Scattering in a waveguide with narrow side channels

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Abstract

We consider an approximate solution based on matched asymptotic expansions to the problem of wave scattering by any number of narrow channels extending perpendicularly to one of the two straight parallel walls defining a uniform waveguide. The matching process results in a system of equations whose size equates to the number of side channels. Particular emphasis is placed on understanding the effect that channel resonance plays in the reflection of incident waves.

1 Introduction

The geometry is illustrated in Fig. 1. A wave is incident along a uniform waveguide from minus infinity and is partially reflected and transmitted by a series of narrow side channels extending in a direction perpendicular to the waveguide. We will consider N such channels each of the same width 2ϵ (presumed to be smaller than other lengthscales in the problem) but having different lengths and placed at arbitrary positions along the waveguide. All walls have a sound-hard (Neumann) condition placed upon them. The waves are supported by a inviscid compressible medium with phase speed c.

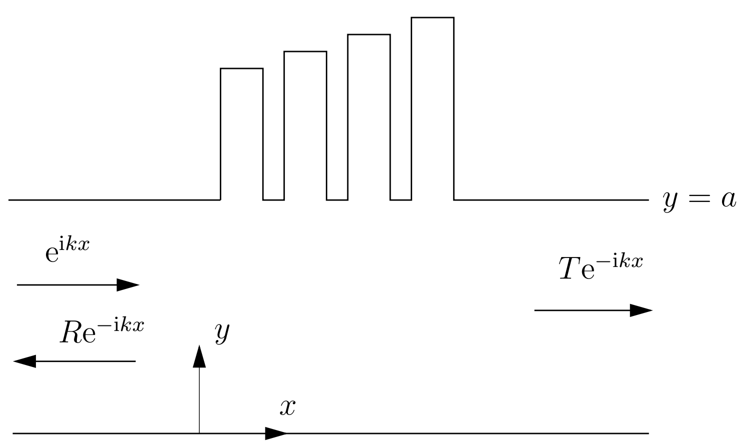

Figure 1: Definition sketch

We approach the solution to this problem using the method of matched asymptotic expansions. The solution in the waveguide is represented away from the side-channel openings as the superposition of an incident wave and wave sources on the waveguide wall, located at the mid-point of each side-channel opening. The solution in each narrow side-channel away from the opening is represented by waves propagating along the channel with no variation across the channel. Solutions, constructed in the vicinity of each opening are used to connect the two outer solutions.

2 Derivation of a wave source on a channel wall

Consider a parallel waveguide with walls along $y = 0$ and $y = a$ for $-\infty < x < \infty$. A wave source is placed on the upper wall at $(x, y) = (0, a)$. The solution, represented by the function $g(x, y)$, satisfies

$$(\nabla^2 + 1)g = 0, \qquad -\infty < x < \infty,\ 0 < y < a.$$

Note that a time-harmonic dependence propotional to $\mathrm{e}^{-\mathrm{i}\omega t}$ has been assumed and that length-scales have non-dimensionalised by the wavenumber $k = \omega/c$, where c is the wave speed. In addition

$$g_y(x, 0) = 0, \qquad g_y(x, a) = \delta(x), \qquad -\infty < x < \infty$$

where the delta function represents a point wave source on the wall. Partly for simplicity and partly because of the underlying assumptions of the method used, we restrict ourselves to the case $0 < a < \pi$ so only one wave mode is able to radiate from the wave source along the waveguide.

