Complementary methods for determining surface wave propagation across a discontinuity in the fluid depth

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Abstract

The scattering of oblique parallel-crested surface waves by a step change in the fluid depth is considered. It is shown that the solution of the resulting boundary-value problem leads to two distinct ways of formulating integral equations. The scattering matrix defining wave propagation across the step depends on 2×2 matrices which are shown, in the two formulations, to be inverses of each other. Approximations to solutions are based on a variational principle equivalent to Galerkin's method and leads to bounds on diagonal elements of these matrices. For one formulation of the problem, accurate numerical results are obtained due to approximations to the solution of the integral equation explicitly incorporating the known inverse cube-root singular behaviour of the flow velocity at the corner of the step. A corresponding refinement has not been possible for the alternative formulation resulting in much slower numerical convergence to exact values. Comments are made about the similarity between the slowly-convergent approximation and a different, widely-used, solution method which doesn't rely on formulating integral equations.

1 Introduction

The scattering of oblique waves over a rectangular step in the bed was the last of a series of problems considered in Porter's (1995) PhD thesis titled "Complementary methods and bounds in linear water waves". All but the last problem involved thin barriers where it was shown that one could always formulate the solution of the problem in terms of the solution of one of *two* different integral equations. These led to approximations to quantities of interest that were reciprocals of each other a property which sometimes allowed upper and lower bounds to be established.

However, the step problem stood out in Porter (1995): only *one* formulation was presented, an integral equation for an unknown function representing the horizontal fluid velocity across the boundary above the corner of the step, formed at the junction between two series expansions of the solution. Recently, Wilks & Meylan (2025) reproduced the solution method of Porter (1995) for a closely-related problem of acoustic wave transmission due to a step discontinuity in the channel width. The purpose of their work was to compare the accuracy of the method of Porter (1995) with a method which they label the "eigenfunction matching method" (following Linton & McIver (2001)). Its use in water wave problems probably dates back to a paper of

Garrett (1971) on water wave scattering by a circular dock. Garrett (1971) had revisited a problem by Miles & Gilbert (1968) formulated in terms of integral equations and approximated using the Schwinger variational principle. It is difficult to trace its subsequent use but certainly the method had become mainstream during the early 1980s. For example, the method was used by Yeung (1981,1982), Kirby & Dalrymple (1983) and Evans & McIver (1984) and is described in the book of Linton & McIver (2001).

The method of Porter (1995) approximated solutions to integral equations using expansions of orthogonal Gegenbauer polynomials as a means of incorporating the known inverse cube root singularity at the corner of the step and was set within the framework of a variational approximation. Unsurprisingly, Wilks & Meylan (2025) found this method superior to the eigenfunction matching mehtod in terms of accuracy and computational efficiency.

The publication of the paper of Wilks & Meylan (2025) has caused me to return to the step problem in Porter (1995). In this paper we describe *two* complementary integral equation formulations for the solution to the problem of surface waves by a discontinuity in the fluid depth. The first formulation is a reproduction of Porter (1995) to a point, but there is some novelty. A variable transformation makes the satisfaction of the free surface condition easier to incorporate in the approximation than in Porter (1995) although the final numerical system is unchanged. A new alternative formulation develops integral equations for the unknown fluid pressure throughout the complete depth at the location of the bed discontinuity, both above the corner of the step and along the front face of the step. The two methods are shown to be connected by 2×2 matrices, fundamental to defining the scattering matrix for wave propagation. Specifically, these matrices are shown to be inverses of each other, establishing the *complementary* nature of the two integral equations.

In both cases, numerical solutions are sought using a variational approximation, equivalent to Galerkin's method. For the new integral equation formulation the set of functions used to expand the unknown pressure are not capable of capturing the detail associated with the complicated behaviour of the potential close to the corner of the step and a simple choice of vertical depth eigenfunctions is used and seems to be the only sensible choice.

It is also shown that there is a close connection between the eigenfunction matching method described in Wilks & Meylan (2025) and the new complementary integral equation formulation and its approximation. However, the method of described in Wilks & Meylan (2025) is not designed to automatically satisfy energy conservation and may exhibit slower convergence than is possible.

2 Definition of the problem

Cartesian coordinates (x, y, z) are used with z directed vertically upwards from an origin in the mean free surface. Two semi-infinite regions of constant depths h_1 $(x < 0)$ and $h_2 < h_1$ $(x > 0)$ are connected by a vertical step at $x = 0$, $-h_1 < z < -h_2$, $y \in \mathbb{R}$. We allow monochromatic waves of angular frequency ω to be obliquely-incident on the step from either depths h_1 and h_2 having wavenumbers k_1 and k_2 and propagating at angles θ_1 and θ_2 (respectively) relative to the positive x-axis. The wavenumbers satisfy

$$\omega^2/g = k_i \tanh k_i h_i, \qquad i = 1, 2 \tag{1}$$

where g is gravitational acceleration. Then θ_2 and θ_1 are related via Snell's law

$$l = k_1 \sin\theta_1 = k_2 \sin\theta_2. \tag{2}$$

It is being assumed that assumptions of linearised theory (e.g. see Mei (1983)) hold in which the motion of the fluid is described by a velocity potential which can be expressed as

$$\Phi(x,y,z,t) = \Re\{\phi(x,z)\mathrm{e}^{\mathrm{i}(ly-\omega t)}\} \tag{3}$$

where the reduced complex potential $\phi(x,z)$ satisfies

$$\phi_{xx} + \phi_{zz} - l^2\phi = 0 \tag{4}$$

in the fluid with

$$\phi_z - (\omega^2/g)\phi = 0, \qquad \text{on } z = 0 \tag{5}$$

and with

$$\phi_z = 0, \qquad \text{on } z = -h_1 \ (x<0) \text{ and } z = -h_2 \ (x>0). \tag{6}$$

