Forces on a submerged circular cylinder moving at constant forward speed

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Abstract

The problem of a circular cylinder submerged, and moving at constant forward speed, in a fluid with a free surface is considered under linearised theory. The method of solution essentially repeats, using modern notation, the method devised by Havelock (1936). In doing so, we highlight the fact that T.H. Havelock pioneered the method of multipole potentials over ten years before Ursell's (1948) work to which the method has since been attributed.

1 Introduction

The problem of determining the steady two-dimensional surface wave pattern formed when an infinitely long uniform cylinder of circular cross-section moves through a fluid below a free surface has a long history. Provided the cylinder is of sufficiently small radius, a, and sufficiently deeply submerged below the free surface of the fluid and that the speed of the cylinder is small enough that the flow can be assumed to be irrotational, a linearisation procedure can be adopted in order to express the solution of the problem in terms of a perturbation series for a velocity potential based on powers of the small parameter a/d where d is the depth of the centre of the cylinder from the undisturbed free surface.

Approaches at attempting to solve the problem to first order in the linearisation was first presented by Lamb (1932) and built upon in papers by Havelock in 1928 and 1936. Lamb (1932) used a submerged dipole to represent the presence of a small circular cylinder. However, as demonstrated by Tuck (1965), the presence of the free surface destroys the closed streamline around the singularity that would exist if the ideal fluid was unbounded and therefore the solution of Lamb (1932) only represents a first-order approximation to a circular cylinder. Havelock (1928) sought to improve upon the solution found in Lamb (1932) by including successive images in the free surface which had the effect of satisfying the cylinder boundary condition up to the next order in the small parameter a/d. Later, in 1936, Havelock revisited the same problem and presented a systematic method for solving the problem for an exactly-circular cylinder. Inspection of the method devised by Havelock (1936) shows it to be identical to what is usually referred to as the multipole method. This method of representing a solution involving circular (or, in three-dimensional problems, spherical) geometries and a plane free surface on which a mixed boundary condition is to be satisfied has historically been attributed to Ursell (1948) who had originally used the method to show that there is zero reflection of surface waves by a submerged circular cylinder. The fact that the contribution of Havelock (1936) has been, as far as the author is aware, overlooked is perhaps even more remarkable given the comments made some years later by Tuck (1965) in a paper which sought to build upon the solution of Havelock (1936). Tuck (a Ph.D. student of Ursell's) states in his paper of 1965 "... Havelock ... in a remarkably ingenious later paper (1936), was able to construct a complete formal solution to the wholly linear problem".

The contribution of Tuck (1965) was to show that exact satisfaction of the cylinder condition, as performed by Havelock (1936), was, at least in some cases, less important than including the effect of non-linearities in the free surface condition. Tuck demonstrated this by considering the linearised problem up to and including second order for the circular cylinder.

The primary focus of the present paper serves to give a modern account of the method of Havelock (1936). It is presented in an style that is arguably easier to follow than Havelock's paper and, despite the fact that essence of the two approaches is the same, there are some differences. We also include more detailed numerical computations that were not possible for Havelock to perform without the aid of a computer. This account also includes a formal expansion of the potential, akin to the form adopted by Tuck (1965) and in doing so confirms Tuck's hypothesis that non-linearities are just as important as satisfying the cylinder condition exactly. There are also new ways of representing various quantities, including the wave resistance and the free surface elevation.

2 Governing equations

The problem is two-dimensional and Cartesian coordinates are chosen with the origin in the undisturbed free surface and y directed vertically upwards. The cylinder is centred at $y = -d$ and is assumed to have radius $a \ll d$. The fluid is inviscid and incompressible and the flow is irrotational and steady. Then there exists a velocity potential $\Phi(x, y)$ which satisfies

$$\nabla^2\Phi = 0, \qquad \text{in the fluid domain} \tag{2.1}$$

There is a uniform stream of velocity U in the positive x direction imposed on the fluid, which translates to the condition

$$\Phi(x, y) \sim Ux, \qquad \text{as } |x| \to \infty,\ y < 0 \tag{2.2}$$

which is to be satisfied to leading order.

