Fluid Dynamics 3 MATH33200

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Course information

- Lecturer: Dr. Richard Porter, Room 1A.17, email `richard.porter@bristol.ac.uk`
- Course timetable: Weeks 1-11: Online lectures recorded on a live-streaming platform twitch.tv on Monday 4-5, Tuesday 10-11 (odd weeks), Tue 1-2, Thu 4-5 (even weeks). Total of 30 recorded lectures.
- Prerequistes: Mechanics 1, APDE2, Multivariable Calculus, Methods of Complex Functions.
- Homework: weekly from set worksheets.
- Problems Classes: Face to face problems classes to go through HW questions (and other queries) in LT2.41 in Fry on Tuesday 10-11 (even weeks), Thu 4-5 (odd weeks). These sessions will be live-streamed on twitch to the other half of the class.
- Assessment: 10% from 3 assessed homeworks on problem sheets 3,6,9 (exam questions from last year's paper) and 90% exam in January.
- Office hours: TBA.
- Teaching materials will be made available from course web page

  `http://people.maths.bris.ac.uk/~marp/fluids3/`

  to include anonymous feedback, solutions, and past papers. Online videos may have to be put on Blackboard.

Recommended texts

1. A.R. Paterson, *A First Course in Fluid Dynamics*, Cambridge University Press. (The recommended text to complement this course - costs ≈ £75 from Amazon; there are 6 copies in Queen's building Library and 3 copies in the Physics Library)
2. L.D. Landau and E.M. Lifshitz, Fluid Mechanics. Butterworth Heine- mann.

3. D.J. Acheson, *Elementary Fluid Dynamics.* Oxford University Press

4. G.K. Batchelor, *An Introduction to Fluid Dynamics*, Cambridge University Press. (*The* definitive book, written by *the* expert, but not for the faint-hearted)

1 Introduction & Kinematics

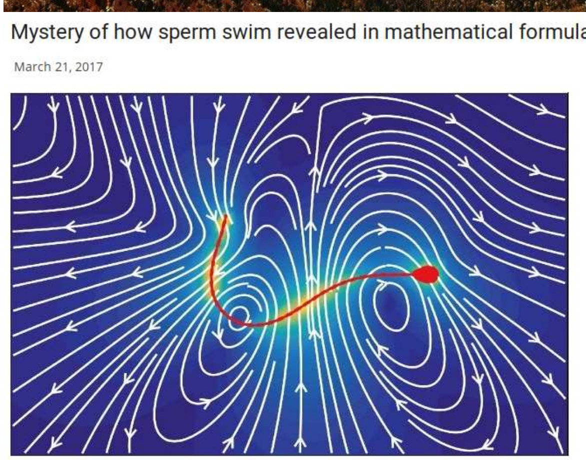
1.1 Opening remarks

Fluid dynamics is an extremely important and practical area of applied mathematics which is used, for example, in studying the weather, waves (e.g. on water, in air), aerodynamics of aircraft/hydrodynamics of ships, blood flow, oil extraction, chemical reactions, sport...

It is an extraordinarily broad subject area which ranges from the study of very large scales (e.g. formation of galaxies/stars/planets or the motion of magma inside planets) to very small scales (cells in biology or swimming organisms). You only need to observe some everyday examples of fluid motion to realise that fluids normally possess very complex behaviour (e.g. smoke rising from a fire, breaking waves on a beach, the splash of a drop in a puddle).

It is quite remarkable then that all fluid motion is governed by the same physical laws (conservation of mass and momentum – or Newton's law) and that all these different complex behaviours are solutions of the same fundamental equation (the Navier-Stokes equation).

What can we acheive in this course ? The reality is that the relatively basic tools of undergraduate mathematics are insufficient to allow us to consider the complete range of fluid dynamics topics nor to explain these complex fluid phenomena; in practice one often resorts to computational methods (so-called CFD) to do this.



What we hope to show in this course is that by making some good assumptions and approximations about real flows, the mathematical tools that you've learnt in Mechanics, PDEs, ODEs, MVC, MCF can be applied to gain some insight and understanding of some fundamental properties of fluid dynamics.

This includes, for example, how lift is generated by wings of an aircraft, how ground effect works in racing cars, how a fluid flows over a wier or how a wind turbine can extract power from a background flow.

You can find many fascinating short videos explaining in current research topics in fluid mechanics here:
`https://gfm.aps.org/meetings` in the American Physical Society's Gallery of Fluid Motion.

1.2 What is a fluid ?

A fluid is a material whose loose packing of molecules which have no structure. This includes both liquid, gases (as well as plasma) but not solids. A gas is easy to compress, a liquid much less so.

Within this definition there is are important sub divisions or classifications of fluids. For e.g. honey, water & saliva would appear to all be fluids, yet they appear to react to forces in different ways.

In this course we shall consider only **ideal** fluids. These are fluids which have negligible[1] **viscosity** (a frictional resistence to sliding forces). This includes water, but not honey (which is viscous) and saliva (which is visco-elastic).

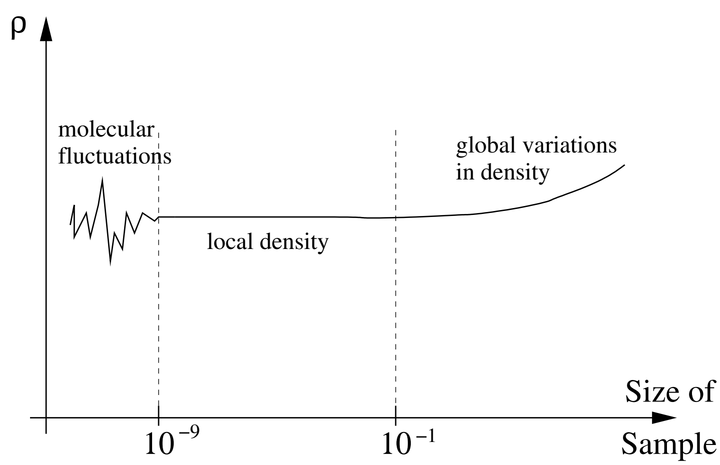
1.3 What are we interested in ?

Things that we observe or measure easily: e.g. velocity, pressure, density, forces ...

... But what exactly do we mean by *measure* and *observe* ?

1.4 The continuum hypothesis

Consider *measuring* (for e.g.) the density ρ at a particular point in a fluid:



Our sample has volume δV and so

$$\rho = \frac{\delta m}{\delta V}, \qquad \delta m \text{ is mass of fluid in } \delta V$$

If the sample is *too small* (e.g. typical molecule size is $\sim 10^{-9}$m or typical mean free path in a gas is $\sim 10^{-7}$m) we will record microscopic fluctuations of the fluid.

[1]This statement is typical in Applied Mathematics but is inherently ambiguous. E.g. I have negligible mass when compared to the mass of the earth, but not compared to the mass of an ant. So in this statement, I've already made some assumptions about the fluid problem I intend to consider: in fact I'm assuming the Reynolds number is not small

If the sample is *too large* we average out detail we want to capture.

Between these two limits we can assign the local average to a point at the centre, say, of δV in space.

This sampling approach is the basis of the **continuum hypothesis** and leads to a broad area of applied mathematics called **continuum mechanics** (e.g. traffic flow models in APDE2 are formulated on this principle.)

I.e. we can define continuous functions of position $\mathbf{r} = (x, y, z)$ and time, t, to describe locally-averaged quantities.

E.g.: density, $\rho(\mathbf{r}, t)$; velocity, $\mathbf{u}(\mathbf{r}, t)$.

Note: *Differentiation* requires, for example, taking the limit as $\delta V \to 0$, an infinitesimal volume, tends to zero. In reality, makes no sense, but does according to the continuum model.

Note: The continuum hypothesis breaks down if one tries to use it to describe features at molecular lengthscales. E.g. later on, singularities where the fluid speed is predicted to be infinite at a point in space is simply a manifestation of continuum assumption.

1.5 Lagrangian and Eulerian descriptions of the flow

We now consider what it means to *observe* a fluid flow. There are two (sensible) ways of doing this.

1. **Eulerian**: What the stationary observer sees.

   Choose fixed points, $\mathbf{r}$, to measure, for e.g. the 'Eulerian velocity' $\mathbf{u}(\mathbf{r}, t)$.

   This provides a spatial distribution of the flow at each instant in time. If the flow is **steady** then $\mathbf{u}$ does not depend on time, t: $\mathbf{u} = \mathbf{u}(\mathbf{r})$.

   E.g. Weather stations.

2. **Lagrangian**: The observer moves with the fluid.

   Choose fluid particles and follow them through the fluid. Measuring their velocity at a given time, t gives its 'Lagrangian velocity'.

   So if at time t_0, the fluid particle is at position $\mathbf{a}$, then at time $t > t_0$, it is at $\mathbf{r}(t)$[2] so that $\mathbf{r}(t_0) = \mathbf{a}$. The Lagrangian velocity is $\mathbf{v}(t) = \dfrac{d\mathbf{r}}{dt}$

   E.g. Ballons, buoys in the ocean.

[2]The vector $\mathbf{r}$ should be interpreted according to the context of its use. In the Eulerian framework, $\mathbf{r}$ is a position vector of a fixed point in space and does not depend upon time; in the Lagrangian framework, we have assumed that $\mathbf{r}$ is a particular path in space which depends upon time t.

The two are related (they have to be, right ?)

$$\mathbf{v}(t) = \mathbf{u}(\mathbf{r}(t), t)$$

(Read as: "*The velocity of a fluid particle following the path* $\mathbf{r}(t)$ *is the same as velocity observed by a stationary observer positioned instantaneously at the point* $\mathbf{r}(t)$")

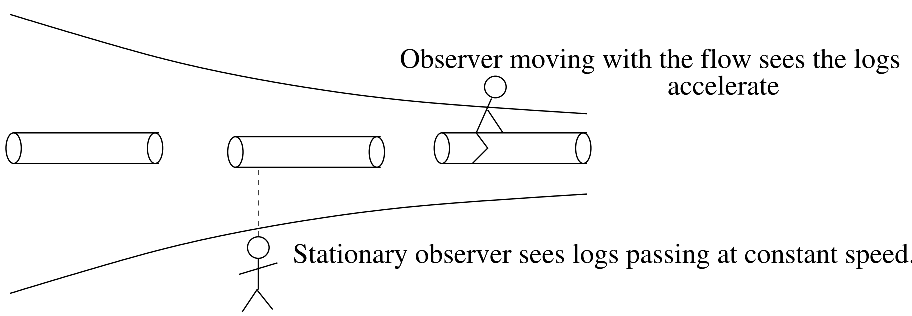
Q: Can a steady flow be accelerating ?

A: Yes !

E.g. consider logs flowing along the centre of a narrowing section of river – take $\mathbf{u} = (kx, 0, 0)$ for some constant $k > 0$.

Fixed observers at, say $x = x_1, x_2, \ldots, x_N$ see logs passing the same points at the same speed. Observers sitting on the logs pass the points $x = x_1, x_2, \ldots, x_N$ at speeds $kx_1, kx_2, \ldots, kx_N$ and the see the logs accelerating.

Note: Since in this example $\mathbf{u}_t = (0, 0, 0)$, we have illustrated that acceleration under the Eulerian description isn't just $\partial\mathbf{u}/\partial t$ as you might expect.



Using a fixed coordinate system (Eulerian) rather than one which moves with the flow (often the thing you are trying to find !) is far far easier in most practical problems, so most of the time we stick to Eulerian. And we will sort out the acceleration issue later...

1.6 2D vs 3D

Defn: A flow is **two-dimensional** if it is independent of one of its components (in some fixed frame of reference).

E.g. $\mathbf{u} = (u(x, y, t), v(x, y, t), 0)$.

This means there is no dependence of the flow on z **and** that there is no component of flow in the $\hat{\mathbf{z}}$ direction.

Note: Whenever we write $\mathbf{u} = (u, v)$, 2D is to be assumed.

1.7 Visualising flows: Particle paths, streamlines and streaklines

Assume the Eulerian velocity field $\mathbf{u}(\mathbf{r}, t)$ is given.

1.7.1 Particle Paths

Defn: Particle paths (**pathlines**) are curves $\mathbf{r}(t)$ in space followed by individual particles. In an experiment, pathlines found by following tracer particles in the flow.

Release particle from $\mathbf{r} = \mathbf{a}$ at $t = t_0$ and then its path, denoted by $\mathbf{r}(t)$, is determined from earlier considerations as the solution to:

$$\boxed{\frac{d\mathbf{r}}{dt} = \mathbf{u}(\mathbf{r}, t), \qquad \text{with initial condition } \mathbf{r}(t_0) = \mathbf{a} = (a_1, a_2, a_3)} \tag{1}$$

Explicitly, writing $\mathbf{r}(t) = (x(t), y(t), z(t))$ and $\mathbf{u}(\mathbf{r}, t) = (u(x, y, z, t), v(x, y, z, t), w(x, y, z, t))$, then (1) is

$$\left.\begin{aligned} \frac{dx}{dt} &= u(x, y, z, t) \\ \frac{dy}{dt} &= v(x, y, z, t) \\ \frac{dz}{dt} &= w(x, y, z, t) \end{aligned}\right\}$$

with $x(t_0) = a_1$, $y(t_0) = a_2$, $z(t_0) = a_3$.

Note: For some (simple) $\mathbf{u}$ these can be integrated using elementary methods, but in general not.

E.g. 1.1: Logs on a river: $\mathbf{u} = (kx, 0, 0)$ with $\mathbf{r}(0) = (1, 0, 0)$, say. So

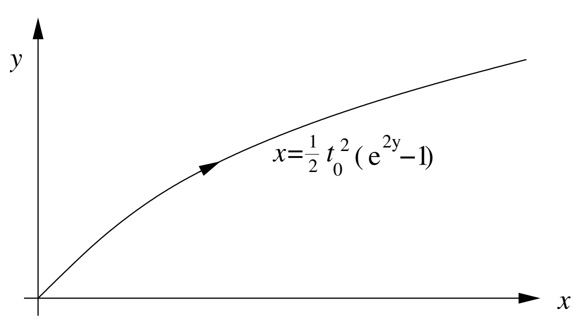
$$\frac{dx}{dt} = kx \qquad \Rightarrow \qquad x(t) = \mathrm{e}^{kt}$$

(and acceleration is manifest.)

E.g. 1.2: 2D flow, $\mathbf{u} = (t, 1/t, 0)$ (so $u = t$, $v = 1/t$, $w = 0$) with $\mathbf{r}(t_0) = (0, 0, 0)$. Then

$$\left\{\begin{aligned} \frac{dx}{dt} &= t, &&\Rightarrow x = \frac{1}{2}t^2 + c_1 \\ \frac{dy}{dt} &= 1/t &&\Rightarrow y = \log t + c_2 \\ \frac{dw}{dt} &= 0 &&\Rightarrow z = c_3 \end{aligned}\right.$$

for constants of integration c_1, c_2, and c_3 determined by the initial condition. So $c_1 = -\frac{1}{2}t_0^2$, $c_2 = -\log t_0$, $c_3 = 0$. It follows that $t = t_0\mathrm{e}^y$ so that $x = \frac{1}{2}t_0^2(\mathrm{e}^{2y} - 1)$ with $z = 0$.



E.g. 1.3: 2D flow, $\mathbf{u} = (y, -x, 0)\cos t$ with $\mathbf{r}(t_0) = (1, 0, 0)$. Then

$$\frac{dx}{dt} = y\cos t, \ (*) \qquad \frac{dy}{dt} = -x\cos t,$$

Soln ?

Method 1:

Divide one by the other to get

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-x}{y}$$

separate and integrate

$$\int y\,dy = -\int x\,dx \qquad \Rightarrow \qquad \tfrac{1}{2}y^2 + \tfrac{1}{2}x^2 = C$$

for some constant C. Since at $t = t_0$, $x = 1$ and $y = 0$, we end up with $x^2 + y^2 = 1$.

Method 2:

Similar, but a bit slicker. Combine two ODEs as

$$0 = x\frac{dx}{dt} + y\frac{dy}{dt} = \frac{d}{dt}(\tfrac{1}{2}x^2 + \tfrac{1}{2}y^2)$$

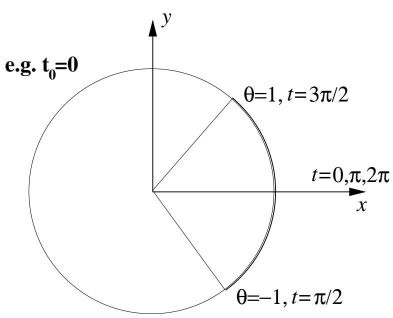
so that the solution satisfying the initial conditions is $x^2 + y^2 = 1$.

Problem: Don't know $x(t)$ and $y(t)$...

Soln: We can parametrise the curve formed by the pathline using by $x(t) = \cos\theta(t)$, $y(t) = \sin\theta(t)$ where $\theta(t)$ is a new unknown.

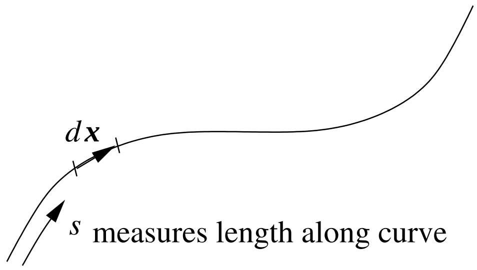
Then $dx/dt = -\sin\theta\dot{\theta} = -y\dot{\theta}$ and from $(*)$ must have $\dot{\theta} = -\cos t$ which integrates to $\theta = -\sin t + \mathit{const}$. The constant is $\sin t_0$, since $\theta = 0$ at $t = t_0$.

So full solution is $x(t) = \cos(\sin t_0 - \sin t)$ and $y(t) = \sin(\sin t_0 - \sin t)$



1.7.2 Streamlines

Defn: A **streamline** of a flow $\mathbf{u}(\mathbf{r}, t)$ at a *given instant in time*, t_0 say, is a curve which is everywhere parallel to $\mathbf{u}(\mathbf{r}, t_0)$.



Let $\mathbf{r} = \mathbf{r}(s)$ describe a curve in space[3] (the streamline) defined by its arclength, s, along the curve.