Additionally, a zero flux condition holds on the front of the step

$$\phi_x = 0, \qquad \text{on } x = 0 \ (-h_1 < z < -h_2) \tag{7}$$

We also make careful note of the behaviour of ϕ in the vicinity of the corner in the bed at the top of the step. Using polar coordinates (r,θ) based on the top of the step so that $x = r\cos\theta$, $z + h_2 = r\sin\theta$, a local expansion of the potential $\tilde{\phi}(r,\theta) = \phi(x,z)$ satisfying Laplace's equation and the no flow conditions on $\theta = 0, 3\pi/2$ but ignoring the influence of boundaries far from the corner gives a solution

$$\phi \sim \tilde{\phi}(r,\theta) = \sum_{n=0}^{\infty} C_n r^{2n/3}\cos(2n\theta/3), \qquad \text{as } r \to 0 \tag{8}$$

for undetermined coefficients C_n. Thus $\phi \to C_0$, a non-zero constant in general, whereas

$$\phi_x(0,z) \sim -\frac{1}{r}\frac{\partial}{\partial\theta}\tilde{\phi}(r,\theta)\bigg|_{\theta=\pi/2} \sim C_1\frac{\sqrt{3}}{3}(z+h_2)^{-1/3} \tag{9}$$

as $z \to -h_2^-$ whilst it may also be useful to consider

$$\phi_z(0,z) \sim \frac{\partial}{\partial r}\tilde{\phi}(r,\theta)\bigg|_{\theta=\pi/2} \sim C_1\frac{1}{3}(z+h_2)^{-1/3} \tag{10}$$

as $z \to -h_2^-$ and

$$\phi_z(0,z) \sim \frac{\partial}{\partial r}\tilde{\phi}(r,\theta)\bigg|_{\theta=3\pi/2} \sim -C_1\frac{2}{3}(-z-h_2)^{-1/3} \tag{11}$$

as $z \to -h_2^+$. That is, velocities are weakly singular at the corner of the step and the function $\phi(0,z)$ has infinite and discontinuous gradients at the corner even though it is bounded there.

We define depth eigenfunctions (e.g. Mei (1983)) appropriate to the two depths h_i $(i = 1,2)$ by

$$\psi_{i,n}(z) = N_{i,n}^{-1/2}\cos k_{i,n}(z+h_i) \tag{12}$$

whose definitions include the normalising factors

$$N_{i,n} = \frac{1}{2}\left(1 + \frac{\sin 2k_{i,n}h_i}{2k_{i,n}h_i}\right) \tag{13}$$

where $k_{i,n}$ are a sequence of increasing positive real roots of $k_{i,n} \tan k_{i,n} h_i = -\omega^2/g$ for $n = 1, 2, \ldots$ and we can extend the definitions above to $n = 0$ using $k_{i,0} = -\mathrm{i}k_i$. Then

$$\frac{1}{h_i} \int_{-h_i}^{0} \psi_{i,n}(z) \psi_{i,m}(z) \, dz = \delta_{mn} \tag{14}$$

for $m, n = 0, 1, 2, \ldots$ ($i = 1, 2$) where δ_{mn} is the Kronecker delta symbol. In $x < 0$ we can write

$$\phi(x, z) = (A_1 \mathrm{e}^{\mathrm{i}\alpha_1 x} + B_1 \mathrm{e}^{-\mathrm{i}\alpha_1 x}) \psi_{1,0}(z) + \sum_{n=1}^{\infty} a_{i,n} \mathrm{e}^{\alpha_{1,n} x} \psi_{1,n}(z) \tag{15}$$

and in $x > 0$ we have

$$\phi(x, z) = (A_2 \mathrm{e}^{\mathrm{i}\alpha_2 x} + B_2 \mathrm{e}^{-\mathrm{i}\alpha_2 x}) \psi_{2,0}(z) + \sum_{n=1}^{\infty} a_{2,n} \mathrm{e}^{-\alpha_{2,n} x} \psi_{2,n}(z) \tag{16}$$

where A_i, B_i are the amplitudes, respectively, of right- and left-propagating waves and $a_{i,n}$ are coefficients associated with evanescent waves decaying away from the step. In the above

$$\alpha_i = \sqrt{k_i^2 - l^2} = k_i \cos\theta_i, \qquad \text{and} \qquad \alpha_{i,n} = \sqrt{k_{i,n}^2 + l^2}. \tag{17}$$

The unknown coefficients are determined by applying the remaining step condition (7) and by ensuring that ϕ and ϕ_x are continuous across the gap $-h_2 < z < 0$, $x = 0$ above the step where the two expansions overlap.

This matching procedure can be done in different ways and it is the purpose of this paper to explore these approaches.

3 Formulation of integral equations relating to the velocity at $x = 0$

We let

$$U(z) = \phi_x(0, z), \qquad -h_2 < z < 0 \tag{18}$$

Then it follows from (15)–(16) and using the condition (14), that

$$\mathrm{i}\alpha_i h_i (A_i - B_i) = \int_{-h_2}^{0} U(z) \psi_{i,0}(z) \, dz, \qquad (i = 1, 2) \tag{19}$$

where (7) has been used for $i = 1$ to reduce the integral in both cases to the interval above the step. Similarly, for $n \geq 1$,

$$\pm \alpha_{i,n} h_i a_{i,n} = \int_{-h_2}^{0} U(z) \psi_{i,n}(z) \, dz \tag{20}$$

where $\pm$ corresponds to $i = 1, 2$. Finally, we match ϕ from $x < 0$ and $x > 0$ at $x = 0$ to get

$$\begin{aligned} -(A_1 + B_1)\psi_{1,0}(z) + (A_2 + B_2)\psi_{2,0}(z) &= \sum_{n=1}^{\infty} a_{1,n} \psi_{1,n}(z) - \sum_{n=1}^{\infty} a_{2,n} \psi_{2,n}(z) \\ &= \int_{-h_2}^{0} U(z') K(z, z') \, dz' \end{aligned} \tag{21}$$

for $-h_2 < z < 0$ where

$$K(z,z') = \sum_{j=1,2} \sum_{n=1}^{\infty} \frac{\psi_{j,n}(z)\psi_{j,n}(z')}{\alpha_{j,n} h_j}. \tag{22}$$