The elevation of the free surface is defined as $y = \eta(x)$, $-\infty < x < \infty$ and the kinematic boundary condition is then expressed as

$$\frac{\partial\Phi}{\partial x}\frac{\partial\eta}{\partial x} = \frac{\partial\Phi}{\partial y} \qquad \text{on } y = \eta(x),\ -\infty < x < \infty \tag{2.3}$$

whilst the dynamic boundary condition on $y = \eta(x)$ is

$$-\frac{P_a}{\rho} = g\eta + \frac{1}{2}|\nabla\Phi|^2 + C, \qquad -\infty < x < \infty \tag{2.4}$$

where P_a is the atmospheric pressure at the free surface, ρ is the density of the fluid, g is gravitational acceleration and C is an arbitrary constant which will be fixed by the requirement that $\eta = 0$ describes the undisturbed free surface.

In order to satisfy a no-flow condition on the cylinder, we also require

$$\frac{\partial \Phi}{\partial r} = 0, \qquad \text{on } r = a \tag{2.5}$$

where $r^2 = x^2 + (y+d)^2$.

Since the submerged cylinder is symmetric about the vertical line $x = 0$, we note that reversing the stream velocity to $-U$ and the direction of the x-axis simultaneously must result in the same boundary-value problem for $\Phi(x, y)$. Thus, in an obvious notation, we must impose the condition

$$\Phi(x, y; U) = \Phi(-x, y; -U) \tag{2.6}$$

on the solution. Without loss of generality, we will assume that $U > 0$ herein.

An approximate solution is sought by a linearisation procedure. Hence, we express Φ and η in terms of a perturbation expansion in powers of $\epsilon = a/d \ll 1$ writing

$$\Phi(x, y) = U(x + \epsilon^2 \phi_1(x, y) + \epsilon^4 \phi_2(x, y) + \ldots) \tag{2.7}$$

which ensures that (2.2) holds and

$$\eta(x, y) = U(\epsilon^2 \eta_1(x) + \epsilon^4 \eta_2(x) + \ldots) \tag{2.8}$$

Then $\phi_1, \phi_2, \ldots$ also must satisfy (2.1). On physical grounds, there can be no waves at $x \to -\infty$; that is upstream of the cylinder. Thus, we also require that

$$\phi_j(x, y) \to 0, \qquad \text{as } x \to -\infty, \quad j = 1, 2, \ldots \tag{2.9}$$

Linearising the boundary conditions (2.3) and (2.4) about $y = 0$ gives, to $O(\epsilon^2)$,

$$U \frac{\partial \eta_1}{\partial x} = \frac{\partial \phi_1}{\partial y} \qquad \text{on } y = 0, \ -\infty < x < \infty \tag{2.10}$$

and

$$\eta_1 = -\frac{U}{g} \frac{\partial \phi_1}{\partial x}, \qquad \text{on } y = 0, \ -\infty < x < \infty \tag{2.11}$$

whilst the constant C in (2.4) is fixed by $C = -P_a/\rho - \frac{1}{2}U^2$ to ensure that $\eta(x) = 0$ represents the undisturbed free surface.

Combining the two linearised surface equations (2.10), (2.11) to eliminate η_1 gives

$$\kappa \frac{\partial \phi_1}{\partial y} = -\frac{\partial^2 \phi_1}{\partial x^2} = \frac{\partial^2 \phi_1}{\partial y^2}, \qquad \text{on } y = 0, \ -\infty < x < \infty. \tag{2.12}$$

where

$$\kappa = g/U^2 \tag{2.13}$$

and κ plays the role of a wavenumber.

Instead of writing the perturbation potential ϕ_1 as a single dipole plus a correction term for the free surface as in Lamb (1932), we shall expand ϕ_1 in terms of sets of multipoles each satisfying (2.12) and (2.9) before imposing (2.5) on Φ as the final condition of the problem. This is essentially the method employed by Havelock (1936).

3 Derivation of multipole potentials

First consider the integral, for $n \geq 1$

$$I_n(z) = \frac{1}{(n-1)!} \int_0^\infty k^{n-1} \mathrm{e}^{-kz} dk \tag{3.1}$$

where it is assumed that $\Re\{z\} > 0$. It is straightforward to verify that $I_1(z) = 1/z$. For $n \geq 2$, it follows using integration by parts that $I_n(z) = I_{n-1}(z)/z$ and hence

$$I_n(z) = \frac{1}{z^n}. \tag{3.2}$$

Sets of symmetric ($\{\psi_n^s(x,y)\}$) and antisymmetric ($\{\psi_n^a(x,y)\}$) multipole potentials will now be constructed, each singular at $(x,y) = (0,-d)$ whilst satisfying (2.1) away from this point, (2.9) and (2.12). Here, the descriptions antisymmetric and symmetric refer to the properties of fundamental singular potentials obtained from taking real and imaginery parts of (3.2) and on which the multipoles are based.