Then $d\mathbf{r}/ds$ is a vector parallel to that curve. and it follows that

$$\frac{d\mathbf{r}}{ds} \propto \mathbf{u}(\mathbf{r}, t_0) \qquad \text{or} \qquad \frac{d\mathbf{r}}{ds} = \lambda(s)\mathbf{u}(\mathbf{r}, t_0)$$

for some (real) $\lambda(s)$. Writing $\mathbf{r}(s) = (x(s), y(s), z(s))$ and matching component-by-component gives $dx = \lambda u ds$, $dy = \lambda v ds$, $dz = \lambda w ds$, or

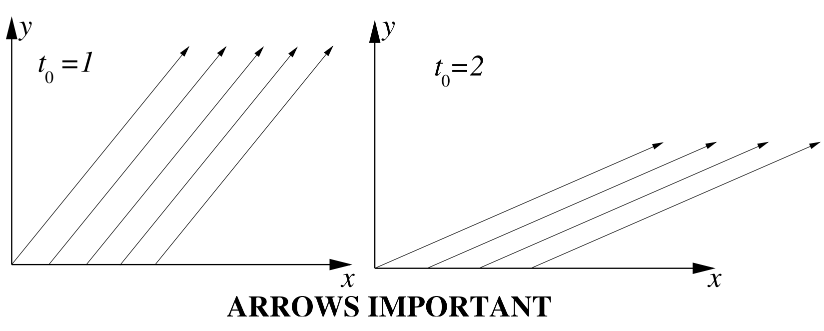
$$\boxed{\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} (= \lambda ds), \qquad t = t_0.} \tag{2}$$

Streamlines are found by solving these equations. For example, $dy/dx = v/u$ and $dx/dz = u/w$ (and a third relation is implied by the solution to the other two).

Note: In general, streamlines vary with time. If the flow is steady streamlines of the flow are fixed (the opposite is not always true.)

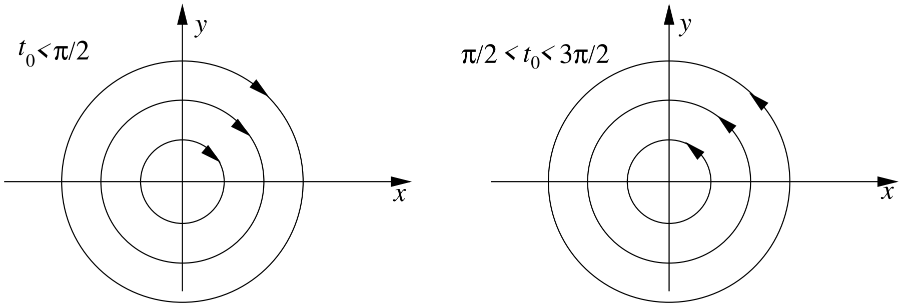
Streamlines can be visualised in an experiment by taking a short-time exposure of illuminated particles in a flow.

E.g. 1.4: $\mathbf{u} = (t, 1/t, 0)$. At $t = t_0$, $\dfrac{dx}{t_0} = t_0 dy$, so that, integrating up, $x = t_0^2 y + const.$



E.g. 1.5: $\mathbf{u} = (y, -x, 0)\cos t$. At $t = t_0$, $\dfrac{dx}{y\cos t_0} = -\dfrac{dy}{x\cos t_0}$ so that $\dfrac{dx}{dy} = -\dfrac{y}{x}$ and then integrating, $x^2 + y^2 = const.$

[3]Now $\mathbf{r}$ is used differently to the Eulerian and Lagrangian uses and measures the position of a point on a line



Note: Streamlines should indicate the direction of the flow.

Note: When solving for streamlines, one does not specify a single point through which a particle passes (as for pathlines) and the effect of integrating the equations defining the streamlines produces a constant whose variation provides a family of streamlines.

1.7.3 Streaklines

Defn: A **streakline** is the locus, at a time t, of particles which have been released continuously from a fixed point $\mathbf{a}$ during a time interval $[\tau, t]$.

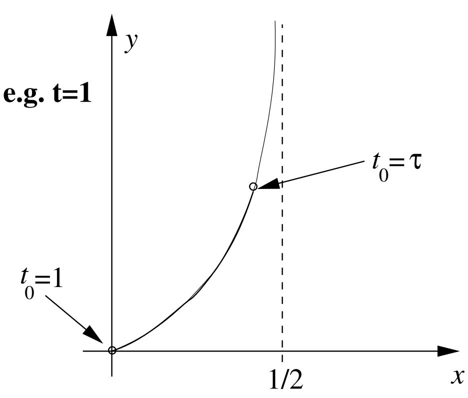
In an experiment it corresponds to the line formed by the continuous release of a dye from a fixed point over a particular period of time. It depends on the history of the fluid back in time to $t = \tau$.

To find the streaklines, find the paths of fluid particles released from a position $\mathbf{a}$ at time t_0 for all $\tau < t_0 < t$. This gives the streakline in parametric form.

E.g. 1.6: $\mathbf{u} = (t, 1/t, 0)$ passing through $(0, 0, 0)$ at $t = t_0$. Then from earlier, we found

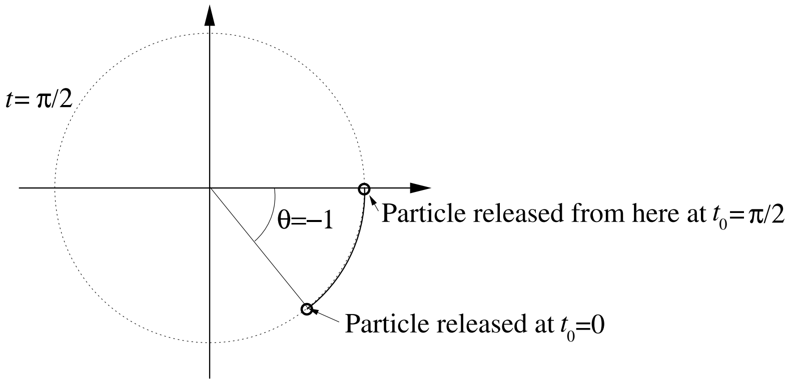
$$x(t) = \tfrac{1}{2}(t^2 - t_0^2), \qquad y(t) = \log(t/t_0).$$

E.g. At $t = 1$, the streakline $(x_s(t_0), y_s(t_0))$ is defined by $x_s(t_0) = \frac{1}{2}(1 - t_0^2)$, $y_s(t_0) = \log(1/t_0)$ and here it is possible to eliminate t_0 so that $x_s = \frac{1}{2}(1 - \mathrm{e}^{-2y_s})$ is the explicit equation of the streakline. It runs from $(0, 0, 0)$ (when $t_0 = 1$) to $(\frac{1}{2}(1 - \tau^2), \log(1/\tau), 0)$ (when $t_0 = \tau$) for some τ in $0 < \tau < 1$.



E.g. 1.7: $\mathbf{u} = (y, -x, 0)\cos t$ passing through $(1, 0, 0)$ at $t = t_0$. From earlier particle paths are given by $x(t) = \cos\theta$, $y(t) = \sin\theta$ with $\theta = \sin t_0 - \sin t$.

E.g. Choose $\tau = 0$, $t = \frac{1}{2}\pi$. Now $\theta = \sin t_0 - 1$. The streakline is arc of a circle, $r = 1$, which runs from $-1 < \theta < 0$.



Note: For **steady flows**, streamlines, streaklines and particle paths all coincide.

1.8 The Lagrangian derivative

(a.k.a. the **convective** derivative, the **material** derivative).

We know how to measure the time derivative of a physical quantity associated with the fluid – temperature $T(\mathbf{r}, t)$ say – at a fixed point in space (the "Eulerian derivative"). It's just

$$\left.\frac{\partial T}{\partial t}\right|_{\mathbf{r}\ fixed}$$

This definition misses useful information. For example, parcels of air over Bristol might pass at a constant 15 degrees, but have warmed to 20 degrees by the time they reach Oxford. According to the Eulerian derivative which observes changes at fixed locations the temperature everywhere is constant. Useful, but it doesn't tell you that parcels of air moving with the fluid have increased in temperature.

Defn: Thus an important idea needed to describe the flow is the *rate of change following a fluid particle*. This is called the **Lagrangian derivative**.

The key is to no longer regard $\mathbf{r}$ as fixed, but $\mathbf{r}(t)$ will describe the path of a particle of fluid (its initial position is not needed as we shall see).

So, for example, $T = T(\mathbf{r}(t), t)$ and

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt} + \frac{\partial T}{\partial t}$$

by the chain rule.

Note: The partial derivatives on the RHS assume all other variables are fixed (this is the conventional definition of partial derivatives) and are hence refer to the *Eulerian* description.

Notation: We use D/Dt to describe the Lagrangian derivative and so the above can be written compactly as

$$\boxed{\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} T} \qquad (3)$$

Defn: Since D/Dt can be applied to the scalar components u, v, w of a velocity field $\mathbf{u}$ and thus to $\mathbf{u}$ itself, the **acceleration** of a fluid particle is

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} \tag{4}$$

Defn: The term $\mathbf{u} \cdot \boldsymbol{\nabla}$ is called the **advective term**. It is the *scalar* operator $u\partial/\partial x + v\partial/\partial y + w\partial/\partial z$ (in Cartesians).

Note: The brackets in $(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u}$ are important and $\mathbf{u} \cdot (\boldsymbol{\nabla}\mathbf{u})$ makes no sense. It is the parallel of the vector operation $(\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ which is nonsense if rewritten as $\mathbf{a} \cdot (\mathbf{bc})$.

E.g. 1.8: Consider log problem from §1.5 where $\mathbf{u} = (kx, 0, 0)$. Then: (i)

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{0}$$

so the flow is steady; and (ii) the **advective** term is

$$(\mathbf{u}.\boldsymbol{\nabla})\mathbf{u} = \left(kx\frac{\partial}{\partial x}\right)(kx, 0, 0).$$

Hence the acceleration of the log is $\dfrac{D\mathbf{u}}{Dt} = (k^2 x, 0, 0)$.

Note: Recall from E.g. 1.1 that we computed a particle path to be $x(t) = \mathrm{e}^{kt}$. Thus, the acceleration in the x-direction is $d^2x/dt^2 = k^2\mathrm{e}^{kt} = k^2 x$, which matches Du/Dt above.

Key points: The continuum hypothesis allows us to consider a flow, not by tracking individual molecules but by describing its properties in terms of local averages.

There are two ways of observing the flow, in fixed and moving frames of reference. These are connected. It is most usual to work in a fixed frame (Eulerian) of reference but an important notion is that of rate of change moving with the fluid which gives rise to the Lagrangian derivative, $D/Dt = \partial/\partial t + (\mathbf{u} \cdot \boldsymbol{\nabla})$.

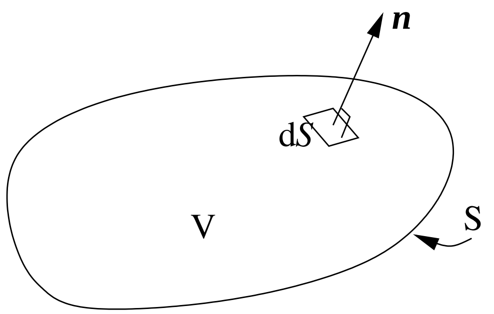
There are three ways of visualing a fluid: particle paths (the unique curve followed by single release of a tracer into the flow), streamlines (families of curves which capture the instantaneous direction of the flow everywhere) and streaklines (finite length curves formed by the continuous release of tracer particles over an interval of time). These are identical for steady flows.

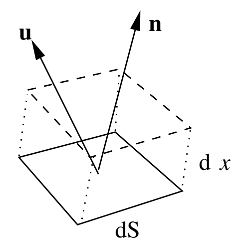
2 Continuity and some basic incompressible flows

2.1 Mass conservation

Consider an *arbitrary* finite volume, V, which is fixed in a fixed frame of reference. V is bounded by the surface S and $\hat{\mathbf{n}}$ represents a unit normal on S outward from V.

A fluid occupies the space of which V is a subset. The fluid has velocity $\mathbf{u}(\mathbf{r}, t)$ and density $\rho(\mathbf{r}, t)$. Fluid can flow in and out of V.





The mass of fluid contained in V is

$$\int_V \rho(\mathbf{r}, t) dV$$

In a short time δt the mass leaving a section, area δS, of the surface of V is $\delta m = \rho \delta x \delta S$ where $\delta x = \mathbf{u} \cdot \hat{\mathbf{n}} \delta t$. So the *rate* at which mass is leaving V (the **mass flux**) through δS is $\rho \mathbf{u} \cdot \hat{\mathbf{n}} \delta S$

Therefore the total flux of mass *into* V is

$$-\int_S \rho \mathbf{u} \cdot \hat{\mathbf{n}} dS$$

By the principle of the conservation of mass (fluid is neither created or destroyed in a closed system):

"the rate of change of mass in V must equal the rate of change of mass into of V through S"

In other words

$$\frac{d}{dt} \int_V \rho(\mathbf{r}, t) dV = -\int_S \rho \mathbf{u} \cdot \hat{\mathbf{n}} dS \tag{5}$$

Since V does not change with t and using the divergence theorem for the RHS we get

$$\int_V \frac{\partial \rho}{\partial t} dV = -\int_V \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) dV$$

or

$$\int_V \left(\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) \right) dV = 0.$$

This is true for *any* fixed V, so it must be

$$\boxed{\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0} \tag{6}$$

at every point in the fluid.

(6) is called the **mass conservation equation** or the **continuity equation**.

2.2 Incompressibility

Defn: A fluid is said to be **incompressible** if the density of each fluid 'particle' is constant, (i.e. $\dfrac{D\rho}{Dt} = 0$).

From (6) we have

$$0 = \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \rho + \rho \boldsymbol{\nabla} \cdot \mathbf{u} = \frac{D\rho}{Dt} + \rho \boldsymbol{\nabla} \cdot \mathbf{u}$$

Thus an incompressible fluid satisfies

$$\boxed{\boldsymbol{\nabla} \cdot \mathbf{u} = 0} \tag{7}$$

That is, the condition (7) implies and hence replaces (6) for an incompressible fluid.

Remark: No fluid is completely incompressible. Gases can compress easily, liquid much less so.

Because of the simplicity of (7) and the way it helps later calculations be made it is often useful to make the approximation of incompressibility, even when considering gases[4].

Note: If ρ is constant the LHS of (5) is zero and conservation of mass can be invoked using

$$\boxed{\int_S \mathbf{u} \cdot \hat{\mathbf{n}} dS = 0} \tag{8}$$

(I.e. the net flux through the surface of a volume is zero.)

E.g. 2.1: Consider $\mathbf{u} = (y, x, 0)$. Then

$$\boldsymbol{\nabla} \cdot \mathbf{u} = \frac{\partial y}{\partial x} + \frac{\partial x}{\partial y} = 0$$

E.g. 2.2: Consider $\mathbf{u} = (x, -y, 0)$. Then

$$\boldsymbol{\nabla} \cdot \mathbf{u} = \frac{\partial x}{\partial x} + \frac{\partial (-y)}{\partial y} = 1 - 1 = 0$$

I.e. both flows are incompressible.

Exercise: Prove (8) holds for both E.g.'s 2.1/2.2 with S defined as a circle of radius a centred on the origin.

[4]Usually OK for very subsonic flows.

2.3 Streamfunctions (for incompressible flows)

Theorem: If $\nabla \cdot \mathbf{u} = 0$, then there exists a *vector field* $\mathbf{A}(\mathbf{r}, t)$ s.t.

$$\mathbf{u} = \nabla \times \mathbf{A} \tag{9}$$

Defn: The vector field $\mathbf{A}$ is called the **vector potential**.

Remark: $\mathbf{A}$ is not unique since if $\mathbf{A}$ is replaced by $\mathbf{A} + \nabla f$ for any scalar potential f, the same $\mathbf{u}$ results from (9). This is because $\nabla \times \nabla f = \mathbf{0}$ for any f (null identities in MVC).

Remark: How easy is it to find $\mathbf{A}$ given $\mathbf{u}$? If you try writing out each component of (9) you'll see things look very complicated – and are, in general – unless the flow has a simplified structure to it.

2.4 Two-dimensional flows

We will consider this description of 2D flows in two different useful coordinate systems.

2.4.1 Cartesians

Consider $\mathbf{A} = (0, 0, \psi(x, y, t))$. Then

$$\nabla \times \mathbf{A} = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, 0 \right) = (u(x, y, t), v(x, y, t), 0) = \mathbf{u}$$

(and clearly $\nabla \cdot \mathbf{u} = 0$).

Key point: In other words, we can represent *both* velocity components of a two-dimensional incompressible flow in terms of a *single* function as

$$\boxed{u(x, y, t) = \frac{\partial \psi}{\partial y}, \qquad v(x, y, t) = -\frac{\partial \psi}{\partial x}} \tag{10}$$

Remark: Why does this make sense ? Because incompressibility means that the flow component in the x-direction has to be related to the component in the y-direction.

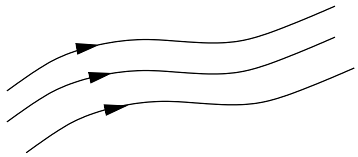
In addition ψ has a very useful physical interpretation. Let $(x(s), y(s), 0)$ be a streamline with s measuring arclength along a streamline as before. Then consider

$$\frac{d\psi}{ds} = \frac{\partial \psi}{\partial x}\frac{dx}{ds} + \frac{\partial \psi}{\partial y}\frac{dy}{ds} = -v\lambda(s)u + u\lambda(s)v = 0$$

using the chain rule and the definition of a streamline.

It follows that $\psi(x, y, t) = const$ on a streamline of the flow.

Defn: The function $\psi(x, y, t)$ is called the **streamfunction**.