We let $U_i(z)$ $(i = 1,2)$ satisfy

$$\int_{-h_2}^{0} U_i(z) K(z,z')\, dz' = \psi_{i,0}(z), \qquad -h_2 < z < 0. \tag{23}$$

It follows that

$$U(z) = -(A_1 + B_1)U_1(z) + (A_2 + B_2)U_2(z) \tag{24}$$

satisfies (21). Using (24) in (19) with $i = 1,2$ gives

$$\mathrm{i}\alpha_i h_i (A_i - B_i) = -(A_1 + B_1)\mathsf{Q}_{i1} + (A_2 + B_2)\mathsf{Q}_{i2} \tag{25}$$

$(i = 1,2)$ where we have written

$$\mathsf{Q}_{ij} = \int_{-h_2}^{0} U_j(z)\psi_{i,0}(z)\, dz, \qquad (i,j = 1,2) \tag{26}$$

as the elements of a 2×2 matrix Q. The pair of equations (25) can be arranged into either a scattering matrix or transfer matrix formulation. Choosing the former, in which outgoing wave amplitudes are written in terms of incoming ones, gives

$$\begin{pmatrix} B_1 \\ A_2 \end{pmatrix} = \mathsf{S} \begin{pmatrix} A_1 \\ B_2 \end{pmatrix} \tag{27}$$

which relates outgoing waves to incoming waves, where

$$\mathsf{S} = (\mathsf{D} + \mathrm{i}\mathsf{Q})^{-1} (\mathsf{D} - \mathrm{i}\mathsf{Q}) \tag{28}$$

and

$$\mathsf{D} = \begin{pmatrix} \alpha_1 h_1 & 0 \\ 0 & \alpha_2 h_2 \end{pmatrix}. \tag{29}$$

Various properties can be shown. Thus Q_{ij} are all real and

$$\mathsf{Q}_{11} \geq 0, \qquad \mathsf{Q}_{22} \geq 0, \qquad \mathsf{Q}_{21} = \mathsf{Q}_{12} \tag{30}$$

are immediate consequences of the definitions of (26) and the integral equations (23) satisfied by $U_i(z)$ and the fact that the integral operator $\mathcal{K}$ is real, self-adjoint and positive-definite.

On account of D being diagonal and Q being symmetric it follows that

$$\begin{aligned} (\mathsf{D} - \mathrm{i}\mathsf{Q})^{-1} \mathsf{D} (\mathsf{D} + \mathrm{i}\mathsf{Q})^{-1} &= \left[(\mathsf{D} + \mathrm{i}\mathsf{Q}) \mathsf{D}^{-1} (\mathsf{D} - \mathrm{i}\mathsf{Q})\right]^{-1} \\ &= \left[(\mathsf{D} - \mathrm{i}\mathsf{Q}) \mathsf{D}^{-1} (\mathsf{D} + \mathrm{i}\mathsf{Q})\right]^{-1} \\ &= (\mathsf{D} + \mathrm{i}\mathsf{Q})^{-1} \mathsf{D} (\mathsf{D} - \mathrm{i}\mathsf{Q})^{-1} \end{aligned} \tag{31}$$

which can be used to show

$$(B_1^*, A_2^*)\mathsf{D} \begin{pmatrix} B_1 \\ A_2 \end{pmatrix} = (A_1^*, B_2^*)\mathsf{D} \begin{pmatrix} A_1 \\ B_2 \end{pmatrix}. \tag{32}$$

This implies that

$$\alpha_1 h_1 \left(|A_1|^2 - |B_1|^2\right) = \alpha_2 h_2 \left(|A_2|^2 - |B_2|^2\right), \tag{33}$$

a relation which states conservation of energy. That is, any approximation which preserves the properties of Q (real and symmetric) will also satisfy energy conservation.

3.1 Transformation of variables

The integral equations (23) require approximation and it is desirable the representation of the unknown $U_i(z)$ satisfies any known conditions over the interval $-h_2 < z < 0$. That is, they should incorporate an inverse cube root behaviour as $z \to -h_2$ and satisfy the free surface conditions $U_i'(0) - (\omega^2/g)U_i(0) = 0$. The second of these is not straightforward and, because of this, below we make a transformation of variables in the integral equations such that the boundary condition at $z = 0$ is easier to satisfy.

Thus, we first note that

$$\psi_{i,n}(z) = \chi_{i,n}(z) + (\omega^2/g) \int_0^z \chi_{i,n}(\zeta)\, d\zeta \tag{34}$$

after making use of the dispersion relation, where

$$\chi_{i,n}(z) = N_{i,n}^{-1/2} \cos k_{i,n} h_i \cos k_{i,n} z \tag{35}$$

and when $n = 0$, $k_{i,0} = -\mathrm{i}k_i$. We also define a new variable

$$\mathcal{U}_i(z) = U_i(z) - (\omega^2/g) \int_{-h_2}^z U_i(\zeta)\, d\zeta \tag{36}$$

such that

$$\mathcal{U}_i'(0) = U_i'(0) - (\omega^2/g)U_i(0) = 0 \tag{37}$$

on account of $U(z) = \phi_x(0, z)$ and ϕ satisfying the free surface condition (5). It also follows from (37) that $\mathcal{U}(z) \sim O((z + h_2)^{-1/3})$ as $z \to -h_2^-$.