We shall first describe the construction of the antisymmetric multipoles $\psi_n^a(x,y)$, $n = 1, 2, \ldots$ and then state the corresponding result for $\psi_n^s(x,y)$, which follows using an identical method. Thus with

$$z = (y+d) + ix = r\mathrm{e}^{i\theta}, \qquad \text{such that} \qquad r^2 = x^2 + (y+d)^2, \quad \tan\theta = \frac{x}{y+d} \tag{3.3}$$

we write

$$\psi_n^a(x,y) = \frac{\sin n\theta}{r^n} + \chi_n^a(x,y) \tag{3.4}$$

where $\chi_n^a(x,y)$ is a bounded harmonic function in $y < 0$ which will be determined to ensure that ψ_n^a satisfies the free surface condition (2.12) and the radiation condition (2.9).

It is noted that

$$\frac{\sin n\theta}{r^n} = \Re\left\{\frac{i}{z^n}\right\} = \frac{1}{(n-1)!} \int_0^\infty k^{n-1} \mathrm{e}^{-k(y+d)} \sin kx dk, \qquad \text{for } y > -d. \tag{3.5}$$

Bearing this relation in mind and anticipating later developments we choose to express $\chi_n^a(x,y)$ in terms of the sum of a Fourier sine integral and a symmetric standing wave term, each satisfying (2.1) by writing

$$\chi_n^a(x,y) = \int_0^\infty B_n(k) \mathrm{e}^{ky} \sin kx dk + C_n(\kappa) \mathrm{e}^{\kappa y} \cos \kappa x \tag{3.6}$$

for $y < 0$ where the function $B_n(k)$ and the amplitude $C_n(\kappa)$ are to be determined. So now,

$$\psi_n^a(x,y) = \frac{1}{(n-1)!} \int_0^\infty k^{n-1} \mathrm{e}^{-k(y+d)} \sin kx dk + \int_0^\infty B_n(k) \mathrm{e}^{ky} \sin kx dk + C_n(\kappa) \mathrm{e}^{\kappa y} \cos \kappa x. \tag{3.7}$$

Applying the free surface condition (2.12), automatically satisfied by the final term in (3.7) furnishes the relation

$$-\frac{k^n \mathrm{e}^{-kd}}{(n-1)!} + kB_n(k) = -\frac{U^2}{g} \left(-\frac{k^{n+1} \mathrm{e}^{-kd}}{(n-1)!} - k^2 B_n(k) \right) \tag{3.8}$$

which is rearranged to give

$$B_n(k) = -\frac{k^{n-1}\mathrm{e}^{-kd}}{(n-1)!}\left(\frac{k+\kappa}{k-\kappa}\right) \tag{3.9}$$

and κ is given by (2.13). So the multipole potentials satisfying (2.1), (2.9) and (2.12) may be written

$$\psi_n^a(x,y) = \frac{\sin n\theta}{r^n} - \frac{1}{(n-1)!}\int_0^\infty \left(\frac{k+\kappa}{k-\kappa}\right) k^{n-1}\mathrm{e}^{k(y-d)}\sin kx dk + C_n(\kappa)\mathrm{e}^{\kappa y}\cos\kappa x \tag{3.10}$$

where the integral is principal-valued. There is a contribution to the behaviour of $\psi_n^a(x,y)$ for large $|x|$ not only from the final term in (3.10) but also from the pole at $k=\kappa$ in the integral. By deforming above, for $x>0$, and below, for $x<0$, the pole at $k=\kappa$ we find that

$$\psi_n^a(x,y) \sim \left(C_n(\kappa) \mp \frac{2\pi\kappa^n\mathrm{e}^{-\kappa d}}{(n-1)!}\right)\mathrm{e}^{\kappa y}\cos\kappa x, \qquad x\to\pm\infty. \tag{3.11}$$

In order that (2.9) be satisfied we require that

$$C_n(\kappa) = -\frac{2\pi\kappa^n\mathrm{e}^{-\kappa d}}{(n-1)!} \tag{3.12}$$

and so as $x\to\infty$

$$\psi_n^a(x,y) \sim -\frac{4\pi\kappa^n\mathrm{e}^{-\kappa d}}{(n-1)!}\mathrm{e}^{\kappa y}\cos\kappa x. \tag{3.13}$$