The solution can be found using Fourier transforms and expressed in the form

$$g(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\cosh \gamma y}{\gamma \sinh \gamma d} \mathrm{e}^{\mathrm{i}lx}\,\mathrm{d}l$$

where $\gamma = \sqrt{l^2 - 1} = -\mathrm{i}\sqrt{1 - l^2}$. There are poles at real values of $l = \pm 1$ where $\gamma = 0$, and the contour of integration is defined to pass above the pole at $l = -1$ and below the pole at $l = 1$ in order to to satisfy the radiation condition (waves generated by the source are outgoing). Thus deforming the inverse contour of integration to infinity in upper and lower half planes, depending on the sign of x, results in the series representation

$$g(x, y) = \frac{\mathrm{i}}{2d} \mathrm{e}^{\mathrm{i}|x|} + \sum_{n=1}^{\infty} \frac{\cos p_n(a - y)}{\gamma_n a} \mathrm{e}^{-\gamma_n |x|}$$

where $\gamma_n = \sqrt{p_n^2 - 1}$ and $p_n = n\pi/a$. This expression can be derived independently using separation solutions as a starting point. Thus, we see that

$$g(x, y) \sim \frac{\mathrm{i}}{2a} \mathrm{e}^{\mathrm{i}|x|}$$

as $|x| \to \infty$. We also need to determine the behaviour of $g(x, y)$ as the source point $(x, y) = (0, a)$ is approached. With this purpose in mind we write

$$g(x, y) = \frac{\mathrm{i}}{2a} \mathrm{e}^{\mathrm{i}|x|} + \sum_{n=1}^{\infty} \left\{ \frac{\mathrm{e}^{-\gamma_n |x|}}{\gamma_n a} - \frac{\mathrm{e}^{-p_n |x|}}{n\pi} \right\} \cos p_n(a - y) + \frac{S(x, y)}{\pi}$$

where

$$S(x, y) = \sum_{n=1}^{\infty} \frac{\cos p_n(a - y))}{n} \mathrm{e}^{-p_n |x|}.$$

We let $z = |x| + \mathrm{i}(a-y) = r\mathrm{e}^{\mathrm{i}\theta}$ where $r = \sqrt{x^2 + (a-y)^2}$ and $\theta = \tan^{-1}((a-y)/|x|)$ and then

$$S(x,y) = \Re\left\{\sum_{n=1}^{\infty} \frac{\mathrm{e}^{-n\pi z/a}}{n}\right\} = -\Re\{\ln(1-\mathrm{e}^{-\pi z/a})\} = -\Re\left\{-\frac{\pi z}{2a} + \ln(2\sinh(\pi z/2a))\right\}.$$

This gives us

$$S(x,y) = \frac{\pi|x|}{2a} - \ln\left(\frac{\pi r}{a}\right) - \Re\left\{\ln\left(\frac{\sinh(\pi z/2a)}{\pi z/2a}\right)\right\}.$$

Therefore as $r \to 0$

$$g(x,y) \sim -\frac{1}{\pi}\ln\left(\frac{\pi r}{a}\right) + C$$

retaining constant terms defined by

$$C = \frac{\mathrm{i}}{2a} + \sum_{n=1}^{\infty}\left(\frac{1}{\gamma_n a} - \frac{1}{n\pi}\right).$$

3 Multiple sources and incident waves

Consider the case where multiple sources are placed along the wall $y = a$ at $x = x_j$, $j = 1, 2, \ldots, N$ in the presence of an incident wave. The full solution in the waveguide guide is given by

$$\phi(x,y) = \mathrm{e}^{\mathrm{i}x} + \sum_{j=1}^{N} m_j g(x - x_j, y)$$

where m_j, $j = 1, 2, \ldots, N$ are as yet undetermined source strengths. Far along the waveguide we have supposed that

$$\phi(x,y) \sim \begin{cases} \mathrm{e}^{\mathrm{i}x} + R\mathrm{e}^{-\mathrm{i}x}, & x \to -\infty \\ T\mathrm{e}^{\mathrm{i}x}, & x \to \infty \end{cases}$$

so that the reflection and transmission coefficients R and T are given by

$$R = \frac{\mathrm{i}}{2a}\sum_{j=1}^{N} m_j \mathrm{e}^{\mathrm{i}x_j}, \qquad T = 1 + \frac{\mathrm{i}}{2a}\sum_{j=1}^{N} m_j \mathrm{e}^{-\mathrm{i}x_j}.$$