Substituting (34) into (23) and integrating by parts results in

$$\mathcal{F}(z) + (\omega^2/g) \int_0^z \mathcal{F}(\zeta)\, d\zeta = 0 \tag{38}$$

where

$$\mathcal{F}(z) \equiv \int_{-h_2}^0 \mathcal{U}_i(z')\kappa(z, z')\, dz' - \chi_{i,0}(z) \tag{39}$$

and we have defined

$$\kappa(z, z') = \sum_{j=1,2} \sum_{n=1}^{\infty} \frac{\chi_{j,n}(z)\chi_{j,n}(z')}{\alpha_{j,n} h_j}. \tag{40}$$

We note that (38) is equivalent to $\mathcal{F}'(z) + (\omega^2/g)\mathcal{F}(z) = 0$ with $\mathcal{F}(0) = 0$ whose solution is $\mathcal{F}(z) \equiv 0$ for $-h_2 < z < 0$. Thus, we have transformed the original integral equations (23) into

$$(\mathcal{K}\mathcal{U}_i)(z) \equiv \int_{-h_2}^0 \mathcal{U}_i(z')\kappa(z, z')\, dz' = \chi_{i,0}(z) \tag{41}$$

for $i = 1, 2$, $-h_2 < z < 0$ in which $\mathcal{U}_i'(0) = 0$, a far simpler boundary condition to apply than for $U(z)$. Likewise, we easily find using integration by parts that

$$\mathsf{Q}_{ij} = \langle \mathcal{U}_j, \chi_{i,0} \rangle \equiv \int_{-h_2}^0 \mathcal{U}_j(z)\chi_{i,0}(z)\, dz \tag{42}$$

in terms of the transformed variables.

3.2 Variational principle and Galerkin's method

To summarise the last two results of the previous section, the integral equation can be written

$$(\mathcal{K}\mathcal{U}_i)(z) = \chi_{i,0}(z), \qquad z \in (-h_2, 0) \tag{43}$$

$(i = 1, 2)$ and $\mathcal{K}$ is real, self-adjoint, positive operator on $L_2(-h_2, 0)$ with

$$\mathsf{Q}_{ij} = \langle \mathcal{U}_j, \chi_{i,0} \rangle \tag{44}$$

$(i, j = 1, 2)$ and the angled brackets denoting the real inner product. We define the real functional

$$J_{ij}(\mathcal{U}, \mathcal{V}) = \langle \mathcal{U}, \chi_{j,0} \rangle + \langle \mathcal{V}, \chi_{i,0} \rangle - \langle \mathcal{K}\mathcal{V}, \mathcal{U} \rangle. \tag{45}$$

It follows immediately that

$$J_{ij}(\mathcal{U}_i, \mathcal{U}_j) = \mathsf{Q}_{ij}. \tag{46}$$

Let $\tilde{\mathcal{U}}_i(z) \approx \mathcal{U}_i(z)$ for $i = 1, 2$ be an approximation to the exact solution of the integral equations. Then it is not difficult to show that

$$\tilde{\mathsf{Q}}_{ij} \equiv J_{ij}(\tilde{\mathcal{U}}_i, \tilde{\mathcal{U}}_j) = J_{ij}(\mathcal{U}_i + (\tilde{\mathcal{U}}_i - \mathcal{U}_i), \mathcal{U}_j + (\tilde{\mathcal{U}}_j - \mathcal{U}_j)) = \mathsf{Q}_{ij} - \langle \mathcal{K}(\tilde{\mathcal{U}}_i - \mathcal{U}_i), (\tilde{\mathcal{U}}_j - \mathcal{U}_j) \rangle. \tag{47}$$

That is to say J_{ij} is stationary at Q_{ij}, the exact values of the elements of Q. Approximations $\tilde{\mathcal{U}}_i(z)$ to the exact values of $\mathcal{U}_i(z)$ therefore lead to approximations $\tilde{\mathsf{Q}}_{ij}$ to exact values Q_{ij} which are second-order accurate with the error proportional to $\|\tilde{\mathcal{U}}_i - \mathcal{U}_i\| \|\tilde{\mathcal{U}}_j - \mathcal{U}_j\|$. Moreover, since $\mathcal{K}$ is a positive operator it follows from (47) that

$$\tilde{\mathsf{Q}}_{ii} \leq \mathsf{Q}_{ii} \tag{48}$$

for $i = 1, 2$. That is, under variational approximation, approximations to the the diagonal elements of the matrix Q are bounded above by their true values. We are not able to provide similar information about the off-diagonal terms or the determinant.

Galerkin's method follows from assuming a truncated expansion in a set of basis functions $\{u_p(z)\}$, say, in $L_2(-h_2, 0)$

$$\tilde{\mathcal{U}}_i(z) = \sum_{p=0}^{P} c_p^{(i)} u_p(z). \tag{49}$$

Using this in (45) and making J_{ij} stationary with respect to the coefficients $c_p^{(i)}$ results in the system of equations

$$\sum_{p=0}^{P} c_p^{(i)} \langle \mathcal{K}u_p, u_q \rangle = \langle \chi_{i,0}, u_q \rangle, \qquad q = 0, 1, \ldots, P \tag{50}$$

and the resulting value of $\tilde{J}_{ij}$ gives us

$$\tilde{\mathsf{Q}}_{ij} = \sum_{p=0}^{P} c_p^{(j)} \langle \chi_{i,0}, u_p \rangle \tag{51}$$

as the approximations to the elements of Q. This implementation of the variational approximation coincides with Galerkin's method (see, for example, Porter & Stirling (1990, §7.2)).