The terms in the integrand are now to be expanded into polar coordinates by considering the result of taking the real and imaginary parts of

$$\mathrm{e}^{k(y+d)}\mathrm{e}^{ikx} = \mathrm{e}^{kz} = \sum_{m=0}^{\infty}\frac{(kr)^m}{m!}\mathrm{e}^{im\theta} \tag{3.14}$$

so that we may now write

$$\psi_n^a(x,y) = \frac{\sin n\theta}{r^n} - \sum_{m=1}^{\infty} B_{mn}r^m\sin m\theta - \sum_{m=1}^{\infty} C_{mn}r^m\cos m\theta \tag{3.15}$$

(the $m=0$ term in (3.14) can be omitted as it only contributes a constant to the potential) where

$$B_{mn} = \frac{1}{m!(n-1)!}\int_0^\infty \left(\frac{k+\kappa}{k-\kappa}\right) k^{m+n-1}\mathrm{e}^{-2kd}dk \tag{3.16}$$

and

$$C_{mn} = \frac{2\pi\mathrm{e}^{-2\kappa d}}{m!(n-1)!}\kappa^{n+m} \tag{3.17}$$

Note that the expansion in (3.15) is valid provided that $r<2d$ (see Thorne (1953) for example).

Applying a similar procedure to derive the symmetric multipoles yields

$$\psi_n^s(x,y) = \frac{\cos n\theta}{r^n} - \sum_{m=1}^{\infty} B_{mn}r^m\cos m\theta - \sum_{m=1}^{\infty} C_{mn}r^m\sin m\theta \tag{3.18}$$

for $n = 1, 2, \ldots$ where B_{mn} and C_{mn} are just those factors already defined in (3.16) and (3.17). Each multipole, $\psi_n^s(x,y)$, defined by (3.18) for $r < 2d$ satisfies (2.1) away from $r = 0$, (2.9) and (2.12). Also, as $x \to \infty$

$$\psi_n^s(x,y) \sim -\frac{4\pi\kappa^n \mathrm{e}^{-\kappa d}}{(n-1)!} \mathrm{e}^{\kappa y} \sin \kappa x. \tag{3.19}$$

4 Solution

The potential $\phi_1(x,y)$ may now be written as a linear combination of both sets of multipoles, so that

$$\phi_1(x,y) = \sum_{n=1}^{\infty} \frac{a^{n-1}d^2}{n} \Big(b_n^a \psi_n^a(x,y) + b_n^s \psi_n^s(x,y) \Big) \tag{4.1}$$

and where b_n^a, b_n^s, $n = 1, 2, \ldots$ are dimensionless coefficients to be determined from application of (2.5), the cylinder boundary condition.

Now from (2.4) up to $O(\epsilon^2)$ we have

$$\Phi = Ur\sin\theta + U\sum_{n=1}^{\infty} \frac{a^{n+1}}{n} \Big(b_n^a \psi_n^a(x,y) + b_n^s \psi_n^s(x,y) \Big) \tag{4.2}$$

whence application of (2.5) gives

$$\begin{aligned} 0 = \left.\frac{\partial \Phi}{\partial r}\right|_{x=a} &= U\sin\theta + U\sum_{n=1}^{\infty} b_n^a \left[-\sin n\theta - \sum_{m=1}^{\infty} \frac{m}{n} a^{m+n} \Big(B_{mn}\sin m\theta + C_{mn}\cos m\theta \Big) \right] \\ &+ U\sum_{n=1}^{\infty} b_n^s \left[-\cos n\theta - \sum_{m=1}^{\infty} \frac{m}{n} a^{m+n} \Big(B_{mn}\cos m\theta + C_{mn}\sin m\theta \Big) \right] \end{aligned} \tag{4.3}$$

We choose to introduce a new notation here, being

$$\widetilde{B}_{mn} = \frac{m}{n} B_{mn} a^{m+n} = \frac{\mu^{m+n}}{n!(m-1)!} \int_0^{\infty} \left(\frac{t+1}{t-1} \right) t^{m+n-1} \mathrm{e}^{-2\lambda t} dt \tag{4.4}$$

and

$$\widetilde{C}_{mn} = \frac{m}{n} C_{mn} a^{m+n} = \frac{2\pi\mu^{m+n}\mathrm{e}^{-2\lambda}}{n!(m-1)!} \tag{4.5}$$

where

$$\mu = \kappa a = ga/U^2 \qquad \text{and} \qquad \lambda = \kappa d = gd/U^2 \tag{4.6}$$

and $\lambda = 1/Fr^2$ where $Fr = U/\sqrt{gd}$ is the Froude number whilst $\mu = \epsilon\lambda$. Thus the problem may be characterised either by μ and λ or Fr and ϵ.

