J_{2m+1,2m+1} = \tfrac{1}{4}L_{2m+1} - \tfrac{1}{2}L_{2m+1}^2 \tag{2.26}$$

and this is all we need.

Numerically, we compute the eigenvalues and eigenvectors of the four truncated real symmetric matrices with entries

$$K_{2m+\mu,2n+\nu,2p+\mu,2q+\nu}/(C_{2m+\mu}C_{2n+\nu}C_{2p+\mu}C_{2q+\nu}) \tag{2.27}$$

and assign them to $a^4\lambda_i^{(\mu\nu)}$; the corresponding eigenvectors represent $\alpha^{(i)}_{2m+\mu,2n+\nu}$ which can be then used to determine the ith eigenmode $W_i^{(\mu\nu)}(x,y)$ using (2.19).

2.4 Orthogonality of eigenmodes

Let $W_i(x,y)$ and $W_j(x,y)$ be any two eigenmodes belonging to the same set of any one of the four different symmetry classes $\mu,\nu=0,1$ and having eigenvalues λ_i and λ_j respectively. Then consider

$$\begin{aligned}(\lambda_i-\lambda_j)\iint_{\mathcal{D}} W_iW_j\,dxdy &= \iint_{\mathcal{D}}\left(W_j\nabla^2\nabla^2W_i - W_i\nabla^2\nabla^2W_j\right)\,dxdy \\ &= \oint_{\mathcal{C}}\left(W_j\partial_n\nabla^2W_i - W_i\partial_n\nabla^2W_j - \partial_nW_j\nabla^2W_i + \partial_nW_i\nabla^2W_j\right)\,dxdy \end{aligned}\tag{2.28}$$

where $\mathcal{C}$ is the closed boundary of $\mathcal{D}$ after integrating by parts (via Green's identity) twice. Use of the free-edge boundary conditions with zero twisting moments on the corners can be used to show that the right-hand side is zero. Within each symmetry group, the eigenvalues λ_i are distinct and so it follows that

$$\frac{1}{ab}\int_{-b}^{b}\int_{-a}^{a} W_i^{(\mu\nu)}(x,y)W_j^{(\mu\nu)}(x,y)\,dxdy = \delta_{ij}E_i^{(\mu\nu)} \tag{2.29}$$

where

$$E_i^{(\mu\nu)} = \sum_{m=0}^{N}\sum_{n=0}^{N}\{\alpha^{(i)}_{2m+\mu,2n+\nu}\}^2 \tag{2.30}$$

in terms of the eigenvector $\alpha^{(i)}_{2m+\mu,2n+\nu}$ associated with the ith eigenmode.

Eigenmodes belonging to different symmetry groups are clearly orthogonal.

3 Results

In the following we have presented computed values of the dimensionless eigenvalue parameters $\widehat{\lambda} = \lambda a^4 = \rho_s d\omega^2 a^4/D$.

In Tab. 1 the results of computations are displayed for a square plate $a/b=1$ for the first five eigenvalues in each of the symmetry groups. Being square, the results for symmetry in x and antisymmetry in y (SA) are identical to those for antisymmetry in x and symmetry in y (AS). The 'fundamental' eigenvalue for SS, SA and AS corresponding to rigid-plate motions are all zero, as expected.

Using $N=12,24$ and $N=48$ in computations shown in Tab. 1 provide an indication of accuracy of results. They confirm what has previously been observed (e.g. Leissa (1973)) that the the convergence is slow. The results are accurate enough for the purposes intended for this work. Since the matrix size is $(N+1)^2$, computational effort increases significantly with increasing N (presumably $O(N^6)$ or close to this). I.e. doubling N results in 64 times the computational effort.

The results shown in Tab. 1 and accompanying mode shapes (illustrated by nodal lines) in Fig. 1 compare well with those reported in the literature, although the focus of Leissa (1969) and Gander & Wanner (2012) appears to be in reproducing the results of Ritz (1909) rather than seeking to improve on their accuracy.

Mode	SS	SA = AS	AA
1	0.0000	0.0000	12.4576
-	0.0000	0.0000	12.4562
-	0.0000	0.0000	12.4552
2	26.0289	81.0237	321.605
-	26.0032	80.9676	321.214
-	25.9900	80.9341	320.969
3	35.6911	235.804	375.938
-	35.6638	235.606	375.604
-	35.6495	235.500	375.420
4	269.639	731.221	1527.72
-	269.520	730.811	1527.14
-	269.437	730.513	1526.65
5	878.033	1106.35	2693.00
-	877.212	1105.05	2689.43
-	876.747	1104.26	2687.15

Table 1: Computations of the first five values of dimensionless eigenvalues $\widehat{\lambda}_i^{(\mu\nu)}$ for $i = 1, 2, 3, 4$ and $\mu = 0, 1$(S,A), $\nu = 0, 1$(A,S) for $a/b = 1$ (a square plate) and a Poisson ratio of 0.225. Values of $N = 12, 24, 48$ (in order) are recorded against each mode.