3.3 Numerical approximation

The numerical approximation of the transformed version of the integral equations involves us writing

$$\mathcal{U}_i(z) \approx \sum_{p=0}^{P} c_p^{(i)} u_p(-z/h_2) \tag{52}$$

where

$$u_p(t) = \frac{(-1)^p 2^{1/6}(2p)!\Gamma(1/6)}{\pi h_2 \Gamma(2p+1/3)(1-t^2)^{1/3}} C_{2p}^{(1/6)}(t) \tag{53}$$

and $C_{2p}^{(1/6)}(t)$ is an orthogonal Gegenbauer polynomial, even in t. Thus, the expansion is made in terms of functions which correctly model the behaviour at the end points of the interval. Furthermore, the scaling factors in the definition (53) have been chosen to provide maximum simplification in the computation of subsequent integrals. Thus, application of the Galerkin method to the integral equations in (41) results in the system of equations

$$\sum_{p=0}^{P} c_p^{(i)} K_{pq} = F_{0q}^{(i)}, \qquad q = 0, \dots, P; \quad i = 1, 2 \tag{54}$$

(this is just (50)) where

$$K_{pq} = \sum_{j=1,2} \sum_{n=1}^{\infty} \frac{F_{np}^{(j)} F_{nq}^{(j)}}{\alpha_{j,n} h_j} \tag{55}$$

and

$$F_{np}^{(i)} = \int_{-h_2}^{0} u_p(-z/h_2)\chi_{i,n}(z)\,dz = N_{i,n}^{-1/2} \frac{\cos k_{i,n} h_i}{(k_{i,n} h_2)^{1/6}} J_{2p+1/6}(k_{i,n} h_2) \tag{56}$$

for $n \geq 1$ (using integral results which may be found in Porter (1995)) with

$$F_{0p}^{(i)} = \int_{-h_2}^{0} u_p(-z/h_2)\chi_{i,0}(z)\,dz = (-1)^p N_{i,0}^{-1/2} \frac{\cosh k_i h_i}{(k_i h_2)^{1/6}} I_{2p+1/6}(k_i h_2). \tag{57}$$

Using (53) in (42) leads to

$$\tilde{\mathsf{Q}}_{ij} = \sum_{p=0}^{P} c_p^{(j)} F_{0p}^{(i)} \tag{58}$$

which is just (50).

4 Formulation of integral equations relating to the pressure along $x = 0$

Returning to the original expansions for $\phi(x, z)$ in (15) for $x < 0$ ($i = 1$) and (16) for $x > 0$ ($i = 2$) we note that orthogonality (14) gives, in the case of $x = 0$, and letting $\phi(0, z) = P(z)$,

$$h_i(A_i + B_i) = \int_{-h_i}^{0} P(z)\psi_{i,0}(z)\,dz \tag{59}$$

and

$$h_i a_{i,n} = \int_{-h_i}^{0} P(z)\psi_{i,n}(z)\,dz, \qquad (n \geq 1) \tag{60}$$

for $i = 1, 2$. Note that in contrast to the previous formulation the integrals for different i extend over different depths.

To complete the matching process we need to impose continuity of $\phi_x(x, z)$ across $x = 0$ for $-h_2 < z < 0$ in addition to applying (7) and this results in

$$\mathrm{i}\alpha_1(A_1 - B_1)\psi_{1,0}(z) + \sum_{n=1}^{\infty} \alpha_{1,n} a_{1,n} \psi_{1,n}(z) = \begin{cases} \mathrm{i}\alpha_2(A_2 - B_2)\psi_{2,0}(z) - \sum_{n=1}^{\infty} \alpha_{2,n} a_{2,n} \psi_{2,n}(z), \\ \qquad\qquad -h_2 < z < 0, \\ 0, \qquad -h_1 < z < -h_2. \end{cases} \tag{61}$$

We now substitute (60) into (61) and find

$$\begin{aligned} &\mathrm{i}\alpha_1(A_1 - B_1)\psi_{1,0}(z) + \sum_{n=1}^{\infty} \frac{\alpha_{1,n}\psi_{1,n}(z)}{h_1} \int_{-h_1}^{0} P(z)\psi_{1,n}(z)\,dz \\ &\quad = \begin{cases} \mathrm{i}\alpha_2(A_2 - B_2)\psi_{2,0}(z) - \sum_{n=1}^{\infty} \frac{\alpha_{2,n}\psi_{2,n}(z)}{h_2} \int_{-h_2}^{0} P(z)\psi_{2,n}(z)\,dz, & -h_2 < z < 0, \\ 0, \qquad -h_1 < z < -h_2. \end{cases} \end{aligned} \tag{62}$$

We let $P_1(z)$ satisfy

$$\begin{aligned} &-h_1^{-1}\psi_{1,0}(z) + \sum_{n=1}^{\infty} \frac{\alpha_{1,n}\psi_{1,n}(z)}{h_1} \int_{-h_1}^{0} P_1(z)\psi_{1,n}(z)\,dz \\ &\qquad = \begin{cases} -\sum_{n=1}^{\infty} \frac{\alpha_{2,n}\psi_{2,n}(z)}{h_2} \int_{-h_2}^{0} P_1(z)\psi_{2,n}(z)\,dz, & -h_2 < z < 0, \\ 0, \qquad -h_1 < z < -h_2, \end{cases} \end{aligned} \tag{63}$$

and $P_2(z)$ satisfy

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{\alpha_{1,n}\psi_{1,n}(z)}{h_1} \int_{-h_1}^{0} P_2(z)\psi_{1,n}(z)\,dz \\ &\qquad = \begin{cases} h_2^{-1}\psi_{2,0}(z) - \sum_{n=1}^{\infty} \frac{\alpha_{2,n}\psi_{2,n}(z)}{h_2} \int_{-h_2}^{0} P_2(z)\psi_{2,n}(z)\,dz, & -h_2 < z < 0, \\ 0, \qquad -h_1 < z < -h_2. \end{cases} \end{aligned} \tag{64}$$