Using the orthogonality of the functions $\sin n\theta$ and $\cos n\theta$ over the interval $[0, 2\pi]$ gives the following coupled system of equations for the coefficients b_m^a, b_m^s

$$b_m^a + \sum_{n=1}^{\infty} \Big(\widetilde{B}_{mn} b_n^a + \widetilde{C}_{mn} b_n^s \Big) = \delta_{m1}, \qquad m = 1, 2, \ldots \tag{4.7}$$

and

$$b_m^s + \sum_{n=1}^{\infty} \left(\widetilde{B}_{mn} b_n^s + \widetilde{C}_{mn} b_n^a \right) = 0 \qquad m = 1, 2, \ldots \tag{4.8}$$

where δ_{mn} is the Kronecker delta.

Following the method used in Linton & Evans (1990), we may re-insert (4.7) and (4.8) back into (4.2) with (3.15) and (3.18) to provide a simplified expression for the potential in the vicinity of the cylinder. It is found that

$$\Phi(x, y) = Ua \sum_{m=1}^{\infty} \frac{1}{m} \left(b_m^a \sin m\theta + b_m^s \cos m\theta \right) \left[\left(\frac{a}{r} \right)^m + \left(\frac{r}{a} \right)^m \right] \tag{4.9}$$

provided $r < 2d$.

We now make the connection with Lamb's (1932) solution. First, we note that we may write

$$\widetilde{B}_{mn} = \epsilon^{m+n} \widehat{B}_{mn}, \qquad \widetilde{C}_{mn} = \epsilon^{m+n} \widehat{C}_{mn} \tag{4.10}$$

where $\widehat{B}_{mn}$, $\widehat{C}_{mn}$ are independent of ϵ. Then it is clear to see from (4.7) and (4.8) that

$$b_m^a = \delta_{m1} + O(\epsilon^2), \qquad b_m^s = O(\epsilon^2) \tag{4.11}$$

Ignoring all contributions of $O(\epsilon^2)$ gives

$$\Phi(x, y) \approx Ur \sin\theta + Ua^2 \psi_1^a(x, y) = U \sin\theta \left(r + \frac{a^2}{r} \right) + Ua^2 \chi_1^a(x, y). \tag{4.12}$$

which is the expression used by Lamb (1932). Inspection of the coefficients in (4.11) reaffirms the observation made by Tuck (1965) that apart from the multipole $\psi_1^a(x, y)$ – the antisymmetric dipole term – all other terms contribute to the total potential at $O(\epsilon^4)$; that is, to the same order as ϕ_2, the second-order potential.

5 Free surface elevation

The free surface elevation is given by (2.8) with (2.11) and requires calculation of, partial derivatives of ψ_n^a, ψ_n^s in x. So, for example,

$$\left. \frac{\partial \psi_n^a}{\partial x} \right|_{y=0} = -\frac{2\kappa}{(n-1)!} \int_0^{\infty} \frac{k^n \mathrm{e}^{-kd} \cos kx}{k - \kappa} dk - C_n(\kappa) \kappa \sin \kappa x \tag{5.1}$$

where the right-hand side has come from substitution of (3.9) into (3.7). Then we note the expansion

$$\frac{k^n}{k - \kappa} = k^{n-1} + \kappa k^{n-2} + \ldots + \kappa^{n-2} k + \kappa^{n-1} + \frac{\kappa^n}{k - \kappa} \tag{5.2}$$

which can be used to write

$$\left. \frac{\partial \psi_n^a}{\partial x} \right|_{y=0} = -\frac{2\kappa}{(n-1)!} \left[\sum_{m=0}^{n-1} \kappa^m \int_0^{\infty} k^{n-m-1} \mathrm{e}^{-kd} \cos kx dk + \kappa^n \int_0^{\infty} \frac{\mathrm{e}^{-kd} \cos kx}{k - \kappa} dk \right] - C_n(\kappa) \kappa \sin \kappa x \tag{5.3}$$