E.g. 2.3: Consider $\mathbf{u} = (y, x, 0)$. Then

$$\frac{\partial\psi}{\partial y} = y, \qquad \Rightarrow \qquad \psi(x,y) = \tfrac{1}{2}y^2 + f(x)$$

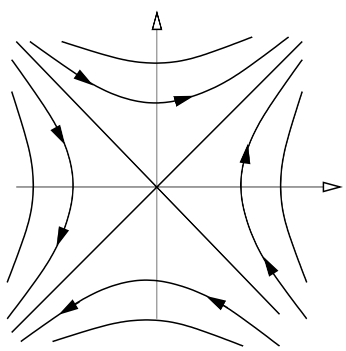
and

$$\frac{\partial\psi}{\partial x} = -x, \qquad \Rightarrow \qquad \psi(x,y) = -\tfrac{1}{2}x^2 + g(y)$$

for some arbitrary functions f, g. These can work together if

$$\psi(x,y) = \tfrac{1}{2}(y^2 - x^2)$$

(an additive constant does not alter the streamlines)



Defn: Points in the flow where $\mathbf{u} = \mathbf{0}$ are called **stagnation points**. For steady flows, streamlines can only cross at stagnation points.

2.4.2 Polars

Consider $\mathbf{A} = \psi(r, \theta, t)\hat{\mathbf{z}}$ in polars ($x = r\cos\theta$, $y = r\sin\theta$). Then (look up definition of curl in polar coordinates)

$$\boldsymbol{\nabla} \times \mathbf{A} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\mathbf{r}} - \frac{\partial\psi}{\partial r}\hat{\boldsymbol{\theta}}$$

It follows that if we write express the 2D flow in polars as $\mathbf{u}(r, \theta, t) = u_r\hat{\mathbf{r}} + u_\theta\hat{\boldsymbol{\theta}}$ then the two scalar components of the velocity in the radial and angular directions are given by

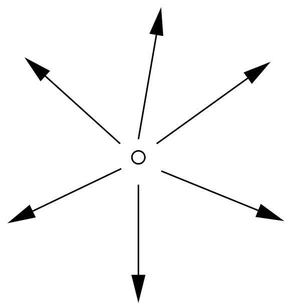
$$\boxed{u_r(r,\theta,t) = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \qquad u_\theta(r,\theta,t) = -\frac{\partial\psi}{\partial r}} \tag{11}$$

Note: Using definition of divergence in polars

$$\boldsymbol{\nabla} \cdot \mathbf{u} = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} = 0$$

after using (11), confirming incompressibility (but we didn't need to of course !)

In polars coordinates, (2) is replaced by $dr/u_r = rd\theta/u_\theta$ and $\psi(r, \theta, t) = const$ define streamlines.[5]



E.g. 2.3: (Rotational flows.) Fluid rotates around the origin. So $u_r = 0$ and u_θ is independent of θ, or:

$$\left.\begin{aligned} u_r &= 0 = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \qquad \Rightarrow \qquad \psi \equiv \psi(r) \\ u_\theta &= f(r) = -\frac{\partial \psi}{\partial r} \end{aligned}\right\}$$

This allows for a variety of flows all with the same streamlines. E.g. if $\psi(r, \theta) = -Ar$ then $u_\theta = A$ (the fluid exhibits **solid body rotation**); if $\psi(r) = -A \ln r$ then $u_\theta = A/r$ (we will come to know this as a **point vortex** or **line vortex**.)

E.g. 2.4: (A **two-dimensional point source** also known as a **line source**.) Fluid flows radially from the origin. So $u_\theta = 0$ and u_r is independent of θ, or:

$$\left.\begin{aligned} u_r &= f(r) = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \\ u_\theta &= 0 = -\frac{\partial \psi}{\partial r} \end{aligned}\right\}$$

2nd eqn. implies $\psi = \psi(\theta)$ but, from 1st eqn. $\partial\psi/\partial\theta$ must be independent of θ. So $\psi = A\theta$ where A constant. Then $f(r) = A/r$.

Defn: The **source strength** is the flux of fluid from the source point.

Since the flow is incompressible, the flux at the origin equals the flux through any closed boundary surrounding the origin (this is just a restatement of (8)).

Most easily calculated by measuring the flux through a circle C, radius r centred at the origin:

$$\text{Source strength, } m = \int_C \mathbf{u} \cdot \hat{\mathbf{n}}\, ds = \int_0^{2\pi} \mathbf{u} \cdot \hat{\mathbf{r}}\, r d\theta = \int_0^{2\pi} u_r \hat{\mathbf{r}} \cdot \hat{\mathbf{r}}\, r d\theta = \int_0^{2\pi} \frac{A}{r}\, r d\theta = 2\pi A.$$

and $A = m/2\pi$ (see line integrals in MVC). Confirms mass flow through a circle surrounding the origin is independent of the radius of the circle.

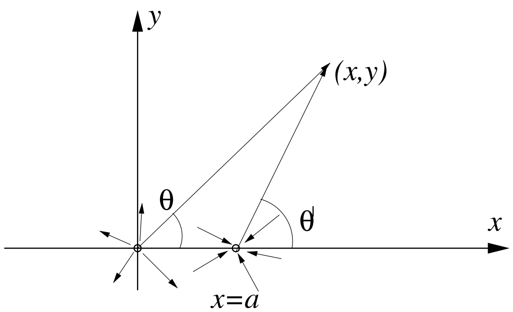
[5]The details of how to show this is a bit tricky. Start with describing the streamline by the curve $\mathbf{r} = r(s)\hat{\mathbf{r}}(s)$ in polars where s is arclength. Then $d\mathbf{r}/ds = (dr/ds)\hat{\mathbf{r}} + r(d\theta/ds)\hat{\boldsymbol{\theta}} = \lambda(s)\mathbf{u} = \lambda(s)(u_r\hat{\mathbf{r}} + u_\theta\hat{\boldsymbol{\theta}})$. This means $dr/ds = \lambda u_r$ and $d\theta/ds = u_\theta/r$ and we can divide one by the other to determine a relation defining streamlines in polars. We can also confirm, using the chain rule, that $d\psi/ds = (\partial\phi/\partial r)(dr/ds) + (\partial\phi/\partial\theta)(d\theta/ds) = 0$ after substituting for each term, from which we assert that ψ is constant along a streamline.

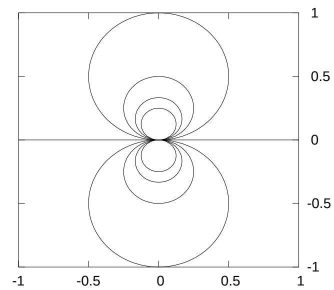
Summary: The flow $\mathbf{u} = \dfrac{m}{2\pi r}\hat{\mathbf{r}}$ is generated by the streamfunction $\psi(r,\theta) = \dfrac{m\theta}{2\pi}$.

Note: converting to Cartesians, $\psi = \dfrac{m}{2\pi}\tan^{-1}(y/x)$.

E.g. 2.5: A (horizontal) **dipole**.

Put a source of strength m at $x = 0$ and an equal **sink** (a source of $-$ve strength) at $x = a$ and let $a \to 0$.





Intuitive result: Nothing ! In some sense correct, but...

$$\psi = \lim_{a\to 0}\left[\frac{m}{2\pi}\theta - \frac{m}{2\pi}\theta'\right] = \lim_{a\to 0}\frac{m}{2\pi}\left[\tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{y}{x-a}\right)\right]$$

Combine this by noting that $\tan(A-B) = (\tan A - \tan B)/(1 + \tan A \tan B)$. Then

$$\begin{aligned}\psi &= \lim_{a\to 0}\frac{m}{2\pi}\tan^{-1}\left(\frac{y/x - y/(x-a)}{1 + y^2/x(x-a)}\right) = \lim_{a\to 0}\frac{m}{2\pi}\tan^{-1}\left(\frac{-ay}{x^2+y^2-ax}\right)\\ &= -\mu\frac{y}{2\pi(x^2+y^2)}\end{aligned}$$

using $\tan^{-1} A \approx A$ when $A \to 0$ and defining $\mu = \lim_{a\to 0}\{ma\}$ be the **dipole strength** s.t. $0 < \mu < \infty$.

Note: Converting to polars, $\psi = -\mu\dfrac{r\sin\theta}{2\pi r^2} = -\mu\dfrac{\sin\theta}{2\pi r}$.

Q: What is ψ for a *vertical* dipole ?

2.5 Three-dimensional (axisymmetric) flows

Again we will use simplified vector potentials now in two different 3D coordinate systems to describe certain flows having some restrictive property – axisymmetry in this case.

2.5.1 Cylindrical polars

Use cylindrical coordinates (r,θ,z) and choose $\mathbf{A} = (\Psi(r,z,t)/r)\hat{\boldsymbol{\theta}}$, then

$$\boldsymbol{\nabla}\times\mathbf{A} = -\frac{1}{r}\frac{\partial\Psi}{\partial z}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\Psi}{\partial r}\hat{\mathbf{z}}$$

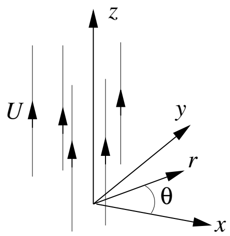
(curl in cylindrical polars). So an incompressible flow $\mathbf{u} = u_r\hat{\mathbf{r}} + u_\theta\hat{\boldsymbol{\theta}} + u_z\hat{\mathbf{z}}$ satisfies

$$u_r(r,z,t) = -\frac{1}{r}\frac{\partial\Psi}{\partial z}, \qquad u_z(r,z,t) = \frac{1}{r}\frac{\partial\Psi}{\partial r}$$

are both independent of θ and $u_\theta = 0$ and the flow is said to be **axisymmetric**.

Can be shown streamlines determined by $dr/u_r = dz/u_z$ or by $\Psi = const$ on streamlines (actually **stream surfaces** or **stream tubes** as 3D flow.)

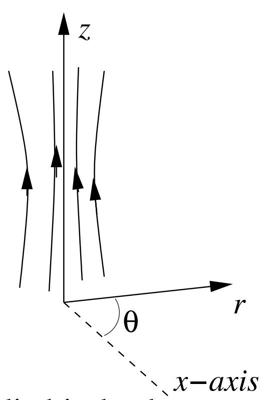
Note: For 3D flows, Ψ is called the **Stokes' streamfunction**.



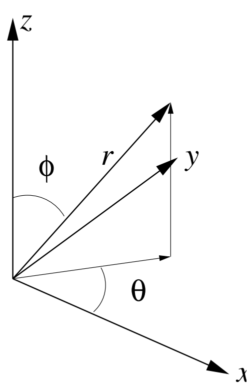
E.g. 2.6: Let $\mathbf{u} = U\hat{\mathbf{z}}$ (uniform flow in direction of z-axis). Then

$$u_r = -\frac{1}{r}\frac{\partial\Psi}{\partial z} = 0, \qquad u_z = \frac{1}{r}\frac{\partial\Psi}{\partial r} = U$$

First eqn implies $\Psi \equiv \Psi(r)$ and 2nd eqn then gives $\Psi = \frac{1}{2}Ur^2$.



Cylindrical polars



Spherical Polars

2.5.2 Spherical polars

Coordinates (r, φ, θ) and with $\mathbf{A} = \dfrac{\Psi(r,\varphi)}{r\sin\varphi}\hat{\boldsymbol{\theta}}$ then

$$\boldsymbol{\nabla}\times\mathbf{A} = \frac{1}{r^2\sin\varphi}\frac{\partial\Psi}{\partial\varphi}\hat{\mathbf{r}} - \frac{1}{r\sin\varphi}\frac{\partial\Psi}{\partial r}\hat{\boldsymbol{\varphi}}$$

from the definition of curl in spherical polars.

Matching to the axisymmetric flow $\mathbf{u} = u_r(r,\varphi)\hat{\mathbf{r}} + u_\varphi(r,\varphi)\hat{\boldsymbol{\varphi}}$ with $u_\theta = 0$ we have

$$u_r = \frac{1}{r^2\sin\varphi}\frac{\partial\Psi}{\partial\varphi}, \qquad u_\varphi = -\frac{1}{r\sin\varphi}\frac{\partial\Psi}{\partial r}$$

Can show again that setting $\Psi = const$ gives stream surfaces.

E.g. 2.7: (a **3D point source**.) Clearly $u_\theta = 0$ and flow does not depend on azimuthal angle θ, so can use streamfunction $\Psi(r, \varphi)$ in spherical polars. Also

$$u_\varphi = 0, \qquad \Rightarrow \qquad \Psi \equiv \Psi(\varphi)$$

whilst

$$u_r = f(r) = \frac{1}{r^2 \sin\varphi} \frac{\partial \Psi}{\partial \varphi}$$

means we must have $\Psi = A - B\cos\varphi$, for A, B constants, and so $f(r) = B/r^2$.

The **source strength** found from measuring flux through *sphere* surrounding origin:

$$\text{Source strength, } m = \int_S \mathbf{u} \cdot \hat{\mathbf{n}}\, dS = \int_0^\pi \int_0^{2\pi} \frac{B}{r^2} r^2 \sin\varphi d\theta d\varphi = 4\pi B$$

since $\mathbf{u} = u_r \hat{\mathbf{r}} = f(r)\hat{\mathbf{r}} = (B/r^2)\hat{\mathbf{r}}$, and $\hat{\mathbf{n}} = \hat{\mathbf{r}}$, $dS = r^2 \sin\varphi d\theta d\varphi$, $S = \{0 < \theta < 2\pi, 0 < \varphi < \pi\}$ (from 1st year Calculus & MVC).

Defining, for e.g., $\Psi = 0$ on $\varphi = 0$ to determine A we get

$$\Psi(\varphi) = \frac{m}{4\pi}(1 - \cos\varphi)$$

Note: We can convert this solution into cylindrical polars (r, θ, z) (where r means a different thing now !) using geometry to get

$$\Psi = \frac{m}{4\pi}\left(1 - \frac{z}{\sqrt{z^2 + r^2}}\right).$$

Key points: One of the two main principles of fluid motion is conservation of mass; this leads to the continuity equation linking changes in density to flow velocity.

If the flow is incompressible then $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$. For such flows we can always find a vector potential $\mathbf{A}$ such that $\mathbf{u} = \boldsymbol{\nabla} \times \mathbf{A}$. This is only really useful for simplified flows, which are 2D flows and 3D flows which are axisymmetric about the z-axis. In these cases it is possible to express the velocity components of the flow in terms of a single 'streamfunction' – this is the key: two unknown flow components plus the conditions of incompressibility are used together to express the flow in terms of a single unknown. The bonus of this approach is that the flow can easily be visualised by sketching lines on which the streamfunction is constant.

We have introduced some basic flows: streaming flows, rotational flows including point vortices and 2D and 3D point sources and dipoles with extensive use of vector calculus in cylindrical and spherical coordinates.

3 Flow dynamics for an incompressible inviscid fluid

3.1 Forces on a fluid

Fluids move in response to the forces that are exerted on each fluid particle or 'parcel'.[6]

These forces are of two types:

- Forces on parcel of fluid, volume δV, that are proportional to δV are known as **body forces**. E.g. gravitational force on δV is $-\delta mg\hat{\mathbf{z}} = -\rho g \delta V \hat{\mathbf{z}}$.
- Forces that are transmitted across the surface element δS of a fluid parcel are called **surface forces**. These require more thought.

When the fluid is a rest, the surface force *must* be in the direction of the normal, $\hat{\mathbf{n}}$ (the definition of a fluid implies it would deform if a tangential force were applied.)

In this course, we consider fluids for which this statement remains true even if they are in motion. Tangential components of forces on fluids when in motion are due to a molecular property of the fluid called **viscosity**.

Defn: An fluid is said to be **inviscid** (or **ideal**) when surface forces act in the normal direction only, even in motion.

That is for an inviscid fluid, the **surface stress** (defined as the force per unit area) on the surface of a fluid parcel is in the direction $\hat{\mathbf{n}}$ normal to a surface element δS and hence the $\delta \mathbf{F}_s = -p\hat{\mathbf{n}}\delta S =$ is force exerted by exterior fluid on fluid inside δV.

The coefficient of proportionality, $p(\mathbf{r}, t)$, represents the force per unit area in normal direction and is called the **pressure**. It is directed inwards (hence the minus sign) because fluids are normally in a state of compression.

Q: Can $p(\mathbf{r}, t)$ also depends on $\hat{\mathbf{n}}$?

A: It can't (proof is long and not important for this course).

That is to say pressure is **isotropic**: at a point in the fluid pressure is the same in all directions.

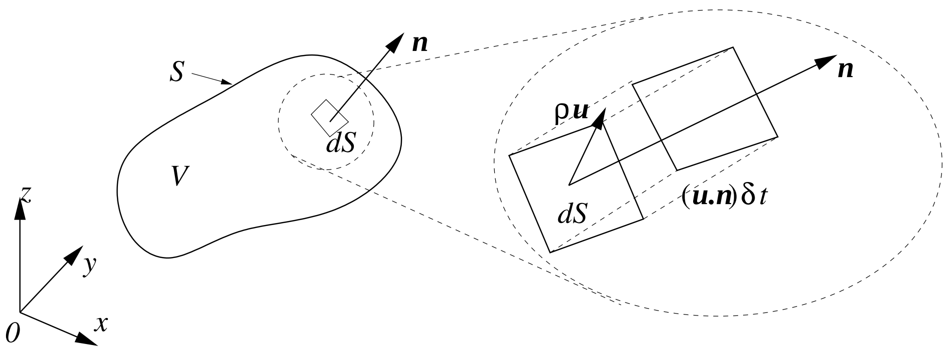
Key points: A fluid responds to forces. There are body forces which act on the bulk (think of gravity). Surface forces act between one layer of a fluid and another. For an ideal fluid, there are no tangential forces between layers (like friction) and forces are normal to surfaces (like reaction forces). The force per unit area in the normal direction is called the pressure and this acts equally in all directions at a point in the fluid.

3.2 Equation of motion

This the equivalent of 'Newton's Law' for an inviscid fluid.

[6] Remember, under the continuum hypothesis a particle is a small finite volume δV with a reference position. As the fluid moves, the shape of δV changes but it contains the same mass.