It follows that the combination

$$P(z) = -\mathrm{i}\alpha_1 h_1(A_1 - B_1)P_1(z) + \mathrm{i}\alpha_2 h_2(A_2 - B_2)P_2(z) \tag{65}$$

will satisfy (62). After using (65) in (59) and defining

$$\mathsf{R}_{1j} = h_1^{-1} \int_{-h_1}^{0} P_j(z)\psi_{1,0}(z)\,dz \tag{66}$$

and

$$\mathsf{R}_{2j} = h_2^{-1} \int_{-h_2}^{0} P_j(z)\psi_{2,0}(z)\, dz, \tag{67}$$

for $j = 1, 2$ as the elements of the 2×2 matrix R, we find

$$A_i + B_i = -\mathrm{i}\alpha_1 h_1 (A_1 - B_1)\mathsf{R}_{i1} + \mathrm{i}\alpha_2 h_2 (A_2 - B_2)\mathsf{R}_{i2} \tag{68}$$

for $i = 1, 2$.

Re-arranging (68) into a scattering matrix formulation (27) as before, gives

$$\mathsf{S} = (\mathsf{RD} + \mathrm{i}\mathsf{I})^{-1} (\mathsf{RD} - \mathrm{i}\mathsf{I}). \tag{69}$$

Comparison with (28) shows that

$$\mathsf{R} = \mathsf{Q}^{-1} \tag{70}$$

and thus we classify the formulation of the problem in terms of $P(z)$ as *complementary* to the formulation of the problem in terms of $U(z)$ (or its transformed version in terms of $\mathcal{U}(z)$). This is the main attraction of presenting this formulation since it mirrors complementary formulations established for thin barriers in Porter & Evans (1995) and subsequent papers by those authors.

It is clear that R is real and symmetric since Q is. Furthermore $\mathsf{R}_{ii} \geq 0$ for $i = 1, 2$ can easily be shown from the definitions of the integral operators and hence $\det(\mathsf{Q}) > 0$ and $\det(\mathsf{R}) > 0$.

4.1 Numerical Approximation

We remark that the methodology described below is underpinned by the same variational framework used for the formulation based on the velocity above the gap.

Since the unknown functions $P_i(z)$, a proxy for $\phi(0, z)$ are defined throughout the depth interval and should satisfy the free surface condition and the bed condition on $z = -h_1$, one obvious choice is to write

$$P_i(z) \approx \sum_{p=0}^{P} d_p^{(i)} \psi_{1,p}(z). \tag{71}$$

We substitute into (63) and (64) for $i = 1, 2$ respectively and then multiply both sides by $\psi_{1,q}(z)$ and integrate over $(-h_1, 0)$ to get

$$\alpha_{1,q} h_1 (1 - \delta_{q0}) d_q^{(i)} + \sum_{p=0}^{P} d_p^{(i)} M_{pq} = G_{0q}^{(i)}, \qquad q = 0, 1, \ldots \tag{72}$$

for $i = 1, 2$ where

$$M_{pq} = \sum_{n=1}^{\infty} \alpha_{2,n} h_2 G_{np}^{(2)} G_{nq}^{(2)} \tag{73}$$

and

$$G_{np}^{(2)} = \frac{1}{h_2} \int_{-h_2}^{0} \psi_{2,n}(z)\psi_{1,p}(z)\, dz \tag{74}$$

whereas

$$G_{0q}^{(1)} = \delta_{q0}. \tag{75}$$

The choice (71) does not capture the details at the corner where $P(z)$ is continuous but contains discontinuous and infinite gradients (see equation (8), (10, (11) for the local behaviour of the potential near the corner). Thus, we imagine that numerical results using this formulation of the problem will not converge as rapidly as for the formulation based on integral equations for the velocity across the gap above the step. It is certainly not an easy task to consider more exotic basis function with which to expand $P_i(z)$.

Using (71) in (66), (67) gives

$$\tilde{\mathsf{R}}_{ij} = \sum_{p=0}^{P} c_p^{(j)} G_{0p}^{(i)}, \qquad i = 1, 2 \tag{76}$$

or, explicitly,

$$\tilde{\mathsf{R}}_{1j} = c_0^{(j)}, \qquad \tilde{\mathsf{R}}_{2j} = \sum_{p=0}^{P} c_p^{(j)} G_{0p}^{(2)} \tag{77}$$

as approximations to R_{ij} and we note that $\tilde{\mathsf{R}}_{ii} \leq \mathsf{R}_{ii}$ is assured by the underpinning theory. The computation of $G_{np}^{(2)}$ is detailed in Appendix A.

5 Some results

In Figs. 1 & 2 we have plotted curves showing various approximations to the modulus of the reflection coefficient, $|R|$, for normally-incident waves ($\theta_1 = \theta_2 = 0$) proagating from $x < 0$ where the depth is h_1 into $x > 0$ where the depth is $h_2 = \frac{1}{4}h_1$ as a function of $k_1 h_1$. In Fig. 1 the velocity formulation has been used and this converges rapidly with P, the truncation size of the system so that only the curves of $P = 0$, corresponding to a one-term approximation, is distinguishable from higher values of P. In Fig. 2 the pressure formulation is used and we see how the curves tend more slowly to the exact values as P is increased.

In order to access $|R|$ we have set $A_1 = 1$, $B_1 = R$, $A_2 = T$ and $B_2 = 0$ and solved (27) for R and T using either (28) or (69) depending on the formulation being used. There are a range of other results which could be shown contained within Porter (1995) and we are not going to replicate these here since the focus is on determining the accuracy of the two complementary methods.

In Tab. 1 we show more precise values of $|R|$ corresponding to the results in Figs. 1 & 2 at three wavenumbers $k_1 h_1 = 1, 2, 3$ for each of the two formulations as a function of P. These illustrate the relatively slow convergence of the results from pressure formulation compared with the velocity formulation. The results suggest the error in the pressure formulation is decaying roughly like $P^{-1.5}$, although this hypothesis has not been rigorously tested. In Wilks & Meylan (2025), their method – a close relative to our method, but with differences – converged like $P^{-1.33}$.