It is straightforward to calculate the final term directly (or see, for example, Lamb (1932)) and so

$$\int_0^\infty \frac{\mathrm{e}^{-kd}\cos kx}{k-\kappa}dk = -\pi\mathrm{e}^{-\lambda}\sin\kappa|x| - E^a(\kappa x,\lambda) \tag{5.4}$$

where

$$E^a(\kappa x,\lambda) = \int_0^\infty \frac{(\sin\lambda u - u\cos\lambda u)}{1+u^2}\mathrm{e}^{-u|\kappa x|}du \tag{5.5}$$

Also, from (3.1) and (3.2) we have the result

$$\frac{1}{(n-m-1)!}\int_0^\infty k^{n-m-1}\mathrm{e}^{-kd}\cos kx dk = \Re\left\{\frac{1}{z^{n-m}}\right\}_{y=0} = \left[\frac{\cos(n-m)\theta}{r^{n-m}}\right]_{y=0} \tag{5.6}$$

which combines with (5.4) to give

$$\begin{aligned}\left.\frac{\partial\psi_n^a}{\partial x}\right|_{y=0} &= -2\kappa\sum_{m=0}^{n-1}\kappa^m\frac{(n-m-1)!}{(n-1)!}\left[\frac{\cos(n-m)\theta}{r^{n-m}}\right]_{y=0} + \frac{2\kappa^{n+1}E(\kappa x,\lambda)}{(n-1)!}\\ &+\frac{2\pi\kappa^{n+1}\mathrm{e}^{-\lambda}}{(n-1)!}\Big(\sin\kappa|x| + \sin\kappa x\Big)\end{aligned} \tag{5.7}$$

for $n = 1, 2, \ldots$. As $|x| \to \infty$, the first and second terms in the above equation tend to zero. A similar result is obtained for the symmetric multipoles. Thus, it is found that

$$\begin{aligned}\left.\frac{\partial\psi_n^s}{\partial x}\right|_{y=0} &= 2\kappa\sum_{m=0}^{n-1}\kappa^m\frac{(n-m-1)!}{(n-1)!}\left[\frac{\sin(n-m)\theta}{r^{n-m}}\right]_{y=0} - \frac{2\mathrm{sgn}(x)\kappa^{n+1}E^s(\kappa x,\lambda)}{(n-1)!}\\ &-\frac{2\pi\kappa^{n+1}\mathrm{e}^{-\lambda}}{(n-1)!}\Big(\mathrm{sgn}(x)\cos\kappa x + \cos\kappa x\Big)\end{aligned} \tag{5.8}$$

for $n = 1, 2, \ldots$ where

$$E^s(\kappa x,\lambda) = \int_0^\infty \frac{(u\sin\lambda u + \cos\lambda u)}{1+u^2}\mathrm{e}^{-u|\kappa x|}du \tag{5.9}$$

Finally using (4.1) in (2.11) with (2.8) up to $O(\epsilon^2)$ shows that

$$\eta(x) = -\frac{U^2}{g}\epsilon^2\sum_{n=1}^\infty\frac{a^{n-1}d^2}{n}\left(b_n^a\left.\frac{\partial\psi_n^a}{\partial x}\right|_{y=0} + b_n^s\left.\frac{\partial\psi_n^s}{\partial x}\right|_{y=0}\right) \tag{5.10}$$

and (5.7) and (5.8) can be used directly in this equation to establish $\eta(x)$ for $-\infty < x < \infty$.

Of particular significance is the free surface elevation for large $|x|$, which is given by taking the limit of $|x| \to \infty$ in (5.7) and (5.8) and inserting into (5.10), whence

$$\eta(x) \sim \begin{cases} -4\pi a\mathrm{e}^{-\lambda}\sum_{n=1}^\infty\frac{\mu^n}{n!}\Big(b_n^a\sin\kappa x - b_n^s\cos\kappa x\Big), & x\to\infty\\ 0, \quad x\to-\infty\end{cases} \tag{5.11}$$

after some algebra.