The solution in the vicinity of the kth source point (x_k, a) is approximated by

$$\phi(x,y) \sim \mathrm{e}^{\mathrm{i}x_k} + m_k\left\{-\frac{1}{\pi}\ln\left(\frac{\pi r_k}{a}\right) + C\right\} + \sum_{j=1,\ j\neq k}^{N} m_j g(x_k - x_j, 0)$$

where $r_k = \sqrt{(x-x_k)^2 + (a-y)^2}$. Here $g(x_k - x_j, 0)$ can be computed most efficiently, using expressions derived in the previous section, by

$$g(x_k - x_j, 0) = -\frac{1}{\pi}\ln\left(\frac{\pi X_{kj}}{a}\right) + \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}X_{kj}}}{2a} + \sum_{n=1}^{\infty}\left(\frac{\mathrm{e}^{-\gamma_n X_{kj}}}{\gamma_n a} - \frac{\mathrm{e}^{-p_n X_{kj}}}{n\pi}\right) + \frac{X_{kj}}{2a} - \frac{1}{\pi}\ln\left(\frac{\sinh(\pi X_{kj}/2a)}{\pi X_{kj}/2a}\right).$$

where $X_{kj} = |x_k - x_j|$. This expression also separates the logarithmically-dominant terms from constants and those terms which tend to zero as $|x_k - x_j| \to 0$.

4 Coupling to narrow side channels

Narrow channels of width 2ϵ and length $b_k - a$ extend from (x_k, a) to (x_k, b_k) for $k = 1, 2, \ldots, N$. Since the channels are narrow, the governing Helmholtz equation in the region between the channel walls is approximated by a one-dimensional wave equation after rescaling in x by 2ϵ. The solution, satisfying a Neumann condition on $y = b_k$, is given

$$\phi_k(x, y) = D_k \cos(b_k - y).$$

In particular we note that, as $y \to a$,

$$\phi_k(x, y) \sim D_k\{\cos(b_k - a) + (y - a)\sin(b_k - a)\} + O((y - a)^2).$$

In order to connect the solution from the waveguide those in the channels we assume each narrow channel has a rectangular opening into the waveguide (as in Fig. 1). After developing a solution in the vicinity of the rectangular opening we will use matched asymptotic expansions to complete the solution.

Referring to Evans, Porter & Chaplin (2018) for a channel of width 2ϵ with motion symmetric about its centreline we make the transformation

$$a - y = \epsilon X, \qquad (x - x_k) = \epsilon Y,$$

and let $Z = X + \mathrm{i}Y$. Accordingly $\phi(x, y) = \Phi(X, Y)$ which, under the assumption that $\epsilon \ll 1$ satisfies $\nabla^2\Phi = 0$. The Schwarz-Christoffel transformation

$$Z = \frac{2}{\pi}(1 - \zeta)^{1/2} + \frac{1}{\pi}\ln\left\{\frac{(1 - \zeta)^{1/2} - 1}{(1 - \zeta)^{1/2} + 1}\right\}$$

maps the domain in the vicinity of the opening into the upper-half ζ-plane. A flow through the opening in the physical plane is represented in the ζ-plane by the complex potential

$$W(\zeta) = \frac{M_k}{2\pi}\ln|\zeta| + \Gamma_k$$

where M_k and Γ_k are constants to be determined.

As $|\zeta| \to \infty$, $Z \to (2/\pi)(1 - \zeta)^{1/2}$ and so $|Z| \sim (2/\pi)|\zeta|^{1/2}$ implying that

$$\Phi(X, Y) \to \frac{M_k}{\pi}\ln(\pi|Z|/2) + \Gamma_k$$

as $|Z| \to \infty$ or, in other words, that

$$\phi(x, y) \sim \frac{M_k}{\pi}\ln\left(\frac{\pi r_k}{2\epsilon}\right) + \Gamma_k$$

as the outer limit into the waveguide of the inner solution.