6 Discussion

The numerical system of equations derived in Wilks & Meylan (2025, equation (15)) is similar to the system (72) here. One difference comes about because of the linear algebra used in this paper to move around the incident wave forcing. Thus, we have assumed forcing from both infinities and developed a scattering matrix formulation to ensure that the kernel of the integral equation is real and self-adjoint. Its consequence is that energy conservation is guaranteed to

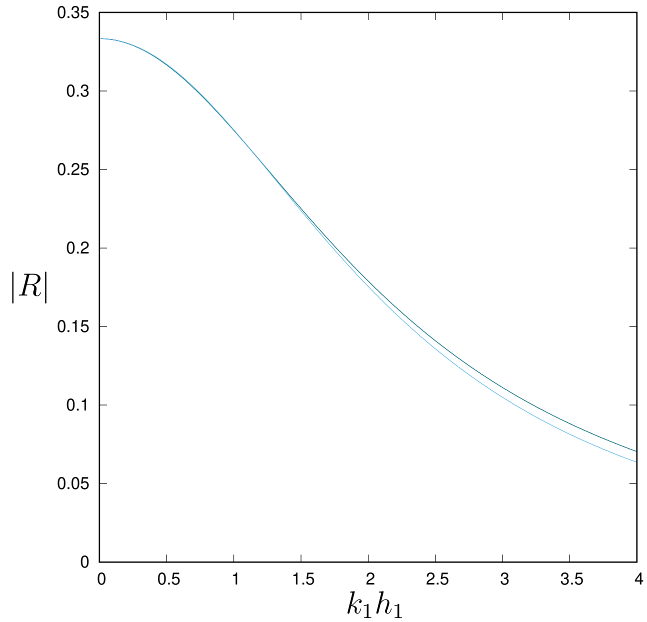

Figure 1: Curves of reflection coefficient for waves propagating from depth h_1 into shallow depth $h_2 = \frac{1}{4}h_1$ as a function of dimensionless wavenumber based on the velocity formulation with $P = 0, 1, 3, 7$ (curves for latter three values are graphically indistinguishable).

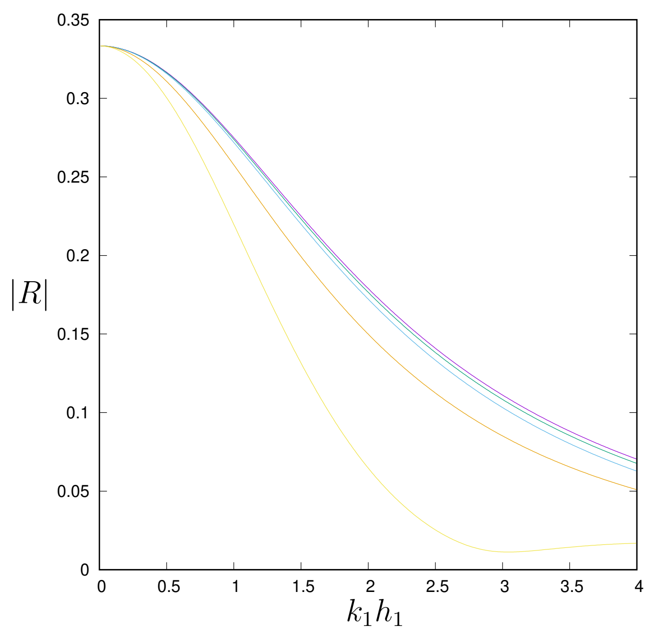

Figure 2: Curves of reflection coefficient for waves propagating from depth h_1 into shallow depth $h_2 = \frac{1}{4}h_1$ as a function of dimensionless wavenumber based on the pressure formulation with $P = 0, 1, 3, 7$ and $|R|$ computed using the velocity formulation with $P = 7$ (magenta).

be satisfied automatically, irrespective of the level numerical truncation employed. It is hard to see the implementation of the Wilks & Meylan (2025) method having this property. A second difference is that we allow the number of terms, $P + 1$, in the approximation to $P_i(z)$ in (71) to be different to the number of terms we take in the infinite series defining M_{pq} (typically much

k_1h_1	$P=0$	$P=1$	$P=3$	$P=7$	$P=15$	$P=31$	$P=63$
1	0.275237	0.274936	0.274920	0.274920	0.274920	0.274920	0.274920
	0.219767	0.257610	0.271888	0.273822	0.274531	0.274778	0.274866
2	0.175518	0.178975	0.178952	0.178952	0.178952	0.178952	0.178952
	0.064719	0.149929	0.172221	0.176549	0.178101	0.178642	0.178835
3	0.104933	0.111172	0.111164	0.111164	0.111164	0.111164	0.111164
	0.011333	0.085142	0.103207	0.108345	0.110168	0.110801	0.111028

Table 1: Values of $|R|$ (rounded to 6 decimal places) for normal incidence, $h_2 = \frac{1}{4}h_1$ and using $N = 10000$ in the infinite series defining K_{pq} or M_{pq} for three different wavenumbers and truncation sizes, P. The upper and lower numbers represent approximations derived from the velocity and pressure formulations (respectively).

much larger than P). We note that in equation (15) of Wilks & Meylan (2025), the system size is the same as the size of their matrix M, equivalent to prescribing identical truncation sizes in our method. We note that there is nothing stopping Wilks & Meylan (2025) equation (15) from employing non-square matrices M.