For the approximately circular cylinder considered by Lamb (1932) we may use (4.11) in (5.10) to show that to $O(\epsilon^4)$

$$\begin{aligned}\eta(x) &\approx -\frac{U^2a^2}{g}\frac{\partial\psi_1^a}{\partial x}\bigg|_{y=0} \\ &= 2a^2\left[\frac{\cos\theta}{r}\right]_{y=0} - 2\pi\kappa a^2 \mathrm{e}^{-\lambda}\Big(\sin\kappa|x| + \sin\kappa x\Big) - 2\kappa a^2 E^a(\kappa x, \lambda) \qquad (5.12)\end{aligned}$$

from (5.7). This expression agrees with Lamb (1932, §247 eqn. 15).

6 Wave resistance and wave lift

The wave resistance on the cylinder is defined to be

$$f_x = -\frac{\rho a}{2}\int_0^{2\pi} |\nabla\Phi|^2_{r=a}\sin\theta d\theta$$

whilst the wave lift on the body is

$$f_y = \frac{\rho a}{2}\int_0^{2\pi} |\nabla\Phi|^2_{r=a}\cos\theta d\theta$$

Using the form of the potential given in (4.11) we can easily evaluate these forces since, using cylindrical polar coordinate definition of $\nabla = (\partial/\partial r, (1/r)\partial/\partial\theta)$

$$(\nabla\Phi)_{r=a} = \left(0, 2U\sum_{m=1}^{\infty}(b_m^a\cos m\theta - b_m^s\sin m\theta)\right)$$

on account of the cylinder condition on $r = a$. Some straightforward algebra leads to the definitions

$$f_x = 2\pi U^2\rho a\sum_{n=1}^{\infty}(b_{n+1}^s b_n^a - b_n^s b_{n+1}^a)$$

which is the analogue of the real part of equation (28) in Havelock (1936).

The wave resistance should also be given by $f_x = \frac{1}{4}g\rho h^2$ where h is the amplitude of the wave train at $x \to \infty$.

Appendix A: Evaluation of $\widetilde{B}_{mn}$

This appendix gives an efficient method for calculating the elements $\widetilde{B}_{mn}$. First, we write

$$\widetilde{B}_{mn} = \frac{\mu^{m+n}}{n!(m-1)!}\Big[J_{m+n}(\lambda) + J_{m+n-1}(\lambda)\Big], \qquad m, n = 1, 2, \ldots \qquad (A.1)$$

where

$$J_p(\lambda) = \int_0^{\infty}\frac{t^p\mathrm{e}^{-2\lambda t}}{t-1}dt = \int_0^{\infty} t^{p-1}\mathrm{e}^{-2\lambda t}dt + J_{p-1}(\lambda) \qquad (A.2)$$

for $p \geq 1$. Applying the recurrence relation (A.2) repeatedly we have

$$J_p(\lambda) = \sum_{s=0}^{p-1} L_{p-1-s}(\lambda) + \int_0^\infty \frac{\mathrm{e}^{-2\lambda t}}{t-1} dt \tag{A.3}$$

where we have defined

$$L_p(\lambda) = \int_0^\infty t^p \mathrm{e}^{-2\lambda t} dt \tag{A.4}$$

It is now straightforward to evaluate $L_p(\lambda)$ using integration by parts to show that, for $p \geq 1$, $L_p(\lambda) = (p/(2\lambda))L_{p-1}(\lambda)$, whilst $L_0(\lambda) = 1/(2\lambda)$ and hence

$$L_p(\lambda) = \frac{p!}{(2\lambda)^{p+1}} \tag{A.5}$$

Also, from Yu & Ursell (1961), it can be shown that

$$Q(\lambda) = \int_0^\infty \frac{\mathrm{e}^{-2\lambda t}}{t-1} dt = -\mathrm{e}^{-2\lambda}(\gamma + \ln(2\lambda)) + \sum_{s=1}^{\infty} \frac{(-2\lambda)^s}{s!} \left(1 + \frac{1}{2} + \ldots + \frac{1}{s}\right) \tag{A.6}$$

where $\gamma = 0.5772\ldots$ is Euler's constant.

Combining all these results into (A.1) we find that

$$\widetilde{B}_{mn} = \frac{\mu^{m+n}}{n!(m-1)!} \left[\frac{(m+n-1)!}{(2\lambda)^{m+n}} + 2 \sum_{s=1}^{m+n-1} \frac{(m+n-s-1)!}{(2\lambda)^{m+n-s}} + 2Q(\lambda) \right], \tag{A.7}$$

for $m, n = 1, 2, \ldots$.

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