Matching with the inner limit of the outer solution in the waveguide gives

$$M_k = -m_k$$

and

$$\Gamma_k = \mathrm{e}^{\mathrm{i}kx_k} + m_k\left\{-\frac{1}{\pi}\ln\left(\frac{2\epsilon}{a}\right) + C\right\} - \sum_{j=1,\ j\neq k}^{N} m_j g(x_k - x_j, 0).$$

Also, as $\zeta \to 0$,

$$Z \sim \frac{1}{\pi}(2(1 - \ln 2) + \ln(-\zeta)) \to -\infty$$

which implies that

$$\frac{1}{\pi} \ln|\zeta| = \Re\left\{\frac{1}{\pi}\ln(-\zeta)\right\} \sim \Re\{Z\} - \frac{2}{\pi}(1 - \ln 2).$$

In other words, as $X \to -\infty$,

$$\Phi(X, Y) \sim \frac{M_k}{2}\left(X - \frac{2}{\pi}(1 - \ln 2)\right) + \Gamma_k$$

or, returning to the physical plane,

$$\phi(x, y) \sim M_k\left(\frac{a - y}{2\epsilon} - \frac{1}{\pi}(1 - \ln 2)\right) + \Gamma_k$$

as the outer expansion along the narrow side channel of the inner solution. Matching with the inner expansion of the outer solution along the side channel gives

$$M_k = -2\epsilon D_k \sin(b_k - a)$$

and

$$-\frac{M_k}{\pi}(1 - \ln 2) + \Gamma_k = D_k \cos(b_k - a).$$

Eliminating D_k we have

$$-M_k\left(\frac{\cot(b_k - a)}{2\epsilon} - \frac{1}{\pi}(1 - \ln 2)\right) = \Gamma_k.$$

Finally we substitute in for M_k and Γ_k in terms of the coefficients m_j to get

$$m_k\left\{\frac{\cot(b_k - a)}{2\epsilon} + \frac{1}{\pi}\left(\ln\left(\frac{4\epsilon}{a}\right) - 1\right) - C\right\} - \sum_{j=1,\ j\neq k}^{N} m_j g(x_k - x_j) = \mathrm{e}^{\mathrm{i}x_k}$$

for $k = 1, 2, \ldots, N$. This is the system of equations we solve for m_j from which R and T are determined.

It is instructive to write

$$m_k = 2\epsilon u_k$$

so that $u_k = D_k \sin(b_k - a)$ represents $\partial\phi_k/\partial y$ evaluated at the opening, x_k. Substituting in the system of equations for U_k gives

$$u_k\left\{\cot(b_k - a) + \frac{2\epsilon}{\pi}\left(\ln\left(\frac{4\epsilon}{a}\right) - 1 - C\pi\right)\right\} - 2\epsilon\sum_{j=1,\ j\neq k}^{N} u_j g(x_k - x_j, 0) = \mathrm{e}^{\mathrm{i}x_k}$$

for $k = 1, 2, \ldots, N$. This system is closely aligned to a numerical discretisation of the integral equation used by Jan & Porter (2018) for a continuous function $u(x)$ and based on a continuum description of a contiguous array of channels of vanishing width.

5 Results

5.1 One channel

If there is just one side channel at $x_1 = 0$ then

$$R = \frac{\mathrm{i}/(2a)}{\Lambda - \mathrm{i}/(2a)}, \qquad T = \frac{\Lambda}{\Lambda - \mathrm{i}/(2a)}$$

where

$$\Lambda = \frac{\cot(b_1 - a)}{2\epsilon} + \frac{1}{\pi}\left(\ln\left(\frac{4\epsilon}{a}\right) - 1\right) - \sum_{n=1}^{\infty}\left(\frac{1}{\gamma_n a} - \frac{1}{n\pi}\right)$$

and energy conservation $|R|^2 + |T|^2 = 1$ is evident.