Consider a fixed volume V with surface S. The momentum within V may change as a result of (i) body forces (ii) surface forces, *and* (iii) fluid momentum transported out of V.



$$\text{Total momentum in } V \; = \int_V \rho \mathbf{u} dV$$

In time δt, the fluid momentum leaving a small section of surface δS is $\rho\mathbf{u}\delta x\delta S$ where $\delta x = (\mathbf{u}\cdot\hat{\mathbf{n}})\delta t$.

So *rate* of momentum transport of momentum (the **momentum flux**) is $\rho\mathbf{u}(\mathbf{u}\cdot\hat{\mathbf{n}})\delta S$.

Thus Newton's Law applied to a volume V of the fluid reads

$$\frac{d}{dt}\int_V \rho\mathbf{u}dV = -\int_S \rho\mathbf{u}(\mathbf{u}\cdot\hat{\mathbf{n}})dS - \int_S p\hat{\mathbf{n}}dS + \int_V \mathbf{f}dV \tag{12}$$

where $\mathbf{f}$ represents the body force *density* or force per unit volume (e.g. for gravity $\mathbf{f} = -\rho g\hat{\mathbf{z}}$).

In each $i = 1, 2, 3$ components,

$$\int_V \frac{\partial}{\partial t}(\rho u_i)dV = -\int_S \rho u_i(u_j n_j)dS - \int_S p n_i dS + \int_V f_i dV.$$

since V is fixed. Now apply the **divergence theorem** to get

$$\int_V \frac{\partial}{\partial t}(\rho u_i)dV = -\int_V \frac{\partial}{\partial x_j}(\rho u_i u_j)dV - \int_V \frac{\partial p}{\partial x_i}dV + \int_V f_i dV.$$

since V is an arbitrary volume within the fluid, we must have

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = -\frac{\partial p}{\partial x_i} + f_i.$$

This is (one form of) the equation of motion, but it can be simplified by expanding as

$$u_i\frac{\partial \rho}{\partial t} + \rho\frac{\partial u_i}{\partial t} + \rho u_j\frac{\partial u_i}{\partial x_j} + u_i\frac{\partial}{\partial x_j}(\rho u_j) = -\frac{\partial p}{\partial x_i} + f_i$$

and using the continuity equation (6) in component form:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0$$

so that we have

$$\rho\frac{\partial u_i}{\partial t} + \rho u_j\frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + f_i.$$

Finally, we can identify this in vector notation as

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla}p + \mathbf{f}$$

or, using the material derivative, as

$$\boxed{\rho\frac{D\mathbf{u}}{Dt} = -\boldsymbol{\nabla}p + \mathbf{f}.} \tag{13}$$

This is the **momentum equation** for the fluid more commonly known as **Euler's equation** (due to Euler 1756).

Remark: (13) contains 3 equations but has four unknowns (the 3 cpts of $\mathbf{u}$ and p.) Mass conservation provides a 4th equation which closes the system.

Remark: Euler's equation can be read as "mass times acceleration equals sum of forces". Note that pressure itself does not act to accelerate a fluid, but pressure *gradients* do.

Remark: The addition of viscosity leads to an additional term on the RHS of (13). The modified equation is called **Navier-Stokes equation**.

E.g. 3.1: From E.g. 2.7 (point source in 3D), $u_r = m/4\pi r^2$, or

$$\mathbf{u} = \frac{m}{4\pi r^2}\hat{\mathbf{r}}$$

in spherical polars. Here, $\boldsymbol{\nabla}\cdot\mathbf{u} = 0$ by construction of solution (but can confirm using divergence in spherical polars:

$$\boldsymbol{\nabla}\cdot\mathbf{u} \equiv \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) = 0 \quad)$$

We also have $\partial\mathbf{u}/\partial t = 0$ whilst

$$\rho(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = \rho\left(u_r\frac{\partial}{\partial r}\right)u_r\hat{\mathbf{r}} = -\frac{\rho m^2}{8\pi^2 r^5}\hat{\mathbf{r}}$$

To find a pressure field able to sustain this flow, equate to $-\boldsymbol{\nabla}p$ from Euler's equation. It follows that $p = p(r)$ so that

$$-\boldsymbol{\nabla}p = -\frac{\partial p}{\partial r}\hat{\mathbf{r}} = -\frac{\rho m^2}{8\pi^2 r^5}\hat{\mathbf{r}}$$

and it follows from equating components of $\hat{\mathbf{r}}$ both sides and integrating up that

$$p = p_0 - \frac{\rho m^2}{32\pi^2 r^4}$$

where p_0 is the pressure at infinity.[7]

Q: Does this make sense ? The pressure increases with r which makes sense because the flow is slowing down in the radial direction, so a force is holding it back.

[7]this is a model problem, so we can do this. Often in practical problems we use infinity as an approximation for 'a long way away'

3.3 Hydrostatics

If $\mathbf{u} = (U_1, U_2, U_3)$ is constant (a uniform streaming flow) then $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$, $\partial \mathbf{u}/\partial t = 0$ and $(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = 0$ and the flow is not accelerating.

Then (13) reduces to

$$\boldsymbol{\nabla} p = \mathbf{f} \tag{14}$$

This means gradients in pressure must balance body forces.

An example we can understand in everyday life is that pressure must increase to support the weight of the fluid above it.

Thus, consider gravity acting as body force. Then $\mathbf{f} = -\rho g \hat{\mathbf{z}}$ and (14) is

$$\boldsymbol{\nabla} p = -\rho g \hat{\mathbf{z}} \tag{15}$$

E.g. 3.3: If ρ is constant $-\rho g \hat{\mathbf{z}} = -\boldsymbol{\nabla}(\rho g z)$ so that (14) becomes

$$\boldsymbol{\nabla}(p + \rho g z) = 0$$

This integrates to

$$p + \rho g z = const \tag{16}$$

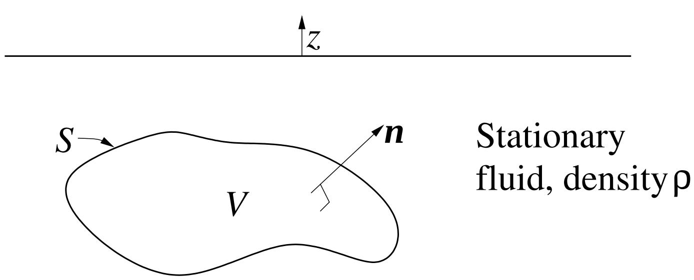
I.e. pressure increases with depth (remember z points upwards).

For a lake, $\rho = 1000\text{kgm}^{-3}$ and $g = 10\text{ms}^{-2}$ so pressure increases by 10KPa for every metre below the surface.

E.g. 3.4: If $\rho = \rho(z)$ (as in the oceans, atmosphere), then (15) can be integrated to give

$$p(z) = p(z_0) - g \int_{z_0}^{z} \rho(z') dz'$$

E.g. 3.5: (Archimedes' principle $\sim$ 250BC)



Consider a static arrangement of a body of volume V with surface S immersed in fluid of density ρ. The forces due to the fluid acting on that body are surface forces (i.e. $-p\hat{\mathbf{n}}\delta S$ integrated over S). Thus

$$\text{Force on submerged body } = -\int_S p\hat{\mathbf{n}} dS = -\int_V \nabla p dV = \hat{\mathbf{z}} \int_V \rho g \, dV$$

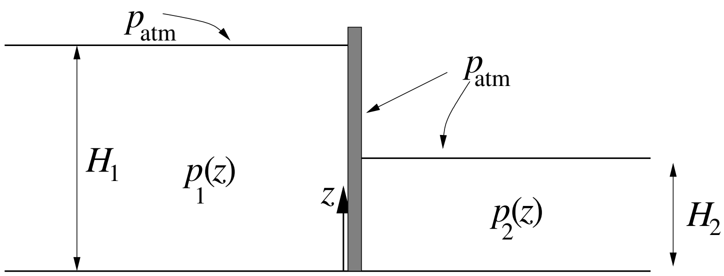
using the divergence theorem and (15).

Therefore force on body of volume V due to the fluid is equal to the weight of fluid displaced.

Q: What about a body intersecting two fluids of different densities (air/water) ?

3.3.1 Problem: Lock gates

Find the force required to hold back a body of fluid behind a lock gate.



In $x < 0$, solve (14) for p_1: $\dfrac{\partial p_1}{\partial x} = \dfrac{\partial p_1}{\partial y} = 0$, $\dfrac{\partial p_1}{\partial z} = -\rho g$ to give $p_1(z) + \rho g z = const$ as in (15). We also have $p = p_{atm}$ (constant atmospheric pressure[8]) on $z = H_1$ so

$$p_1(z) = p_{atm} + \rho g(H_1 - z), \qquad 0 < z < H_1$$

and similarly, in $x > 0$,

$$p_2(z) = \begin{cases} p_{atm}, & z > H_2, \\ p_{atm} + \rho g(H_2 - z), & 0 < z < H_2, \end{cases}$$

Force exerted *by* fluid *on* gate (per unit width of gate) from $x < 0$ is

$$\mathbf{F}_1 = \int_0^{H_1} p_1(z)\hat{\mathbf{n}}dz$$

Here $\hat{\mathbf{n}} = \hat{\mathbf{x}}$ is the vector pointing *out* of the fluid into the gate. Thus the x-component is, say,

$$F_1 = \mathbf{F}_1 \cdot \hat{\mathbf{x}} = \int_0^{H_1} p_1(z)\,dz = p_{atm}H_1 + \tfrac{1}{2}\rho g H_1^2$$

Similarly, the force exerted *by* fluid *on* the gate from $x > 0$ is

$$F_2 = -\int_0^{H_1} p_2(z)\,dz = -p_{atm}H_1 - \tfrac{1}{2}\rho g H_2^2$$

(the minus sign is because $\hat{\mathbf{n}} = -\hat{\mathbf{x}}$ in the fluid to the right of the gate).

Thus the *net* horizontal force is

$$F_1 + F_2 = \frac{1}{2}\rho g(H_1^2 - H_2^2)$$

[8] we can prove/we will show later that pressure is continuous across fluid/fluid interfaces such as water/air.

Checks: If $H_1 > / = / < H_2$ then forces act in right way.

Q: What about torque ? (Problem sheet)

3.4 The momentum integral theorem (for steady flows)

For *steady flows*, LHS of (12) is zero and if we also assume a conservative body force, $\mathbf{f} = -\rho\boldsymbol{\nabla}\Phi$[9] then

$$0 = -\int_S \rho\mathbf{u}(\mathbf{u}\cdot\hat{\mathbf{n}})dS - \int_S p\hat{\mathbf{n}}dS - \int_V \rho\boldsymbol{\nabla}\Phi\, dV$$

Using the corollary to the divergence theorem in Appendix A.9.1 we get

$$\boxed{\int_S \rho\mathbf{u}(\mathbf{u}\cdot\hat{\mathbf{n}}) + (p+\rho\Phi)\hat{\mathbf{n}}\, dS = 0} \qquad (17)$$

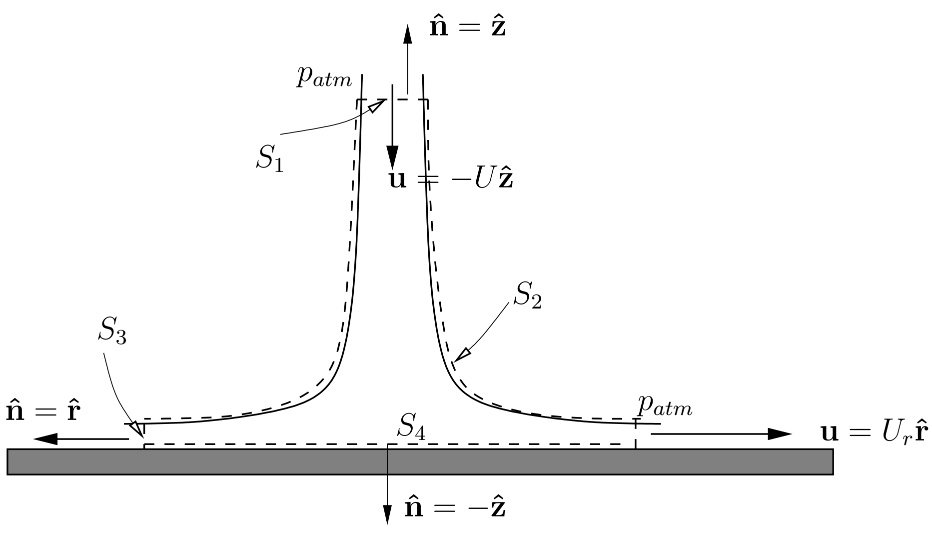
for a fixed closed surface S surrounding V.

Eqn (17) is referred to as the **momentum integral theorem**.[10]

Note: The divergence theorem works in 2D and 3D and so S can be a closed 2D surface surrounding a 3D volume or a closed 1D curve surrounding a 2D surface.

3.4.1 Problem: Jet impinging on a wall

Find the force exterted by axisymmetric jet hitting a the wall.



[9]For gravity $\Phi = gz$

[10]One can regard the relationship of (17) to (13) in the same way as (8) is related to (6). Thus (17) balances momentum flux through fixed surface in the same way that (8) balances mass flux through a fixed surface. The advantage of using either of these is that you need only apply them on the bounding surface of the flow where you have data rather than everywhere in the volume where the detail of the flow is not known or, perhaps, important.

Ignore the effects of gravity (you can make a technical argument that the timescales over which such a flow evolves mean that gravity has a negligible effect on the dynamics, but as students – just accept it.)

Take a **control surface**, S, around the flow cutting the flow far from the point of impingement. Divide S into four segments: $S = S_1 \cup S_2 \cup S_3 \cup S_4$ (see figure). Assume uniform outward flow parallel to wall across S_3, $\mathbf{u} = U_r \hat{\mathbf{r}}$ and uniform input flow across S_1, $\mathbf{u} = -U\hat{\mathbf{z}}$. So U is given and U_r is something to be found.

Also assume pressure on all boundaries apart from wall S_4 is p_{atm} (constant atmospheric). Then (17) with $\Phi = 0$ reads

$$\int_S \rho \mathbf{u}(\mathbf{u} \cdot \hat{\mathbf{n}}) + p\hat{\mathbf{n}}\, dS = 0.$$

Now, here's a neat trick that often helps in this type of problem:

$$\int_S p\hat{\mathbf{n}} dS = \int_S (p - p_{atm})\hat{\mathbf{n}} dS + \int_S p_{atm}\hat{\mathbf{n}} dS = -\int_{S_4} (p - p_{atm})\hat{\mathbf{z}}\, dS$$

since $\int_S p_{atm}\hat{\mathbf{n}} dS = \int_V \boldsymbol{\nabla} p_{atm} dV = 0$ (corollary to divergence theorem) since p_{atm} is constant.

Also,

$$\int_S \rho \mathbf{u}(\mathbf{u} \cdot \hat{\mathbf{n}})\, dS = \int_{S_1} \rho U^2 \hat{\mathbf{z}}\, dS + \int_{S_3} \rho U_r^2 \hat{\mathbf{r}}\, dS.$$

since $\mathbf{u} \cdot \hat{\mathbf{n}} = 0$ on the wall and the fluid surface[11]

Thus we have

$$0 = -\int_{S_4} (p - p_{atm})\hat{\mathbf{z}}\, dS + \int_{S_1} \rho U^2 \hat{\mathbf{z}}\, dS + \int_{S_3} \rho U_r^2 \hat{\mathbf{r}}\, dS$$

and if we dot product with $\hat{\mathbf{z}}$ we get ($\hat{\mathbf{z}} \cdot \hat{\mathbf{r}} = 0$)

$$\int_{S_4} (p - p_{atm}) dS = \rho U^2 A$$

where $A = \int_{S_1} dS$ equals the cross-sectional area of the incoming jet. The left-hand side is the integrated pressure (in excess of atmospheric) exerted by the fluid integrated over the wall which equals the force exterted by wall on fluid and is therefore $\rho U^2 A$.

Q: What about U_r ? Can do this once you know about ...

3.5 Bernoulli's equation for steady flows

We start with the vector identity (see Appendix A.8(3))

$$(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = \boldsymbol{\nabla}(\tfrac{1}{2}\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \times \boldsymbol{\omega}$$

[11] i.e. there is no flow across these surfaces so the component of $\mathbf{u}$ in the direction normal to the fixed surface is zero.

where we have defined $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ (we shall come to know this as the **vorticity**).

Write $|\mathbf{u}|^2 = \mathbf{u} \cdot \mathbf{u}$ and use above in (13)

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times \boldsymbol{\omega} = -\boldsymbol{\nabla}(p/\rho + \Phi + \tfrac{1}{2}|\mathbf{u}|^2) \tag{18}$$

and we have assumed $\mathbf{f} = -\rho\nabla\Phi$ as in §3.4. The flow is steady, so $\dfrac{\partial}{\partial t} = 0$ and taking the dot product with $\mathbf{u}$ on both sides gives

$$\mathbf{u} \cdot (\mathbf{u} \times \boldsymbol{\omega}) = 0 = \mathbf{u} \cdot \nabla(p/\rho + \Phi + \tfrac{1}{2}\mathbf{u}^2)$$

The definition of a streamline is $\dfrac{d\mathbf{r}}{ds} = \lambda(s)\mathbf{u}$, so

$$\frac{d}{ds}(p/\rho + \Phi + \tfrac{1}{2}|\mathbf{u}|^2) = \frac{dx_i}{ds}\frac{\partial}{\partial x_i}(p/\rho + \Phi + \tfrac{1}{2}|\mathbf{u}|^2) = \lambda\mathbf{u} \cdot \boldsymbol{\nabla}(p/\rho + \Phi + \tfrac{1}{2}|\mathbf{u}|^2) = 0$$

Consequently,

$$\boxed{p/\rho + \Phi + \tfrac{1}{2}|\mathbf{u}|^2 = const, \qquad \text{along any streamline in the flow}} \tag{19}$$

This is **Bernoulli's equation** (for steady flows.)

Remark: Ignoring body forces, the Bernoulli equation says that pressure reduces when speed increases. This explains how two pieces of paper attract when you blow between them, contrary to basic intuition.

E.g. 3.6: (Jet against the wall) Follow a streamline along the surface of the flow from the incoming jet to the outgoing jet. There is no gravity so $\Phi = 0$ and $p = p_{atm}$ at all points. So Bernoulli gives us speed is constant and $U_r = U$.