In Tab. 2 we present similar numbers for $a/b = 2$. Now the SA modes are distinct from the AS modes. Computations are now shown only with $N = 24$. Fig. 2 shows the eigenmodes corresponding to the top four lines of Tab. 2.

References

[1] Chladni, E. F. F., (1787), Entdeckungen über die Theorie des Klanges. Leipzig.

[2] Gander, M. J. & Wanner, G., (2012), From Euler, Ritz, and Galerkin to modern computing. SIAM Review **54**(4) 627-666.

[3] Leissa, A. W. (1969), Vibrations of plates. NASA SP-160. Technical report, NATIONAL AERONAUTICS AND SPACE ADMINISTRATION.

[4] Leissa, A. W. (1973), The free vibration of rectangular plates, J. Sound Vib., **31** 257-293.

[5] Ritz, W. (1909), Theorie der transversalschwingungen einer quadratischen platte mit freien rändern. Ann. Phys., **18** 737-807.

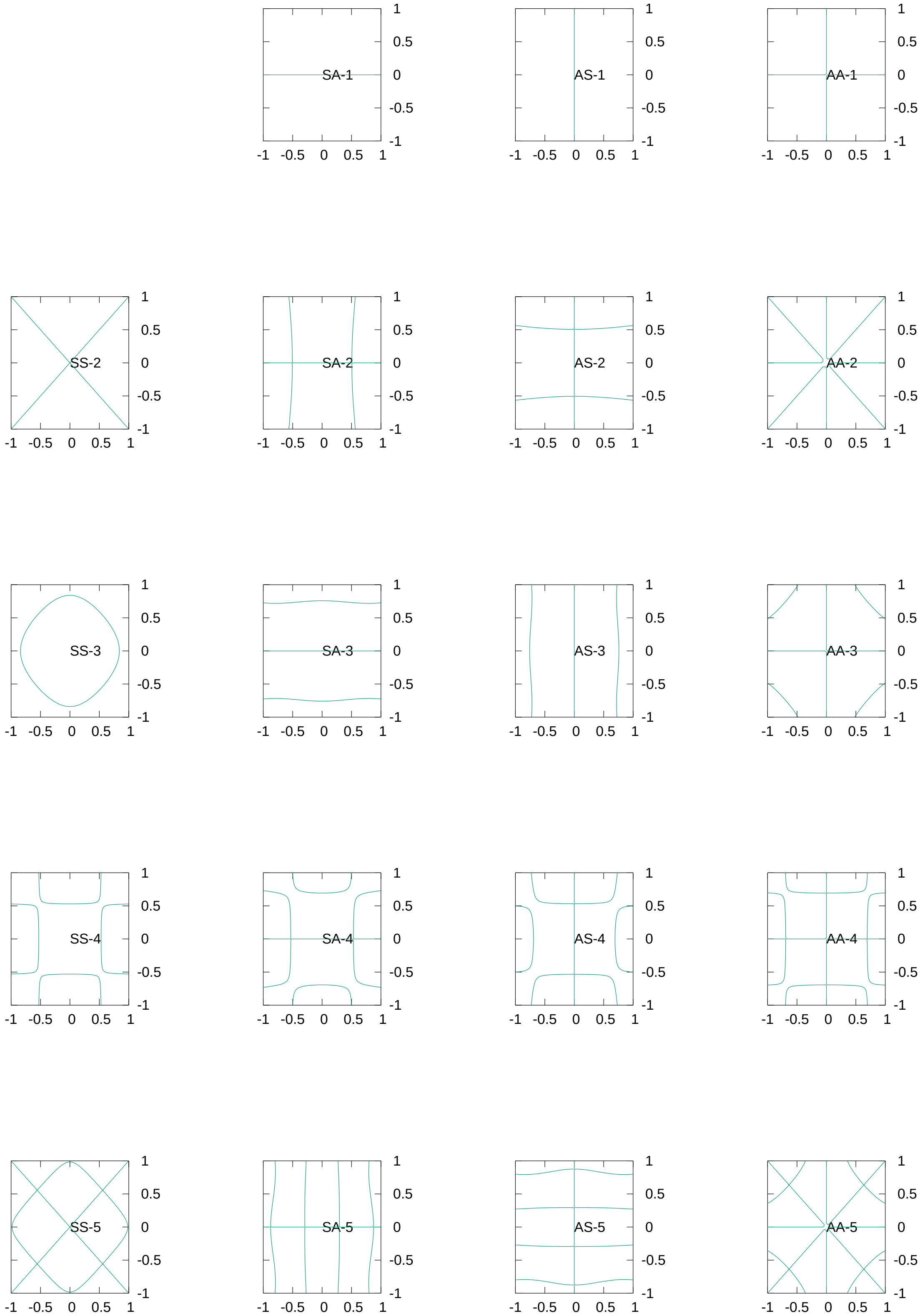

Figure 1: A matrix of nodal lines in the solutions to the first plate 5 eigenmodes (going downwards) in each symmetry group (going across) from SS, SA, AS, AA corresponding to eigenvalues determined in Tab. 1, for $a/b = 1$, $\nu = 0.225$ and with $N = 24$. The axes are non-dimensionalised with respect to a.