In isolation, the side channels have natural resonances. There are two types: the first is where there is a node at the opening of the channel and the second is where there is an anti-node. The side channel is node/anti-node resonant when $(b_1 - a) = \frac{1}{2}\pi$. Consider a small shift away from resonance by writing $b_1 - a = \frac{1}{2}\pi - 2\epsilon\sigma$, where it is assumed that $\epsilon \ll a$ and $\sigma = O(1)$. Then $\Lambda = 0$ when σ satisfies

$$\sigma \approx -\frac{1}{\pi}\left(\ln\left(\frac{4\epsilon}{a}\right) - 1\right) + \sum_{n=1}^{\infty}\left(\frac{1}{\gamma_n a} - \frac{1}{n\pi}\right)$$

(noting that γ_n depends on σ also, so the equation above is not explicit). However, $\sigma > 0$ and so we can infer that total reflection of waves occurs at a frequency close to and just below that for resonance in the side channel.

If there is an anti-node/anti-node resonance in the channel, $b_1 - a \to \pi$ such that $\Lambda \to \infty$ and so $R = 0$ and $T = 1$; the solution in the side channel is decoupled but synchronised to the incident wave. Hereafter we use the term resonance to describe the non-passive node/anti-node resonance.

See Fig. 2(a) for an illustration for $\epsilon = 0.2a$. That is, the side-channel is 40% the width of the waveguide. It confirms both features described above, although there is a second zero of transmission appearing just below $a = \pi$, not present if ϵ/a takes smaller values.

5.2 Equally-spaced channels of tapered length

Suppose that $x_j = 2\epsilon j$, $j = 1, 2, \ldots, N$ and that the length of the channels are tapered linearly with position along the waveguide (as illustrated in Fig. 1). Specifically we choose $x_j = -c + 2(j - \frac{1}{2})\epsilon$ and $b_j = b + mx_j$. This choice is made in order to compare with results presented in Jan & Porter (2018).

We remark that the spacing and channel width being equal means that each opening is neighboured immediately by another opening and the formal basis for the asymptotic approximation is compromised.

In Figs. 2(b-f) we plot results for an increasing number of channels occupying the same interval, $0.4a$, of the channel side-wall. Thus each channel width is set to $2\epsilon = 0.4a/N$. The lengths of the N channels are tapered linearly with distance along the waveguide, so that the smallest channel length is $0.8a$ and the longest is $1.2a$. Fig. 2 demonstrates that there are as many zeros of transmission as there are channels over the range of frequencies shown. The frequencies at which these zeros appear are approximately within the range of discrete resonant frequencies for the side-channels. That is, we infer that each zero of transmission can be attributed to a side-channel resonance.

In Fig. 3 we set the taper to zero, replicating one of the results of Jan & Porter (2018). We have chosen $N = 16$ channels each of width $2\epsilon = 0.025a$ which extend to $b_j = 2a$. We plot the solution over a narrow range of $1.3 < a < 1.6$ to focus on the complicated oscillatory nature of $|R|$ as channel resonance at $a = \frac{1}{2}\pi$ is approached. The results share the same underlying qualitative behaviour as in the work of Jan & Porter (2018).

In Fig. 4 we plot $|u_j|$ against j for an array of $N = 25, 50, 100$ contiguous channels defined by lengths $b_j = 2a + x_j$ for $x_j = -0.2a + 2\epsilon(j - \frac{1}{2})$, $2\epsilon = 0.4a/N$ for the particular dimensionless frequency $a = \frac{1}{2}\pi$ corresponding to resonance at the central channel of the array. These are successively refined discretisations of channels of linearly tapered length between $x = -0.2a$ and $0.2a$. The solution can be seen to oscillate (slowly) before a rapid increase at the resonant channel followed by a rapid decay to almost nothing beyond this.