Note: The solution to the jet problem is not yet complete as haven't used conservation of mass. What would this tell you ? How would you apply it ?

E.g. 3.7: (3D point source). From E.g. 2.7

$$\mathbf{u} = \frac{m}{4\pi r^2}\hat{\mathbf{r}}$$

and the flow moves steadily outwards along radial lines from the origin. Use Bernoulli along one of these streamlines from infinity (where $p = p_0$) to a general point:

$$p_0/\rho + \tfrac{1}{2}0^2 = p/\rho + \tfrac{1}{2}|\mathbf{u}|^2$$

to give

$$p = p_0 - \frac{\rho m^2}{32\pi^2 r^4}$$

the same as the answer in E.g. 3.2

E.g. 3.8: A small rod is fixed vertically in a stream of constant speed U. How high does the water rise up the rod ?

The flow is steady, so can use Bernoulli along a streamline. Choose the streamline on the surface (i.e. $p = p_{atm}$) of the stream which connects a point far upstream (set to $z = 0$) to a stagnation point at the front of the rod, ($z = h$). Along this streamline $p/\rho + gz + \frac{1}{2}|\mathbf{u}|^2$ is constant. So

$$p_{atm}/\rho + g.0 + \tfrac{1}{2}U^2 = p_{atm}/\rho + gh + \tfrac{1}{2}0^2$$

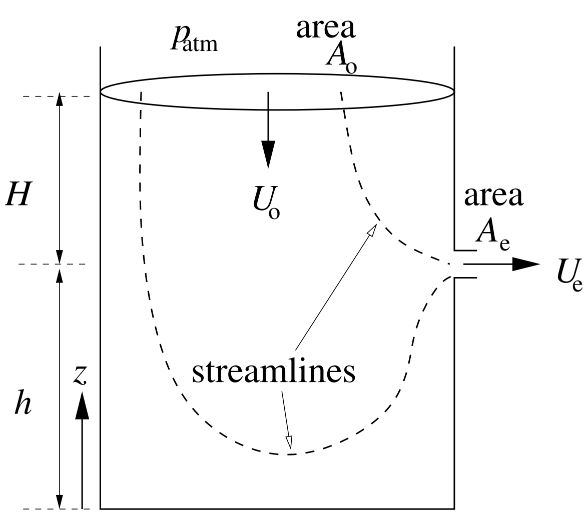
gives

$$h = U^2/2g$$

(about 5cm for $U = 1$m/s)

3.5.1 Problem: Flow out of a tank

A tank of uniform cross section A_0 has a small hole, area A_e, at a height h above the base. The height of the fluid above the hole is H. Find (i) the flow speed from the hole; (ii) the time for the fluid to drain; and (iii) the distance travelled by the jet out of the hole.



Since the hole is small and the level of fluid in the tank falls very slowly reasonable approximation is that the flow is steady at each instant in time.

Here gravity is in play, so $\Phi = gz$. Set $z = 0$ at the bottom of the base (this is an arbitrary choice and doesn't affect the final answer).

Streamlines connect the surface to the exit.

Along any one of these streamlines apply Bernoulli's equation, so

$$p_{atm}/\rho + g(h + H) + \tfrac{1}{2}U_0^2 = p_{atm}/\rho + gh + \tfrac{1}{2}U_e^2$$

or

$$gH = \tfrac{1}{2}(U_e^2 - U_0^2) \tag{20}$$

One equation, but two unknowns (U_0 and U_e).

However, conservation of mass hasn't yet been used ! Using (8) we can write

$$A_0 U_0 = A_e U_e \tag{21}$$

since the fluxes across two boundaries must be equal.

Using (21) in (20) gives

$$U_e = \frac{\sqrt{2gH}}{\sqrt{1-(A_e/A_0)^2}}$$

Since it has been assumed that $A_e \ll A_0$,

$$U_e \approx \sqrt{2gH}.$$

(ii) If $H = H(t)$ is the height of the fluid above the hole, then the velocity of this surface is dH/dt. Since $H(t)$ is measured upwards,

$$\frac{dH}{dt} = -U_0 = -\left(\frac{A_e}{A_0}\right) U_e$$

by (21). I.e.

$$\frac{dH}{dt} \approx -(A_e/A_0)\sqrt{2gH}.$$

If initially $H = H_0$ at $t = 0$ and $H = 0$ when $t = t_d$, the draining time is found by integrating up:

$$(A_e/A_0)\int_0^{t_d} dt = -\int_{H_0}^{0} \frac{dH}{\sqrt{2gH}}$$

or

$$t_d \approx (A_0/A_e)\sqrt{2H_0/g}$$

(iii) Distance travelled by the jet. The flow speed is maximum when $t = 0$ and $H = H_0$. Here $U_e \approx \sqrt{2gH_0}$.

As soon as fluid particles emerge from the hole they are at atmospheric pressure and the only forces they are subject to are gravitational. Thus they obey the same equations as a projectile fired horizontally with speed U_e.

It is a simple to calculate the time taken to hit the ground as $t = \sqrt{2h/g}$ and the distance travelled to be

$$x = U_e t = \sqrt{2gH_0}\sqrt{2h/g} = 2\sqrt{H_0 h}$$

If we assume the filling height is fixed $H_f = H_0 + h$ then

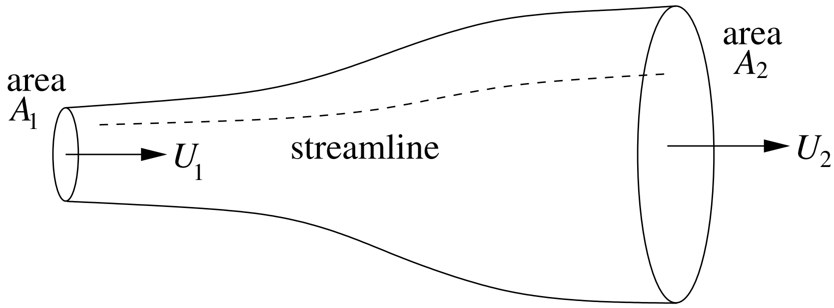
$$x = 2\sqrt{(H_f - h)h}$$

and this is maximised when $dx/dh = 0$, which a simple calculation shows is when $h = \frac{1}{2}H_f$ and whence $x_{max} = H_f$.

Experiment: You can do an experiment with a water bottle to check these predictions !

3.5.2 Problem: Steady flow through a slowly diverging pipe

Find the pressure jump needed to drive a flow through a pipe of varying cross-section.



Mass conservation (equating mass fluxes at two ends of pipe using (8)

$$A_1 U_1 = A_2 U_2$$

Bernoulli's equation (ignoring gravity) along streamline in flow:

$$\tfrac{1}{2}\rho U_1^2 + p_1 = \tfrac{1}{2}\rho U_2^2 + p_2$$

Hence, the pressure difference needed to sustain this flow is

$$\Delta p = p_2 - p_1 = \frac{1}{2}\rho(U_1^2 - U_2^2) = \frac{1}{2}\rho U_1^2(1 - A_1^2/A_2^2) > 0$$

(as the pipe widens, the flow slows down and the pressure increases).

Key points: The motion of ideal fluid is governed by the Euler equation, which expresses conservation of momentum and is the fluid equivalent of Newton's Law for solid body mechanics.

If the flow is uniform and steady or at rest, then the body forces must balance pressure gradients. When there are no body forces, this means the background pressure is constant; under gravity the pressure gradient acts as a force against gravity and pressure increases with depth to support the weight of the fluid above.

There are two alternative manifestations of Euler's equation when the flow is steady: (i) the momentum integral theorem expresses the balance of momentum flux within a closed surface; (ii) Bernoulli's equation also expresses conservation of momentum flux along individual streamlines in the flow.

Without proof, we have asserted that pressure is continuous across interfaces between different surfaces.

To solve a problem, one can use *either* Euler's equation directly, the momentum integral theorem or Bernoulli's equation (but no more than one of these as this will duplicate information). To find the complete solution to a problem one also needs to use conservation of mass as this principle is not integrated into the momentum conservation equations.

4 Vorticity

4.1 Analysis of effects of local fluid motion

Consider a short line segment in the fluid, with ends $\mathbf{r}$ and $\mathbf{r} + \delta\mathbf{r}$. At a small time δt later, the two ends have been advected by the fluid to the points

$$\mathbf{r} + \mathbf{u}(\mathbf{r}, t)\delta t, \qquad \text{and} \qquad \mathbf{r} + \delta\mathbf{r} + \mathbf{u}(\mathbf{r} + \delta\mathbf{r}, t)\delta t$$

Expanding the latter term using multivariable Taylor's theorem (MVC), gives

$$\mathbf{r} + \delta\mathbf{r} + \mathbf{u}(\mathbf{r}, t)\delta t + \delta\mathbf{r} \cdot \boldsymbol{\nabla}\mathbf{u}(\mathbf{r}, t)\delta t + \ldots$$

Thus the two ends of the line segment have undergone rigid-body translation in the flow apart from a term

$$\delta\mathbf{r} \cdot \boldsymbol{\nabla}\mathbf{u}(\mathbf{r}, t) \equiv \delta\mathbf{u}, \text{ say}$$

We continue by investigating this 'relative motion' term:

In suffix notation,

$$\begin{aligned} [\delta\mathbf{u}]_i = \delta x_j \frac{\partial u_i}{\partial x_j} &= \delta x_j \left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)\right] \\ &\equiv \delta x_j \left[e_{ij} + \Omega_{ij}\right] \\ &\equiv [\delta\mathbf{v}]_i + [\delta\mathbf{w}]_i \end{aligned} \tag{22}$$

Note: The **tensors** $e_{ij} = e_{ji}$ and $\Omega_{ij} = -\Omega_{ji}$ are said to be symmetric and antisymmetric (respectively).

4.1.1 The antisymmetric part

Consider from the definition

$$\epsilon_{ijk}\Omega_{ij} = \frac{1}{2}\epsilon_{ijk}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}\left(\epsilon_{ijk}\frac{\partial u_i}{\partial x_j} - \epsilon_{jik}\frac{\partial u_i}{\partial x_j}\right) = -\epsilon_{kji}\frac{\partial u_i}{\partial x_j} = -[\nabla \times \mathbf{u}]_k \tag{23}$$

Let us define

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}$$

Then (23) says $\epsilon_{ijk}\Omega_{ij} = -\omega_k$. Now consider (tricky, but necessary !) multiplying both sides with ϵ_{lmk}:

$$\epsilon_{lmk}\epsilon_{ijk}\Omega_{ij} = -\epsilon_{lmk}\omega_k$$

The LHS is

$$\epsilon_{lmk}\epsilon_{ijk}\Omega_{ij} = \epsilon_{klm}\epsilon_{kij}\Omega_{ij} = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\Omega_{ij} = \Omega_{lm} - \Omega_{ml} = 2\Omega_{lm}$$

(using a double product and antisymmetry). This means that

$$\Omega_{lm} = -\tfrac{1}{2}\epsilon_{lmk}\omega_k$$

or, to make it easier, relabel as

$$\Omega_{ij} = -\tfrac{1}{2}\epsilon_{ijk}\omega_k$$

This means the contribution of the antisymmetric component to the local velocity anomaly is

$$[\delta \mathbf{w}]_i = \delta x_j \Omega_{ij} = -\delta x_j \tfrac{1}{2}\epsilon_{ijk}\omega_k = \tfrac{1}{2}[\boldsymbol{\omega} \times \delta \mathbf{r}]_i$$

and so

$$\delta \mathbf{w} = \tfrac{1}{2}\boldsymbol{\omega} \times \delta \mathbf{r}$$

This corresponds (e.g. Mech 1) to *rigid body rotation* of angular velocity $\frac{1}{2}\boldsymbol{\omega}$.

Defn: The vector $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{u}$ is called the **vorticity** and is a measure of the local rate of rotation in the flow field described by $\mathbf{u}$.

Defn: A flow is called **irrotational** if $\boldsymbol{\omega} = \mathbf{0}$.

4.1.2 The symmetric part

It turns out (details ommitted) that under a particular choice of right-angled axes (called the **principal axes of strain**) that e_{ij} diagonalises so that $e_{ij} = 0$ if $i \neq j$.

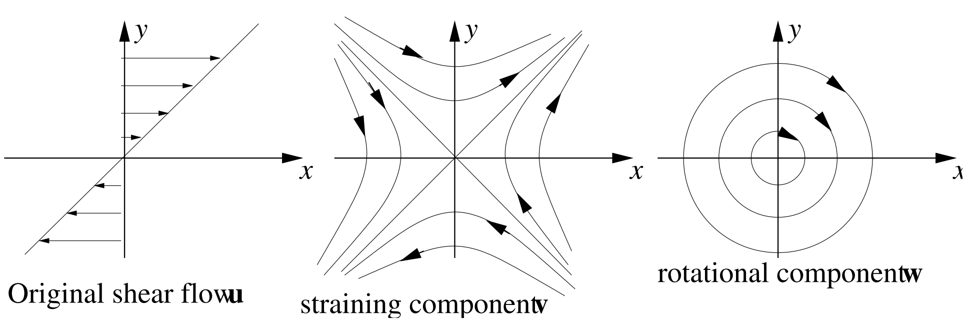
Then the contribution of the symmetric part to the relative motion between the end points of the line segment is

$$\delta \mathbf{v} = \delta \mathbf{r} \cdot \mathbf{e}, \qquad \text{where } \mathbf{e} = (e_{11}, e_{22}, e_{33})$$

This implies a stretching (when $e_{ii} > 0$) and compression (when $e_{ii} < 0$) along respective axes of strain.

Defn: e_{ij} is called the **rate of strain tensor**.

E.g. 4.1: Consider a **shear flow**, $\mathbf{u} = (ky, 0, 0)$.



The vorticity is $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{u} = -k\hat{\mathbf{z}}$ (Check !)

For this simple (linear) example we can write

$$\mathbf{u} = \left(\frac{1}{2} \begin{pmatrix} 0 & k & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & k & 0 \\ -k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \mathbf{r}$$

where $\mathbf{r} = (x, y, z)$.

Thus $\mathbf{u} = \mathbf{v} + \mathbf{w}$ where

- $\mathbf{v} = (\frac{1}{2}ky, \frac{1}{2}kx, 0)$ is called a **straining flow** (see E.g. 2.3) and is irrotational: $\boldsymbol{\nabla} \times \mathbf{v} = 0$.
- $\mathbf{w} = (\frac{1}{2}ky, -\frac{1}{2}kx, 0)$ is a rotating flow (see E.g. 1.5).

E.g. 4.2: In E.g. 2.3 we considered a general rotating flow $\mathbf{u} = f(r)\hat{\boldsymbol{\theta}}$ (in cylindrical polars). According to the definition of curl in cylindrical polars,

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \times (f(r)\hat{\boldsymbol{\theta}}) = \frac{1}{r}\frac{\partial}{\partial r}(rf(r))\hat{\mathbf{z}}$$

Thus rotating flow can only be irrotational if $f(r) = A/r$ or $\mathbf{u} = (A/r)\hat{\boldsymbol{\theta}}$.

Defn: This is a **point vortex** (i.e. all the vorticity is at the origin).

Key points: A general fluid flow observed at a local scale exhibits three separate effects: a rigid body translation, a rigid body roation and stretching/compression along perpendicular axes.

The rotation is attributed to a vorticity field in the flow. If the vorticity is zero through the flow (apart from perhaps at isolated singularites – point vortices), it is said to be irrotational.

4.2 The vorticity equation

We start with Euler's equation written as in (18), namely

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times \boldsymbol{\omega} = -\boldsymbol{\nabla}(p/\rho + \Phi + \tfrac{1}{2}|\mathbf{u}|^2)$$

Taking the curl of this gives

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \boldsymbol{\nabla} \times (\mathbf{u} \times \boldsymbol{\omega}) = \mathbf{0}$$

since $\boldsymbol{\nabla} \times \boldsymbol{\nabla} f = 0$. We can use a vector identity:

$$\boldsymbol{\nabla} \times (\mathbf{u} \times \boldsymbol{\omega}) = \mathbf{u}(\boldsymbol{\nabla} \cdot \boldsymbol{\omega}) - \boldsymbol{\omega}(\boldsymbol{\nabla} \cdot \mathbf{u}) + (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\mathbf{u} - (\mathbf{u} \cdot \boldsymbol{\nabla})\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\mathbf{u} - (\mathbf{u} \cdot \boldsymbol{\nabla})\boldsymbol{\omega}$$

since the flow is incompressible and $\boldsymbol{\nabla} \cdot \boldsymbol{\omega} = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{u}) = 0$. Hence

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\mathbf{u}$$

or

$$\boxed{\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\mathbf{u}} \tag{24}$$

This is called the **vorticity equation**.

E.g. 4.3: Consider a flow of the form $\mathbf{u} = (0, 0, (1 - x^2 - y^2)f(t))$ where $f(t)$ is an arbitrary function of time. Then (check)

$$\boldsymbol{\omega} = \boldsymbol{\nabla}\times\mathbf{u} = (-2yf(t), 2xf(t), 0)$$

which means the RHS of (24) is

$$\left(-2yf(t)\frac{\partial}{\partial x} + 2xf(t)\frac{\partial}{\partial y}\right)(0, 0, (1 - x^2 - y^2)f(t)) = (0, 0, 4xy - 4yx)f^2(t) = \mathbf{0}$$

Equating to the LHS

$$(-2y, 2x, 0)f'(t) + (1 - x^2 - y^2)f(t)\frac{\partial}{\partial z}(-2yf(t), 2xf(t), 0) = \mathbf{0}$$

means $f'(t) = 0$ and so $f(t) = \textit{constant}$ are only possible solutions.

4.2.1 Vorticity in 2D

In 2D flows, $\mathbf{u} = (u(x, y), v(x, y), 0)$ and so, by definition $\boldsymbol{\omega} = (0, 0, \omega(x, y)) \equiv \omega(x, y)\hat{\mathbf{z}}$. It is then clear that the term

$$(\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\mathbf{u} = \left(\omega(x, y)\frac{\partial}{\partial z}\right)\mathbf{u} = \mathbf{0}$$

and so

$$\frac{D\boldsymbol{\omega}}{Dt} = \mathbf{0}$$

That is, **vorticity is conserved as it moves with the flow**.