We also note that in Wilks & Meylan (2025) the two systems of equations for two sets of coefficients can be combined to eliminate either set of coefficients in favour of the other although it is only done one way. This choice is also noted in the account of Linton & McIver (2001, §2.5.2) although they only show how it is done one way because it favours later calculations. Linton & McIver (2001, equation (2.121)) appears to be a system of equations that is closely-related to an approximation to the solution of the integral equation representing the unknown velocity across the gap (below the corner in their problem) performed by expanding in terms of depth eigenfunctions. That is, without incorporating the singularity at the corner as we have done here. A simple calculation shows that the system of equations that results from performing this is

$$\frac{(1-\delta_{q0})}{\alpha_{2,q}h_2}c_q^{(i)} + \sum_{p=0}^{P} c_p^{(i)} L_{pq} = H_{0q}^{(i)}, \qquad q = 0, 1, \ldots, P \tag{78}$$

for $i = 1, 2$ with

$$L_{pq} = \sum_{n=1}^{\infty} \frac{H_{np}^{(1)} H_{nq}^{(1)}}{\alpha_{1,n}h_1} \tag{79}$$

and

$$H_{np}^{(1)} = G_{pn}^{(2)}, \qquad H_{0q}^{(2)} = \delta_{q0}. \tag{80}$$

Then

$$\tilde{Q}_{ij} = \sum_{p=0}^{P} c_p^{(j)} H_{0p}^{(i)}, \qquad i, j = 1, 2 \tag{81}$$

is enough to define the reflection and transmission coefficients. A quick comparison of the convergence of this method with the original method based on incorporating the singularity at the corner of the step is shown in Tab. 2 for the same set of parameters as in Tab. 1. Of

k_1h_1	$P=0$	$P=1$	$P=3$	$P=7$	$P=15$	$P=31$	$P=63$
1	0.275237	0.274936	0.274920	0.274920	0.274920	0.274920	0.274920
	0.274748	0.275181	0.275072	0.274989	0.274949	0.274932	0.274924
2	0.175518	0.178975	0.178952	0.178952	0.178952	0.178952	0.178952
	0.178093	0.179503	0.179287	0.179106	0.179018	0.178979	0.178963
3	0.104933	0.111172	0.111164	0.111164	0.111164	0.111164	0.111164
	0.109097	0.111679	0.111531	0.111339	0.111240	0.111195	0.111177

Table 2: Values of $|R|$ (rounded to 6 decimal places) for normal incidence, $h_2 = \frac{1}{4}h_1$ and using $N = 10000$ in the infinite series defining K_{pq} or L_{pq} for three different wavenumbers and truncation sizes, P. The upper and lower numbers represent approximations derived from the velocity formulation approximated using functions which include singularities and using depth eigenfunctions for the shallower depth (respectively).

particular note is how much better this alternative method, based on eigenfunctions, is than the previous pressure formulation results.

We make an important comment with regards to the implementation of the matching of solutions described in Wilks & Meylan (2025). If one truncates the infinite series defining L_{pq} and M_{pq} to the same value, P, as the number of modes in the expansion of the solution of the integral equations as Wilks & Meylan (2025) have done, then the two methods described by (72)–(76) and (78)–(81) give identical results for R and T (this has been determined numerically, but no doubt it can be proved).

What do we learn from this ? The "eigenfunction matching method" is a method which can be applied to wave scattering by abrupt changes in the width of otherwise constant rectangular domains and which results in two systems of equations following the matching of solutions across the discontinuity. At this point there will be a choice in how one eliminates between the two systems and this offers two distinct ways of computing the solution. We have suggested that each of these is related to a corresponding formulation of the problem in terms of integral equations whose solutions have been approximated by expanding unknowns in terms of the eigenfunctions of the original problem. Although the "eigenfunction matching method" is normally used as a practical tool for solving problems and is applied directly without considering the distinction between propagating waves and evanescent waves, if the linear algebra is done correctly one ought to be able to maintain properties of solutions of the integral equations, such as conservation of energy and bounds on certain fundamental quantities. Having said this the formulation and approximation of integral equations is essentially no more complicated than the "eigenfunction matching method" and it is straightforward to set solutions within the framework that guarantees desirable properties of solutions. The integral equation formulation has more flexibility than the "eigenfunction matching method" since it allows one a choice of how to approximate solutions. Thus, when it can be done, a judicious choice of expansion functions can lead to a significant increase in convergence over the choice of eigenfunctions. Even when depth eigenfunctions are used to approximate solutions of integral equations, we have shown that the existence of two different truncation parameters should be exploited to improve the results.

Appendix A: computational details

From the definition (74), (12) can show that, $n > 0$, $p > 0$,

$$G^{(2)}_{np} = \frac{k_{1,p} \sin k_{1,p}(h_1 - h_2)}{N^{1/2}_{2,n} N^{1/2}_{1,p} h_2(k^2_{2,n} - k^2_{1,p})}. \tag{82}$$

The special cases of $n = 0$, $p = 0$ follow from using $k_{i,0} = -\mathrm{i}k_i$, $(i = 1, 2)$. Specifically, when $n = 0$ and $p > 0$ (82) is written

$$G^{(2)}_{0p} = -\frac{k_{1,p} \sin k_{1,p}(h_1 - h_2)}{N^{1/2}_{2,0} N^{1/2}_{1,p} h_2(k^2_{2} + k^2_{1,p})} \tag{83}$$

when $n > 0$ and $p = 0$ it becomes

$$G^{(2)}_{n0} = -\frac{k_{1} \sinh k_{1}(h_1 - h_2)}{N^{1/2}_{2,n} N^{1/2}_{1,0} h_2(k^2_{2,n} + k^2_{1})} \tag{84}$$

and when $n = 0$ and $p = 0$ we have

$$G^{(2)}_{00} = \frac{k_{1} \sinh k_{1}(h_1 - h_2)}{N^{1/2}_{2,0} N^{1/2}_{1,0} h_2(k^2_{2} - k^2_{1})}. \tag{85}$$

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