References

1 Evans, D.V., Porter, R. & Chaplin, J.R. (2018) Extraordinary transmission past cylinders in channels. In *Proceedings of the 33rd International Workshop of Water Waves and Floating Bodies, Brest, France.*

2 Jan, A. & Porter, R. (2018) Acoustic wave transmission along a waveguide with a metamaterial cavity. *In submission to J. Acoust. Soc. Am.*

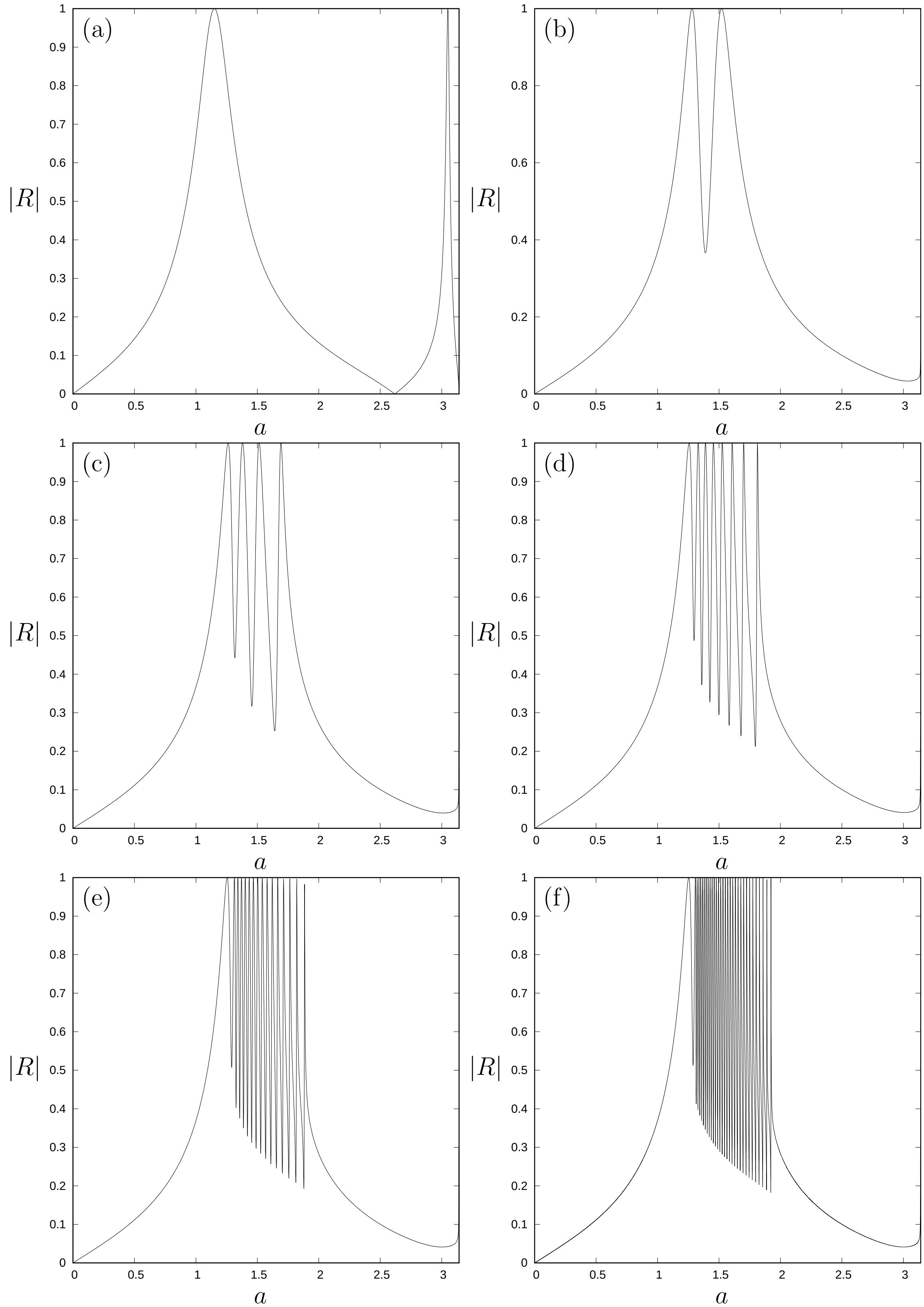

Figure 2: Modulus of the reflection coefficient against dimensionless frequency, a, for $N = 1, 2, 4, 8, 16, 32$ channels with $x_j = -0.2a + 2\epsilon(j - \frac{1}{2})$ with $2\epsilon = 0.4a/N$ and $b_j = 2a + x_j$, $j = 1, 2, \ldots, N$.