Note: If $\boldsymbol{\omega} = 0$ at time $t = 0$, then $\boldsymbol{\omega} = 0$ for all time; vorticity cannot be generated in a 2D flow.

4.3 Kelvin's circulation theorem

In 3D the conservation of vorticity (the fluid analogue of conservation of angular momentum in solid-body mechanics) takes a more subtle form:

Defn: The **circulation** of a velocity field is defined to be

$$\boxed{\Gamma = \int_{C(t)} \mathbf{u}\cdot d\mathbf{r}} \tag{25}$$

where $C(t)$ is a closed loop which moves with the fluid and $d\mathbf{r}$ is an infinitesimal line segment along $C(t)$.

Note: By Stokes' theorem

$$\Gamma = \int_{C(t)} \mathbf{u} \cdot d\mathbf{r} = \int_{S(t)} (\boldsymbol{\nabla} \times \mathbf{u}) \cdot \hat{\mathbf{n}} dS = \int_{S(t)} \boldsymbol{\omega} \cdot \hat{\mathbf{n}} dS$$

where $S(t)$ is any surface whose edges connect with $C(t)$.

Kelvin's theorem: Circulation is conserved as it moves with the flow, or $\boxed{\frac{D\Gamma}{Dt} = 0}$.

Proof:

$$\frac{D\Gamma}{Dt} = \frac{D}{Dt} \int_{C(t)} \mathbf{u} \cdot d\mathbf{r} = \int_{C(t)} \frac{D\mathbf{u}}{Dt} \cdot d\mathbf{r} = \int_{C(t)} -\nabla(p/\rho + \Phi) \cdot d\mathbf{r}$$

where D/Dt has been taken under the integral since the path $C(t)$ moves with the fluid.

Now, by Stokes' theorem

$$\frac{D\Gamma}{Dt} = -\int_{S(t)} \boldsymbol{\nabla} \times \boldsymbol{\nabla}(p/\rho + \Phi) \cdot \hat{\mathbf{n}} dS = 0$$

using $\boldsymbol{\nabla} \times \boldsymbol{\nabla} f = 0$ for any f.

Note: Irrotational flows can have circulation if they have point sources of vorticity.

E.g. 4.4: Consider the point vortex $\mathbf{u} = (A/r)\hat{\boldsymbol{\theta}}$ (E.g. 4.2)

Then $\boldsymbol{\omega} = \nabla \times \mathbf{u} = \mathbf{0}$ apart from at $r = 0$. Choose C to be the circle $r = R$

$$\Gamma = \int_C \mathbf{u}|_{r=R} \cdot d\mathbf{r} = \int_0^{2\pi} \frac{A}{R} \hat{\boldsymbol{\theta}} \cdot R\hat{\boldsymbol{\theta}} d\theta = \int_0^{2\pi} A \, d\theta = 2\pi A.$$

(following MVC for computing line integrals: parametrise C by $\mathbf{r} = R\hat{\mathbf{r}} = R(\cos\theta, \sin\theta)$, so $d\mathbf{r} = (d\mathbf{r}/d\theta)d\theta = R(-\sin\theta, \cos\theta)d\theta = R\hat{\boldsymbol{\theta}}d\theta$.)

Therefore, a point vortex can be defined in terms of its circulation by $\mathbf{u} = (\Gamma/2\pi r)\hat{\boldsymbol{\theta}}$ (in an analogous way to a point source being defined in terms of its source strength).

4.3.1 Problem: Circulation in a shear flow

Consider the shear flow $\mathbf{u} = (ky, 0, 0)$. At time $t = 0$ let C_0 denote the circle centred on the origin of radius a.

(i) Calculate $C(t)$, the locus of this circle as it moves with the flow.

(ii) Calculate the circulation, Γ, around $C(t)$.

(iii) Find the answer to (ii) another way.

(i) Consider a tracer particle placed at $(x_0, y_0) = (a\cos\theta, a\sin\theta)$ at $t = 0$. The curve C_0 is formed from points (x_0, y_0) for $0 < \theta < 2\pi$. To get $C(t)$ we consider tracking the tracer particles as the flow evolves which means integrating the particle paths

$$\begin{cases} \dfrac{dx}{dt} = ky, & x(0) = a\cos\theta \\ \dfrac{dy}{dt} = 0, & y(0) = a\sin\theta \end{cases}$$

which integrates to $y(t) = a\sin\theta$ (a constant) first and then $x = a\cos\theta + kat\sin\theta$. So $C(t)$ is the curve $\mathbf{r} = (a\cos\theta + kat\sin\theta, a\sin\theta, 0)$ for $0 < \theta < 2\pi$.

(ii) The circulation is

$$\Gamma = \int_{C(t)} \mathbf{u}\cdot d\mathbf{r} = \int_0^{2\pi} (ka\sin\theta, 0, 0)\cdot\frac{d\mathbf{r}}{d\theta}d\theta = \int_0^{2\pi} (ka\sin\theta, 0, 0)\cdot(-a\sin\theta + kat\cos\theta, a\cos\theta, 0)\, d\theta$$

(path integrals in MVC). This integrates to give

$$\Gamma = -k\pi a^2$$

(iii) Kelvin's circulation theorem tells us that the value of Γ found in part (ii) is constant (and our answer confirms this). Therefore it is the same as at $t = 0$ when $C(t) = C_0$ a circle. For our flow $\mathbf{u} = (ky, 0, 0)$ a simple calculation gives us $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\mathbf{u} = -k\hat{\mathbf{z}}$. So

$$\Gamma = \int_{C_0} \mathbf{u}\cdot d\mathbf{r} = \int_{S_0} \boldsymbol{\omega}\cdot\hat{\mathbf{z}}dS$$

where S_0 is the interior of the circle, radius a, centre $(0, 0)$, lying in the $x - y$ plane with normal $\hat{\mathbf{z}}$ (by the right-hand thumb rule in MVC). Thus

$$\Gamma = -k\int_{S_0} dS = -k\pi a^2$$

Key points: The vorticity equation is just version of Euler's equation which tells us about conservation of angular momentum, rather than translational momentum, in the flow. It provides information about how vorticity is transported by the flow. For 2D flows, it tells us that vorticity is conserved and so a flow with no initial vorticity will be free of vorticity for all time. More generally, Kelvin's circulation theorem tells us that a the circulation in the flow (related to vorticity) is conserved as you move with the flow.

5 Irrotational flows: potential theory

5.1 The velocity potential

If $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{u} = 0$ throughout the domain (apart from at isolated singularities) – i.e. irrotational – it follows that there exists[12] a $\phi(\mathbf{r}, t)$ s.t.

$$\boxed{\mathbf{u} = \boldsymbol{\nabla}\phi} \tag{26}$$

Defn: ϕ is called the **velocity potential**.

If the fluid is also incompressible then, $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$ and

$$\boxed{\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\phi \equiv \nabla^2\phi = 0} \tag{27}$$

That is, ϕ satisfies **Laplace's equation**.

Remark: Similar concept to streamfunction for incompressible flows in that flow components are determined as derivatives of a *single* function. However, extra condition of irrotationality means potential can be used for 2D and 3D flows.

5.2 Some basic flows

Before exploiting some of the added benefits of potential theory, let us return to consider how some of our basic flows can be expressed in terms of a velocity potential.

5.2.1 A uniform stream

If $\mathbf{u} = \mathbf{U} = (U_1, U_2, U_3)$ then (26) says

$$\begin{cases} \dfrac{\partial \phi}{\partial x} = U_1 \qquad \Rightarrow \qquad \phi = U_1 x + f(y, z) \\ \dfrac{\partial \phi}{\partial y} = U_2 \qquad \Rightarrow \qquad \phi = U_2 y + g(x, z) \\ \dfrac{\partial \phi}{\partial z} = U_3 \qquad \Rightarrow \qquad \phi = U_3 z + h(x, y) \end{cases}$$

where f, g, h are arbitrary functions. These are compatible if

$$\phi = U_1 x + U_2 y + U_3 z + const = \mathbf{U} \cdot \mathbf{r}$$

Note: it is normal to ignore the additive constant on the velocity potential as it does not change $\mathbf{u}$ which is defined by derivatives.

[12] and it is unique – there is a straightforward proof of this, ommitted from the notes

5.2.2 A 2D point source

In E.g. 2.4 we found $\mathbf{u} = \dfrac{m}{2\pi r}\hat{\mathbf{r}}$ (in cylindrical polars) and the streamfunction was found to be $\psi = \dfrac{m\theta}{2\pi}$.

Here

$$\begin{cases} u_\theta = 0 = \dfrac{1}{r}\dfrac{\partial \phi}{\partial \theta} \qquad \Rightarrow \qquad \phi = \phi(r) \\ u_r = \dfrac{m}{2\pi r} = \dfrac{\partial \phi}{\partial r} \qquad \Rightarrow \qquad \boxed{\phi = \dfrac{m}{2\pi}\ln r} \end{cases}$$

5.2.3 A 2D point vortex

In E.g. 4.4 we found $\mathbf{u} = \dfrac{\Gamma}{2\pi r}\hat{\boldsymbol{\theta}}$ so

$$u_r = 0 = \frac{\partial \phi}{\partial r} \qquad \Rightarrow \qquad \phi = \phi(\theta)$$

$$u_\theta = \frac{\Gamma}{2\pi r} = \frac{1}{r}\frac{\partial \phi}{\partial \theta} \qquad \Rightarrow \qquad \phi = \boxed{\frac{\Gamma\theta}{2\pi}}$$

5.2.4 A 2D dipole

Consider a source, represented by the potential $\phi_s = (m/2\pi)\ln r$ at $(0,0)$ and a sink a distance a from the origin at $a\hat{\mathbf{d}}$ where $\hat{\mathbf{d}} = (\cos\alpha, \sin\alpha)$ is a unit vector in the direction α

$$\phi_d(\mathbf{r}) = \lim_{a\to 0}\{\phi_s(\mathbf{r}) - \phi_s(\mathbf{r} - a\hat{\mathbf{d}})\} = \lim_{a\to 0}\{\phi_s(\mathbf{r}) - (\phi_s(\mathbf{r}) - a\hat{\mathbf{d}}\cdot\boldsymbol{\nabla}\phi_s(\mathbf{r}) + O(a^2))\} = \lim_{a\to 0}\{a\hat{\mathbf{d}}\cdot\boldsymbol{\nabla}\phi_s(\mathbf{r})\}$$

using multivariable Taylor's theorem. Letting $\mu = \lim_{a\to 0}\{ma\}$, represent the **dipole strength**; then

$$\boxed{\phi_d = \frac{\mu}{2\pi}\frac{\hat{\mathbf{d}}\cdot\hat{\mathbf{r}}}{r}}$$

where $\hat{\mathbf{r}} = (\cos\theta, \sin\theta)$, $r = \sqrt{x^2 + y^2}$.

E.g. 5.1: if $\alpha = 0$ so that $\hat{\mathbf{d}} = (1, 0)$ then

$$\phi_d = \frac{\mu\cos\theta}{2\pi r}$$

represents a horizontal dipole. In general

$$\phi_d(r,\theta) = \frac{\mu\cos(\theta - \alpha)}{2\pi r}.$$

5.2.5 A 3D point source

In E.g. 2.7 we found $\mathbf{u} = \dfrac{m}{4\pi r^2}\hat{\mathbf{r}}$ (in spherical polars, so $r = \sqrt{x^2 + y^2 + z^2}$ now).

Here

$$\begin{cases} u_\varphi = 0 = \dfrac{1}{r}\dfrac{\partial \phi}{\partial \varphi} \quad \text{and} \quad u_\theta = 0 = \dfrac{1}{r\sin\varphi}\dfrac{\partial \phi}{\partial \varphi} \qquad \Rightarrow \qquad \phi = \phi(r) \\ u_r = \dfrac{m}{4\pi r^2} = \dfrac{\partial \phi}{\partial r} \qquad \Rightarrow \qquad \boxed{\phi = -\dfrac{m}{4\pi r}} \end{cases}$$

5.2.6 A 3D dipole

We can follow §5.2.4 but with $\phi_s = -\dfrac{m}{4\pi r}$ to get

$$\phi_d = \frac{\mu \hat{\mathbf{d}} \cdot \hat{\mathbf{r}}}{4\pi r^2}$$

E.g. 5.2: an axisymmetric dipole aligned with the z-axis has $\hat{\mathbf{d}} = (0, 0, 1)$ and gives

$$\phi_d = \frac{\mu \cos\varphi}{4\pi r^2}$$

where φ is the polar angle.

5.3 Constructing more complex flows by superposition

Because Laplace's equation is linear, if ϕ_1 and ϕ_2 are two solutions then so is $\alpha\phi_1 + \beta\phi_2$ (**linear superposition**).

In fact we have already used this to construct dipoles.

Q: Can we use superposition to create complex flows from basic ones ?

5.3.1 Steady 2D flow past a circular cylinder

Select a potential as the sum of a horizontal stream, speed U in the +ve x-direction and a horizontal dipole, strength μ, at the origin:

$$\phi = Ux + \frac{\mu\cos\theta}{2\pi r}$$

Note: We have mixed coordinate systems in the two terms, so switch both terms to plane polars:

$$\phi = Ur\cos\theta + \frac{\mu\cos\theta}{2\pi r}$$

Consider the radial flow component

$$u_r(a, \theta) = \left.\frac{\partial \phi}{\partial r}\right|_{r=a} = \left(U - \frac{\mu}{2\pi a^2}\right)\cos\theta$$

We can make this zero for all θ if the dipole strength is chosen to be

$$\mu = 2\pi U a^2$$

In other words, the 2D flow past an infinitely-long circular cylinder can be represented by the potential

$$\boxed{\phi = U\left(r + \frac{a^2}{r}\right)\cos\theta} \tag{28}$$

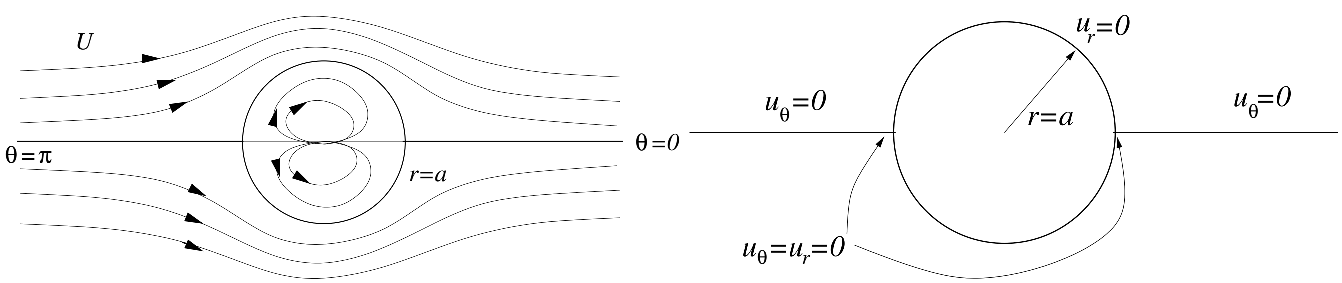
Alternative method: Because the flow is 2D we can also use streamfunctions. So combining streamfunctions for a horizontal stream and dipole (E.g. 2.5) gives

$$\psi = Uy - \frac{\mu \sin\theta}{2\pi r} = Ur\sin\theta - \frac{\mu\sin\theta}{2\pi r}$$

With $\mu = 2\pi U a^2$, we see that $\psi(a,\theta) = 0$ for all θ and because the flow is steady, the circle $r = a$ is a streamline of the flow. By definition, the fluid is everywhere parallel to the streamline and thus no fluid crosses it. This means the streamline around $r = a$ can be replaced by a rigid body.

Note: $\psi(r,\theta) = 0$ on $\theta = 0$ and $\theta = \pi$. Thus the streamline which defines the circle also extends to infinity along the x-axis. At the points $(\pm a, 0)$ front and back of the circle, both $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ components of the flow are zero and we have **stagnation points**. Recall from earlier that dividing streamlines for stready flow imply flow stagnation.

Remark: What about the dipole inside the cylinder ? Are we saying that you need fluid *inside* a cylinder as well ? No, the potential in (28) defines a flow which exists in $r > a$ in the presence of a solid boundary on $r = a$. The fact that we constructed this flow using a stream and a dipole is a mathematical trick. Later we see that the dipole appears as an image (or reflection) of the exterior flow in the cylinder surface.



5.3.2 Steady flow past a sphere

Similar idea to above, but in 3D, we combine potentials for a uniform stream of speed U along the z-axis and a dipole aligned with the z-axis (§5.2.6):

$$\phi = Uz + \frac{\mu\cos\varphi}{4\pi r^2}$$

Writing both terms in spherical polars with $z = r\cos\varphi$ gives

$$\phi = Ur\cos\varphi + \frac{\mu\cos\varphi}{4\pi r^2}$$

Then the radial component of the flow on $r = a$ is

$$\left.\frac{\partial\phi}{\partial r}\right|_{r=a} = \cos\varphi\left(U - \frac{\mu}{2\pi a^3}\right)$$

and with $\mu = 2\pi U a^3$ this is zero and hence we have a solid boundary – a sphere – at $r = a$.

Note: Can also construct with Stokes' streamfunctions.