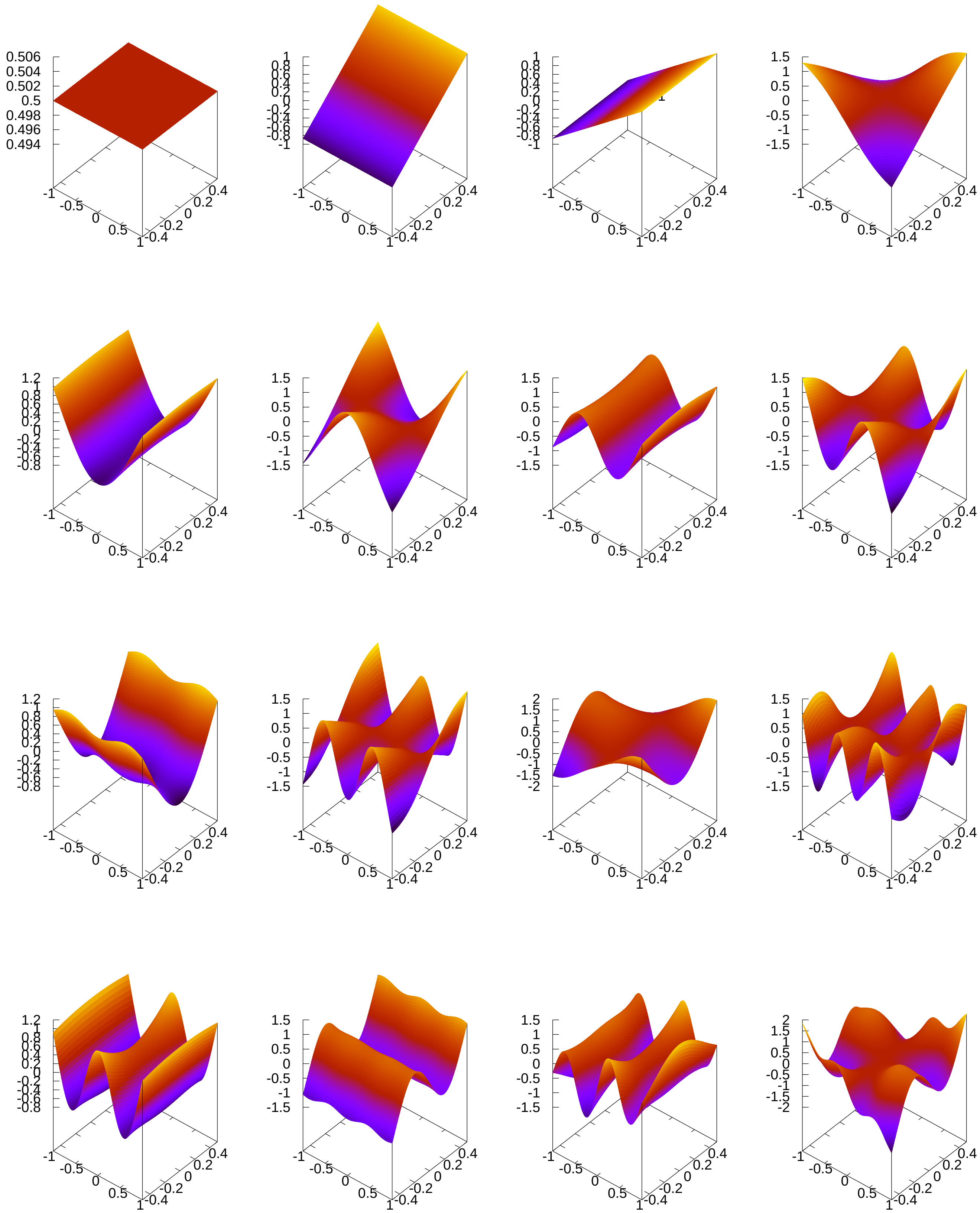

Figure 2: A matrix of the first plate 4 eigenmodes (going downwards) in each symmetry group (going across) from SS, SA, AS, AA corresponding to eigenvalues determined in Tab. 2, for $a/b = 2$, $\nu = 0.225$ and with $N = 24$. The horizontal axes are non-dimensionalised with respect to a.

Mode	SS	SA	AS	AA
1	0.00000	0.00000	0.00000	48.58585
2	29.9240	231.847	229.455	686.9604
3	492.131	1688.91	697.557	3623.084
4	897.781	3781.80	2434.12	4287.587
5	1356.85	5633.82	2632.19	8217.760
6	4694.13	7285.95	8079.79	13115.61
7	5524.65	12177.5	10760.8	17925.72
8	13264.6	22114.2	15329.0	35225.41

Table 2: Computations of the first eight values of dimensionless eigenvalues $\widehat{\lambda}_i^{(\mu\nu)}$ for $i = 1, 2, \ldots, 8$ and $\mu = 0, 1$(S,A), $\nu = 0, 1$(A,S) for $a/b = 2$ (a rectangle), and a Poisson ratio of 0.225. A value of $N = 24$ is used.