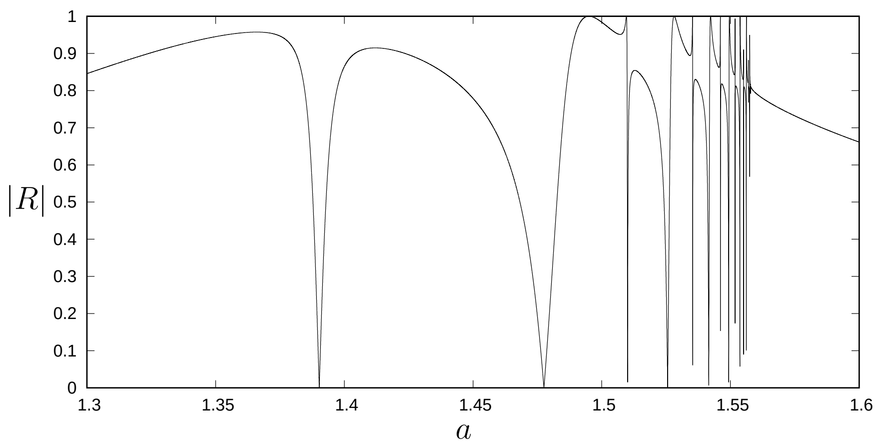

Figure 3: Modulus of the reflection coefficient against dimensionless frequency, a, for $b_k = 2a$ for $-0.2a < x < 0.2a$ and $N = 16$.

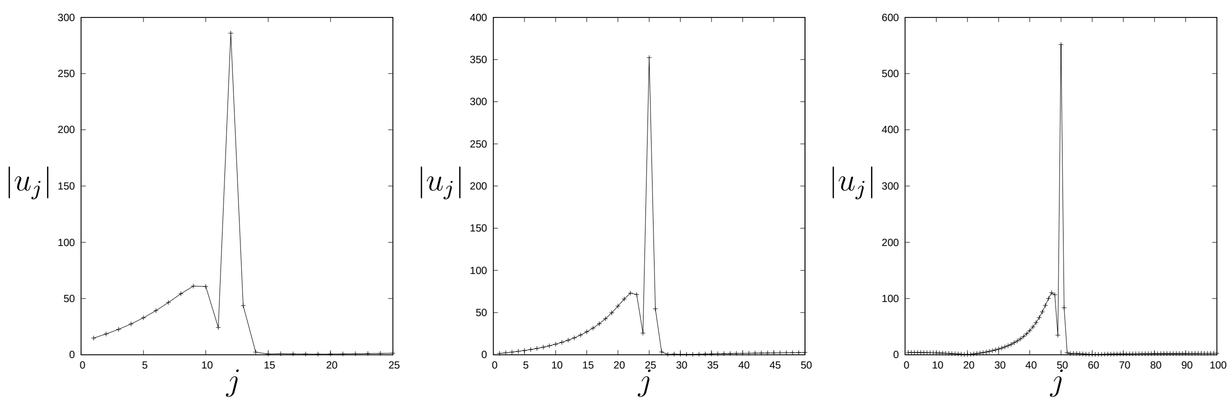

Figure 4: Modulus of the scaled source strength $|u_j|$ against channel number, j, for $b_j = 2a + x_j$ for $x_j = -0.2a + 2\epsilon(j - \frac{1}{2})$, $2\epsilon = 0.4a/N$. In the three plots, $N = 25, 50, 100$.