5.3.3 Flow past a semi-infinite cylinder

Can try lots of different combinations. E.g. Consider axisymmetric 3D flow generated by a stream along z-axis and a 3D point source:

$$\phi = Ur\cos\varphi - \frac{m}{4\pi r}$$

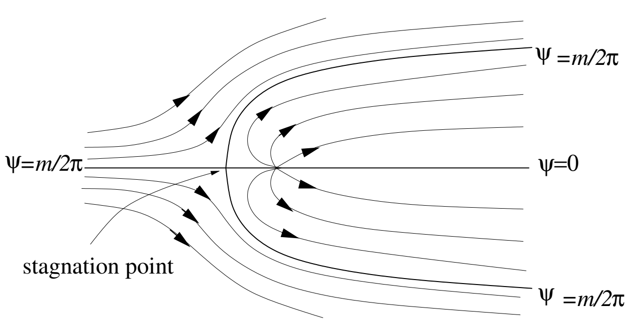
Now less easy to identify a solid body in the flow. So use Stokes' streamfunction (§2.5.1 and §2.5.2)

$$\Psi = \tfrac{1}{2}Ur^2 + \frac{m}{4\pi}(1 - \cos\varphi)$$

but note that first term is in cylindrical polars and the second is in spherical polars. Converting to spherical polars in which $r \to r\sin\varphi$ gives

$$\Psi = \tfrac{1}{2}Ur^2\sin^2\varphi + \frac{m}{4\pi}(1 - \cos\varphi)$$

Need to plot Ψ to see streamsurfaces, but need to analyse how the streamline $\Psi = 0$ along the negative z-axis reaches a stagnation point (Exercise).



5.4 Kinematic Boundary Condition

We have been sort of putting this off and making oblique references to this already. The **kinematic boundary condition** provides information about the flow velocities at the boundary of a fluid.

For a fixed solid boundary with normal $\hat{\mathbf{n}}$, it reads

$$\boxed{\mathbf{u} \cdot \hat{\mathbf{n}} = 0}$$

(the flow component perpendicular to the boundary is zero).

For a boundary which is moving with a prescribed velocity $\mathbf{U}(\mathbf{r}, t)$

$$\boxed{\mathbf{U} \cdot \hat{\mathbf{n}} = \mathbf{u} \cdot \hat{\mathbf{n}}}$$

(the flow component perpendicular to the boundary matches the normal component of the wall velocity).

In both cases the conditions ensure the fluid does not penetrate the boundary and are referred to as **no-flow conditions**.

For a so-called **free surface** which moves in response to the fluid (e.g. the interface between two fluids such as air and water or oil and water) the argument is a bit more subtle. Define such a free surface by a function $S(\mathbf{r}, t) = 0$. A particle which sits on the the fluid remains on the surface as the flow evolves, and mathematically this is expressed as

$$0 = \frac{DS}{Dt} \equiv \frac{\partial S}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} S \tag{29}$$

If there are two fluids, such as oil above water, separated by a free surface described by S then (29) applies to each phase to give:

$$\left.\begin{array}{ll} \dfrac{DS}{Dt} = \dfrac{\partial S}{\partial t} + \mathbf{u}_{oil} \cdot \boldsymbol{\nabla} S = 0, & \text{on } S = 0 \text{ from above} \\ \dfrac{DS}{Dt} = \dfrac{\partial S}{\partial t} + \mathbf{u}_{water} \cdot \boldsymbol{\nabla} S = 0, & \text{on } S = 0 \text{ from below} \end{array}\right\}$$

Now $\boldsymbol{\nabla} S$ points in the direction normal to the surface $S = const$ (Calc 1, MVC), so $\hat{\mathbf{n}} = \boldsymbol{\nabla} S / |\boldsymbol{\nabla} S|$ is the unit normal to the surface $S = 0$. It follows that

$$\boxed{\mathbf{u}_{oil} \cdot \hat{\mathbf{n}} = \mathbf{u}_{water} \cdot \hat{\mathbf{n}}, \qquad \text{on } S = 0} \tag{30}$$

This is the same condition as before, but here the motion of the surface is free and previously it was prescribed.

5.5 Bernoulli's theorem for unsteady, irrotational flows

From (18), the modified version of Euler's equation is

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times \boldsymbol{\omega} = -\boldsymbol{\nabla}(p/\rho + \Phi + \tfrac{1}{2}|\mathbf{u}|^2).$$

For irrotational flows, $\mathbf{u} = \boldsymbol{\nabla}\phi$, and $\boldsymbol{\omega} = 0$ so

$$\boldsymbol{\nabla}\left(\frac{\partial\phi}{\partial t} + \frac{p}{\rho} + \Phi + \tfrac{1}{2}|\mathbf{u}|^2\right) = 0$$

It follows that

$$\boxed{\frac{\partial\phi}{\partial t} + \frac{p}{\rho} + \Phi + \tfrac{1}{2}|\mathbf{u}|^2 = \mathcal{C}(t)} \tag{31}$$

where $\mathcal{C}(t)$ is an arbitrary function of time.

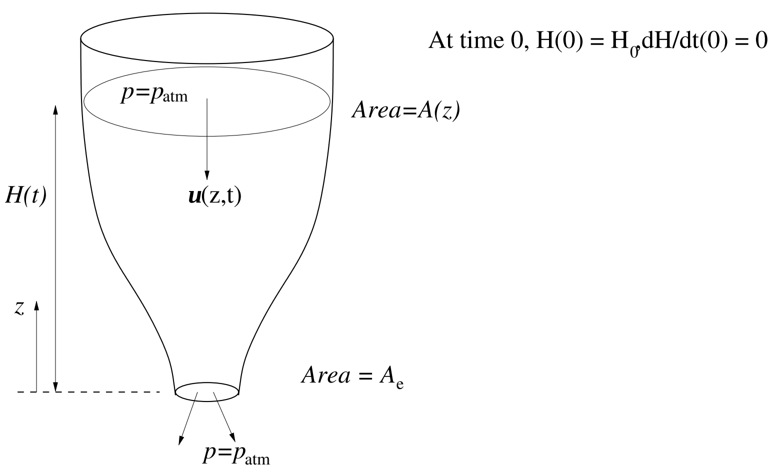
Recall $\Phi = gz$ for gravitational forces.

Notes: (i) (31) applies *throughout* the fluid, not just on a streamline.

(ii) Redefining $\phi \to \phi + \int^t \mathcal{C}(t')\,dt'$ has the effect of setting the RHS of (31) to zero but doesn't change the flow field.

Remark: The connection between pressure and velocity potential in Bernoulli's equation (an embodiment of conservation of momentum) allows to use velocity potentials to solve dynamical fluid problems. Some examples follow.

5.6 Problem: Flow out of a bottle



Assume bottle has slowly-varying cross-sectional area $A(z)$ which varies from $A(H_0) = A_0$ to $A(0) = A_e$ where $H(t)$ measures height of fluid from the exit, and at time $t = 0$, $H(0) = H_0$ and $dH/dt|_{t=0} = 0$.

Assume $A_0/A_e > 1$, but do not assume $A_0/A_e \gg 1$ as problem §3.5.1. So *cannot* approximate to quasi-steady flow. But we *can* assume irrotational flow.

Problem: find time for bottle to empty

Because $A(z)$ varies slowly with z, the velocity is approximately aligned with $\hat{\mathbf{z}}$ and only a function of z (and t). So we write

$$\mathbf{u} = w(z,t)\hat{\mathbf{z}}$$

Conservation of mass (5) implies that the volume flux through $z = H$ is the same as across any other depth in the fluid, or:

$$w(H,t)A(H) = w(z,t)A(z), \qquad \text{for } 0 < z < H(t).$$

The kinematic boundary condition says that the surface of the fluid moves with the fluid, so

$$w(H,t) = \frac{dH}{dt} \equiv \dot{H}$$

Combining, we have

$$\mathbf{u} = w(z,t)\hat{\mathbf{z}}, \qquad \text{where} \qquad w(z,t) = \frac{A(H)\dot{H}}{A(z)}$$

We still need to apply momentum conservation to get a solution and will use unsteady Bernoulli because we know the pressure is p_{atm} on the surface and the exit.

This means we need to derive a velocity potential for the flow. Since $\mathbf{u} = \boldsymbol{\nabla}\phi$ it follows that

$$\frac{\partial \phi}{\partial z} = w(z,t)$$

and so

$$\phi(z,t) = \int_0^z w(z',t)\, dz' = A(H)\dot{H} \int_0^z \frac{1}{A(z')}\, dz'$$

(the lower limit on the integral just sets an additive constant to ϕ and doesn't alter the velocity field.)

Unsteady Bernoulli's equation (gravity clearly in play) is

$$(p/\rho) + \tfrac{1}{2}|\mathbf{u}|^2 + gz + \frac{\partial \phi}{\partial t} = \mathcal{C}(t)$$

everywhere in the fluid. Apply at top surface $z = H(t)$:

$$p_{atm}/\rho + \tfrac{1}{2}\dot{H}^2 + gH + \frac{d}{dt}\left(A(H)\dot{H}\right) \int_0^H \frac{1}{A(z')} dz' = \mathcal{C}(t)$$

Apply on exit ($z = 0$):

$$p_{atm}/\rho + \tfrac{1}{2}\left(\frac{A(H)}{A_e}\dot{H}\right)^2 + 0 + 0.\frac{d}{dt}\left(A(H)\dot{H}\right) = C(t)$$

Subtract the two equations to leave

$$(A(H)\ddot{H} + \dot{H}^2 A'(H)) \int_0^H \frac{dz'}{A(z')} + \tfrac{1}{2}\left(1 - \frac{A^2(H)}{A_e^2}\right)\dot{H}^2 + gH = 0 \qquad (32)$$

Since the bottle is slowly varying $A'(H) \approx 0$.

This is an awkward-looking *2nd order non-linear ODE* for $H(t)$ with initial conditions $H(0) = H_0$, $dH/dt|_{t=0} = 0$. In principle it can be solved numerically (e.g. CompMaths).

However, if we assume $A(z) \approx A_0 = A(H_0)$ for most of the bottle

$$A(H)\int_0^H \frac{dz'}{A(z')} \approx A_0 \int_0^H \frac{dz'}{A_0} = H(z)$$

and (32) is approximated by

$$H\ddot{H} + \tfrac{1}{2}\left(1 - \frac{A_0^2}{A_e^2}\right)\dot{H}^2 + gH = 0$$

The next bit is very fancy:

$$H\ddot{H} = H\frac{d}{dt}\dot{H} = H\frac{dH}{dt}\frac{d}{dH}\dot{H} = H\dot{H}\frac{d}{dH}\dot{H} = H\frac{d}{dH}(\tfrac{1}{2}\dot{H}^2)$$

and if we let $s(H) = \frac{1}{2}\dot{H}^2$ then we get a 1st order ODE

$$\frac{ds}{dH} + \left(1 - \frac{A_0^2}{A_e^2}\right)\frac{s}{H} = -g$$

to solve (c.f. integrating factors – left as an exercise !)

Experiment: If time permits the solution will be tested against experiments.

5.7 Problem: The collapse of a spherical cavity

Suppose that at $t = 0$ we have a spherical cavity, initially at rest, radius R_0 in an infinite fluid. We assume negligible pressure in the cavity (set $p = 0$) and a constant postive background at infinity of $p_0 > 0$. The pressure gradient causes the cavity to collapse; this is assumed to happen quickly enough to ignore the influence of gravity[13].

Problem: find the evolution of the radius of the spherical cavity $R(t) < R_0$ for $t > 0$ and time of collapse.

The flow is spherically symmetric, so

$$\mathbf{u}(r,t) = u_r(r,t)\hat{\mathbf{r}}$$

in spherical coordinates.

The kinematic boundary condition on the surface of the cavity means

$$u_r(R,t) = \dot{R} \tag{33}$$

[13]A problem of practical importance as cavitation bubbles are formed on the surfaces of high speed propellors on ships; their collapse creates engineering difficulties

A velocity potential ϕ is introduced (the flow is assumed to be irrotational) and so $\mathbf{u} = \boldsymbol{\nabla}\phi$ means $\phi = \phi(r,t)$ and

$$\frac{\partial \phi}{\partial r} = u_r(r,t)$$

There are two ways of invoking conservation of mass into the solution. One is say that the flow is generated by a 3D point sink of unknown strength (constructed for incompressible flows).

The other is say that incompressibility means ϕ satisfies $\nabla^2\phi = 0$. The only solution of Laplace's equation in spherical coordinates which is independent of φ and θ is A/r (separation solutions from APDE2) and so

$$\phi(r,t) = \frac{A(t)}{r}$$

Thus

$$u_r = -\frac{A}{r^2}$$

and so (33) implies $\dot{R} = -A(t)/R^2$, or $A(t) = -R^2\dot{R}$ and hence

$$\phi(r,t) = -\frac{R^2\dot{R}}{r}, \qquad u_r = -\frac{R^2\dot{R}}{r^2}.$$

Need to use conservation of momentum and unsteady Bernoulli is good because we know pressure on cavity ($p = 0$) and at infinity ($p = p_0$). Thus:

$$0 + \tfrac{1}{2}\dot{R}^2 + \left.\frac{\partial \phi}{\partial t}\right|_{r=R} = \mathcal{C}(t) = p_0/\rho + 0 + 0$$

since $\phi \to 0$ and $u_r \to 0$ as $r \to \infty$ whilst $|\mathbf{u}|^2 = u_r^2 = \dot{R}^2$ on cavity.

Continuing,

$$\tfrac{1}{2}\dot{R}^2 - \frac{1}{R}\frac{d}{dt}(R^2\dot{R}) = p_0/\rho$$

or

$$\tfrac{1}{2}\dot{R}^2 - \frac{1}{R}\left(2R\dot{R}^2 + R^2\ddot{R}\right) = p_0/\rho$$

or

$$\tfrac{3}{2}\dot{R}^2 + R\ddot{R} = -p_0/\rho$$

which is an ODE for $R(t)$.

Fancy trick: notice that

$$\frac{d}{dt}\left(R^3\dot{R}^2\right) = 3R^2\dot{R}^3 + 2R^3\dot{R}\ddot{R} = -\frac{2p_0}{\rho}R^2\dot{R}$$

using the ODE. Equating LHS and RHS and noticing another trick

$$\frac{d}{dt}\left(R^3\dot{R}^2\right) = -\frac{2p_0}{\rho}\frac{d}{dt}\left(\tfrac{1}{3}R^3\right)$$

we can now integrate up w.r.t. t to get

$$R^3\dot{R}^2 = -\frac{2p_0}{3\rho}R^3 + C$$

where C is a constant, determined by the initial condition: $\dot{R} = 0$ when $R = R_0$. Hence

$$\dot{R}^2 = \frac{2p_0}{3\rho}\left(\frac{R_0^3}{R^3} - 1\right)$$

or

$$\frac{dR}{dt} = -\sqrt{\frac{2}{3}\frac{p_0}{\rho}}\left(\frac{R_0^3 - R^3}{R^3}\right)^{1/2}$$

where the negative sign must be chosen because $\dot{R} < 0$ – by assumption the cavity is collapsing. Integrating up gives

$$\int_{R_0}^{R(t)} \frac{R^{3/2}\,\mathrm{d}R}{(R_0^3 - R^3)^{1/2}} = -\int_0^t \left(\frac{2p_0}{3\rho}\right)^{1/2} \mathrm{d}t$$

Hmmm. Tricky. If the cavity collapses at time $t = t_c$ then $R(t_c) = 0$ implies (making substitution $R = R_0 s$ in integral)

$$\int_0^1 \frac{s^{3/2}\,\mathrm{d}s}{(1 - s^3)^{1/2}} = \frac{t_c}{R_0}\left(\frac{2p_0}{3\rho}\right)^{1/2}$$

and the integral can be evaluated numerically to a value of 0.747. Hence

$$t_c = 0.915\left(\frac{\rho}{p_0}\right)^{1/2} R_0$$

That's a pretty neat result.

5.8 Problem: cylinder moving through a fluid

Suppose a cylinder of radius a moves along the x-axis in an unbounded otherwise stationary fluid. Its position as a function of time is $\mathbf{s}(t) = (s(t), 0)$ such that $\mathbf{U}(t) = (U(t), 0)$, $U(t) = \dot{s}(t)$, is its velocity.

Problem: find force exerted by fluid on cylinder.

We know that $\mathbf{u} \to \mathbf{0}$ as $|\mathbf{r} - \mathbf{s}| \to \infty$ and this is a "boundary condition at infinity".

The kinematic boundary condition the cylinder is

$$\mathbf{u} \cdot \hat{\mathbf{n}} = \mathbf{U} \cdot \hat{\mathbf{n}}, \qquad \text{on } |\mathbf{r} - \mathbf{s}| = a$$

with $\hat{\mathbf{n}} = \hat{\mathbf{r}} = (\cos\theta, \sin\theta)$. That is, we are going to use polar coordinates (r, θ) which are centred on the cylinder and therefore are in a moving frame of reference.

Also assume potential flow, $\mathbf{u} = \boldsymbol{\nabla}\phi$ and incompressibility implies $\nabla^2\phi = 0$ where $\phi = \phi(r, \theta, t)$.

Then we want $\boldsymbol{\nabla}\phi \to \mathbf{0}$ as $r \to \infty$ and

$$\hat{\mathbf{n}} \cdot \boldsymbol{\nabla}\phi = \frac{\partial \phi}{\partial r} = U(t)\cos\theta, \qquad \text{on } r = a. \tag{34}$$

since $\hat{\mathbf{n}} = \hat{\mathbf{r}} = (\cos\theta, \sin\theta)$.

Select separation solutions of $\nabla^2\phi$ in plane polars which fit the conditions at infinity and on the cylinder. Then the only possible solution is

$$\phi = A(t)\frac{\cos\theta}{r}$$

Remark: This is a horizontal dipole potential which is not surprising since we constructed a streaming flow past a cylinder in §5.3.1 with a stream and a dipole. That is, an alternative way of constructing ϕ is from set of basic incompressible flows.

Imposing (34) gives

$$A(t) = -U(t)a^2, \qquad \text{so} \qquad \phi = -U(t)a^2\frac{\cos\theta}{r} \tag{35}$$

What is the force exerted by the fluid on the cylinder ? General formula is:

$$\mathbf{F} = -\int_S p\hat{\mathbf{n}}\,\mathrm{d}S$$

and need $p(r,\theta)$ from Bernoulli[14] (no influence of gravity):

$$(p/\rho) + \tfrac{1}{2}|\mathbf{u}|^2 + \frac{\partial\phi}{\partial t} = \mathcal{C}(t) = (p_0/\rho)$$

where $p \to p_0$, $\mathbf{u} \to 0$ and $\phi \to 0$ as $r \to \infty$.

Elsewhere in the flow,

$$\mathbf{u} = \boldsymbol{\nabla}\phi = \phi_r\hat{\mathbf{r}} + (\phi_\theta/r)\hat{\boldsymbol{\theta}} = U(t)\frac{a^2}{r^2}\cos\theta\hat{\mathbf{r}} + U(t)\frac{a^2}{r^2}\sin\theta\hat{\boldsymbol{\theta}}$$

Now U is a function of time, but so are θ and r since they are in a moving frame of reference.

Points in the fixed frame of reference $\mathbf{r}$ are connected to the moving frame of reference with

$$\mathbf{r} = \mathbf{s}(t) + (r\cos\theta, r\sin\theta)$$

Taking time derivatives of each component gives

$$\begin{cases} 0 = U + \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\ 0 = \dot{r}\sin\theta + r\dot{\theta}\cos\theta \end{cases}$$

[14] since we are regarding $U(t)$ as prescribed, we do not need Bernoulli to *solve* for the flow; in this example, Bernoulli allows us to find the pressure that results from the prescribed motion

since $\dot{s}(t) = U(t)$. Eliminating (e.g. multiplying the top by $\cos\theta$ and the bottom by $\sin\theta$ and adding) gives

$$\dot{r} = -U\cos\theta, \qquad r\dot{\theta} = U\sin\theta. \tag{36}$$

So the time derivative applied to (35)

$$\frac{\partial\phi}{\partial t} = -\dot{U}\frac{a^2}{r}\cos\theta + U\frac{a^2}{r}\dot{\theta}\sin\theta + U\frac{a^2}{r^2}\dot{r}\cos\theta = -\dot{U}\frac{a^2}{r}\cos\theta - U^2\frac{a^2}{r^2}\cos 2\theta$$

simplifying using (36) Applying Bernoulli on the cylinder $r = a$:

$$p = p_0 - \tfrac{1}{2}\rho U^2 + \rho\dot{U}a\cos\theta + \rho U^2\cos 2\theta.$$

Now the force in x-direction is

$$F_x = \mathbf{F}\cdot\hat{\mathbf{x}} = -\int_S p\hat{\mathbf{n}}\cdot\hat{\mathbf{x}}\,dS = -\int_0^{2\pi} p\cos\theta a\,\mathrm{d}\theta = -\pi\rho a^2\dot{U}$$

(remember $\hat{\mathbf{n}} = (\cos\theta, \sin\theta)$ and in y-direction

$$F_y = \mathbf{F}\cdot\hat{\mathbf{y}} = -\int_0^{2\pi} p\sin\theta a\,\mathrm{d}\theta = 0$$

(as expected, by symmetry).

Note: F_x is just mass per unit length of the cylinder times acceleration. It is often called an 'added mass' term.

Note: if $U(t)$ is constant then the force on the cylinder is zero. This is an example of **D'Alembert's Paradox** (explained later) in which all steadily moving rigid bodies experience no force due to the fluid under the ideal fluid assumption.

5.9 Kinetic energy for potential flows

The **kinetic energy** of fluid, constant density ρ, in a volume V with bounding surfaces S is defined as

$$E_{kin} = \frac{\rho}{2}\int_V |\mathbf{u}|^2\,dV = \frac{\rho}{2}\int_V \boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi\,dV = \frac{\rho}{2}\int_V \boldsymbol{\nabla}\cdot(\phi\boldsymbol{\nabla}\phi) - \nabla^2\phi\,dV$$

using a result from vector calculus. Now $\nabla^2\phi = 0$ and the divergence theorem gives

$$\boxed{E_{kin} = \frac{\rho}{2}\int_S \phi\hat{\mathbf{n}}\cdot\boldsymbol{\nabla}\phi\,dS} \tag{37}$$

That is, the kinetic energy in the fluid can be determined from ϕ and the normal component of the gradient of ϕ on the surfaces bounding the fluid.

E.g. 5.3: (translating cylinder poblem of §5.8)

We had in (35)

$$\phi = -U(t)a^2\frac{\cos\theta}{r}$$

giving

$$\boldsymbol{\nabla}\phi = U(t)\frac{a^2}{r^2}\cos\theta\hat{\mathbf{r}} + U(t)\frac{a^2}{r^2}\sin\theta\hat{\boldsymbol{\theta}} = U(t)\cos\theta\hat{\mathbf{r}} + U(t)\sin\theta\hat{\boldsymbol{\theta}}$$

on the cylinder. Putting this in (37)

$$E_{kin} = \frac{\rho}{2}\int_0^{2\pi} U^2(t)a\cos^2\theta\, ad\theta$$

since $\hat{\mathbf{n}} = -\hat{\mathbf{r}}$ as it points out of the fluid and there is no contribution from a bounding surface at infinity since the integral tends to zero there. I.e.

$$E_{kin} = \tfrac{1}{2}\rho\pi a^2U^2 = \tfrac{1}{2}MU^2$$

where $M = \rho\pi a^2$ is the mass (per unit width) of the cylinder.

Note: An extension of Newton's Law in Mechanics states that the rate of change of energy equals the power. In this case

$$\frac{dE_{kin}}{dt} = \frac{d}{dt}(\tfrac{1}{2}MU^2) = MU\dot{U}$$

whilst in §5.8 we found $\mathbf{F} = M\dot{U}$ and so the power is

$$\mathbf{F}\cdot\mathbf{U} = MU\dot{U}$$

and the two agree.

I.e. The kinetic energy method can be used to find the force due to the fluid.

5.9.1 Corollary: A freely falling cylinder

Since the force on the cylinder due to the fluid is $M\dot{U}$ in the direction of travel of the cylinder, we can solve for a freely-falling cylinder of mass (per unit length) M_c by Newton's Law (mass times acceleration equals sum of the forces)

$$M_c\dot{U} = -M\dot{U} + (M_c - M)g$$

accounting for the Achimedian upthrust means

$$\dot{U}(t) = \frac{(M_c - M)g}{M + M_c}, \qquad \Rightarrow \qquad U(t) = \frac{(M_c - M)gt}{M_c + M} \quad (= \dot{s})$$

for a cylinder starting from rest and so cylinder falls ballistically a distance

$$s(t) = \frac{(M_c - M)\frac{1}{2}gt^2}{M_c + M}$$

after time t. In reality drag associated with viscosity provides a resistive force.

Experiment: Repeat calculation above for a sphere and then choose M_c close to M (close to neutral buoyancy) so that the influence of drag is reduced. For e.g. a water bomb balloon filled with salty water.

Key points: For the irrotational motion of an ideal incompressible fluid, the three components of velocity can be expressed as the gradient of a single scalar function – the velocity potential. This is similar to the streamfunction. The streamfunction does not require the motion to be irrotational, but its practical use is limited to 2D or quasi-2D flows. The velocity potential can be used for 2D and 3D flows.

Velocity potentials are easily be constructed for the basic flows introduced for the streamfunction. More complex flows can be constructed by superposition of these basic flows.

A significant result is the alternative version of Bernoulli's equation for unsteady irrotational flows which holds throughout the flow. This allows us to consider a range of unsteady flows. Many such examples include moving surfaces and the kinematic boundary condition tells us how to connect the fluid to fixed or moving surfaces.

Finally, we show that the total kinetic energy of the flow can be expressed in terms of the velocity potential on the boundary of the flow. This not only allows the KE to be easily computed, but gives us information about the force on a body in the flow.

6 Complex potentials for two-dimensional flows

6.1 Definition of the complex potential

Consider 2D flows $\mathbf{u} = (u, v)$ in which the flow is irrotational and the fluid incompressible. Then the flow can either be described by a potential $\phi(x, y)$ (§5) or by a streamfunction $\psi(x, y)$ (§2.4.1) such that

$$\left. \begin{aligned} u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \\ v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \end{aligned} \right\} \tag{38}$$

Defn: Let $z = x + iy$ and define the **complex potential** $W(z) = \phi(x, y) + i\psi(x, y)$. Then $W(z)$ is **analytic** (or **holomorphic**, or complex differentiable) since

$$W'(z) = \frac{\mathrm{d}W}{\mathrm{d}z} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = -i\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial y} \tag{39}$$

and (38) are the **Cauchy-Riemann** equations for the function $W = \phi + i\psi$.

Note: $\nabla^2 \phi = \nabla^2 \psi = 0$ follows directly from (38).

Note: $\boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\psi = \frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial y} = 0$ meaning **equipotential lines** (where $\phi = const$) are perpendicular to streamlines (where $\psi = const$). Sometimes ϕ and ψ are called **harmonic conjugates**.

Defn: The **complex velocity** following from (38) and (39) is defined as

$$\boxed{W'(z) = u - iv \equiv q\mathrm{e}^{-i\chi}} \tag{40}$$

in complex polars where $q = |W'(z)| = \sqrt{u^2 + v^2}$ is the speed of the flow and χ is the angle it makes from the x-axis.

E.g. 6.1: (a straining/stagnation point flow)

Consider $W(z) = kz^2$.

Then $W = k(x + iy)^2 = k(x^2 - y^2) + 2ikxy$ and so $\left\{ \begin{array}{l} \phi = \Re\{W\} = k(x^2 - y^2) \\ \psi = \Im\{W\} = 2kxy \end{array} \right.$

Also, $W'(z) = 2kz = 2kx + i2ky$ so $u = 2kx$ and $v = -2ky$ or $q = 2k\sqrt{x^2 + y^2} = 2kr$ and $\chi = \tan^{-1}(-y/x) = -\theta$.

Note: We can obviously do this in reverse: e.g. if we have ϕ we can find ψ and hence $W(z)$.

6.2 Some common potentials

Flows we've seen before now defined as complex potentials and then separated into real and imaginary parts; the associated $\phi(x, y)$ and $\psi(x, y)$.

(i) Uniform stream along x axis:

$$\boxed{W(z) = Uz} = Ux + iUy$$

(ii) 2D source at the origin, strength m:

$$\boxed{W(z) = \frac{m}{2\pi}\log(z)} = \frac{m}{2\pi}\log(re^{i\theta}) = \frac{m}{2\pi}(\log(r) + \log(e^{i\theta})) = \frac{m}{2\pi}\log(r) + i\frac{m\theta}{2\pi}$$

(iii) 2D vortex of strength Γ:

$$\boxed{W(z) = -\frac{i\Gamma}{2\pi}\log(z)} = \ldots = \frac{\Gamma\theta}{2\pi} - i\frac{\Gamma}{2\pi}\log r$$

Note: The difference between a source and a vortex is just a factor of i.

(iv) Horizontal dipole of strength μ:

$$\boxed{W(z) = \frac{\mu}{2\pi z}} = \frac{\mu\bar{z}}{2\pi|z|^2} = \frac{\mu x}{2\pi(x^2 + y^2)} - i\frac{\mu y}{2\pi(x^2 + y^2)}$$

which can easily be written in (r, θ).

6.3 Flow at a corner

Consider $W(z) = kz^\alpha$, where $k, \alpha \in \mathbb{R}$.

Writing $z = re^{i\theta}$ we find $\phi = kr^\alpha \cos\alpha\theta$, $\psi = kr^\alpha \sin\alpha\theta$.

Streamlines given by $\psi = const$. In particular, $\psi = 0$ on $\theta = 0, \pi/\alpha$ and represents flow in a 'corner' of angle π/α

Notes:

(i) If $\alpha = 1$, corner angle is π (a straight wall) and flow is a horizontal stream of speed k.

(ii) If $\alpha > 1$ then flow is in a corner, angle $< \pi$.

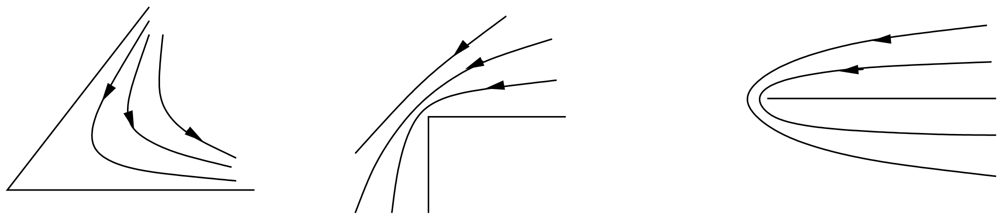
(iii) $\alpha \geq \frac{1}{2}$, since the largest angle of a corner is 2π.

(iv) Speed of the flow is given by

$$W'(z) = k\alpha z^{\alpha-1} = k\alpha r^{\alpha-1}e^{i(\alpha-1)\theta} = u - iv = qe^{-i\chi}.$$

So if $\alpha > 1$, and the corner angle $< \pi$, flow speed tends to zero as $r \to 0$.

If $\frac{1}{2} < \alpha < 1$ (reflex angles) there is a **singularity** (infinite velocity at $r = 0$) in the flow speed:



6.4 Translation, rotation and superposition

The complex potential $W_1(z) \equiv W(z - z_0)$ represents the flow generated by W, centred around the origin $z_0 = x_0 + iy_0$ in the complex plane.

The complex potential $W_1(z) \equiv W(z\mathrm{e}^{-i\alpha})$ represents the flow generated by W rotated an angle α about the origin (since $z\mathrm{e}^{-i\alpha} = r\mathrm{e}^{i(\theta-\alpha)}$.)

Linear superposition of complex potentials $\alpha W_1(z) + \beta W_2(z)$ forms a new complex potential.

E.g. 6.2: The complex potential

$$W(z) = Uz\mathrm{e}^{-i\alpha} + \frac{m}{2\pi}\log(z - i)$$

represents a source of strength m placed at $(x, y) = (0, 1)$ in a uniform stream of speed U flowing at an angle α w.r.t. the positive x-axis.

6.5 Mass flux and circulation

Theorem: The mass flux/circulation of $W(z)$ through/around a closed loop C is

$$\boxed{\int_C W'(z)dz = \Gamma + im} \tag{41}$$

Proof:

$$\begin{aligned}\int_C W'(z)\,\mathrm{d}z = \int_C (u - iv)(\,\mathrm{d}x + i\,\mathrm{d}y) &= \int_C (u\,\mathrm{d}x + v\,\mathrm{d}y) + i\int_C (u\,\mathrm{d}y - v\,\mathrm{d}x) \\ &= \int_C \mathbf{u}\cdot\,\mathrm{d}\mathbf{r} + i\int_C \mathbf{u}\cdot\hat{\mathbf{n}}\,\mathrm{d}s \\ &= \Gamma + im\end{aligned}$$

since $\hat{\mathbf{n}}\,\mathrm{d}s = (\,\mathrm{d}y, -\,\mathrm{d}x)$ (by geometrical arguments).

E.g. 6.3: Consider the source $\frac{m}{2\pi}\log(z)$. Then

$$\int_C W'(z)\,\mathrm{d}z = \frac{m}{2\pi}\int_C \frac{1}{z}\,\mathrm{d}z = \begin{cases} im, & \text{if } C \text{ encloses the origin} \\ 0, & \text{otherwise}\end{cases}$$

by **Cauchy's Residue Theorem**. I.e. captures a source of mass flux at the origin.

E.g. 6.4: Consider the dipole $\dfrac{\mu}{2\pi z}$. Then

$$\int_C W'(z)\ \mathrm{d}z = -\frac{\mu}{2\pi}\int_C \frac{1}{z^2}\,\mathrm{d}z = 0$$

(since $1/z^2$ is not a simple pole). So a dipole produces no mass flux or circulation.

6.6 Method of Images: flow next to a wall

Theorem: Let $f(z)$ be a complex potential for a flow in an infinite domain. Then

$$\boxed{W(z) = f(z) + \overline{f}(z)} \tag{42}$$

represents the complex potential of a flow in $y > 0$ in the presence of a rigid wall along $y = 0$.

Notation: $\overline{f}(z) \equiv \overline{f(\overline{z})}$ and means insert $\overline{z}$ as the argument of the function and then conjugate everything. What this actually does is conjugate everything apart from z (and so the function is still analytic !)

Proof: Consider

$$\psi(x,0) = \Im\{W(x+i0)\} = \Im\{f(x) + \overline{f(x)}\} = 0$$

Therefore $\psi = 0$ on $y = 0$ and $y = 0$ is a streamline and can be interpreted/replaced by a rigid wall.

Remark: This is no good if the **image potential** $\overline{f}(z)$ has introduced singularities into $y > 0$. But if $f(z)$ has a singularity at $z = z_0$, the image potential has a singularity at $z = \overline{z}_0$. So if all the singularities of $f(z)$ lie in $y > 0$ then all the singularities of the image potential lie in $y < 0$ and outside the new domain $y > 0$ above the wall.

Corollary: The complex potential

$$\boxed{W(z) = f(z) + \overline{f}(-z)} \tag{43}$$

represent a flow flow in $x > 0$ in the presence of a rigid wall along $x = 0$; all singularities of $\overline{f}(-z)$ are hidden in $x < 0$.

Proof: Consider

$$\psi(0,y) = \Im\{W(0+iy)\} = \Im\{f(iy) + \overline{f(\overline{-iy})}\} = \Im\{f(iy) + \overline{f(iy)}\} = 0$$

and so $\psi = 0$ along $x = 0$ and there is a streamline along $x = 0$ which can be replaced by a wall.

6.6.1 Problem: Vortex next to a wall

Find the path of a point vortex above a wall.

Point vortex (§6.2) at $z = z_0 = x_0 + iy_0$ with $y_0 > 0$ given by $f(z) = -\frac{i\Gamma}{2\pi}\log(z - z_0)$.

Next to a wall at $y = 0$, use (42) to give

$$W(z) = -\frac{i\Gamma}{2\pi}\log(z - z_0) + \frac{i\Gamma}{2\pi}\log(z - \overline{z_0})$$

The motion of the vortex at $z = z_0$ is determined by the flow field generated by everthing other than itself.

Thus, the vortex in $y > 0$ moves due to the presence of the image vortex in $y < 0$ (and vice versa). Specifically, the velocity field created by the image vortex is

$$\frac{d}{dz}(W(z) - f(z)) = \frac{i\Gamma}{2\pi}\frac{1}{(z - \overline{z_0}))}$$

and so

$$\frac{d}{dz}(W(z) - f(z))\bigg|_{z=z_0} = \frac{i\Gamma}{2\pi}\frac{1}{(z_0 - \overline{z_0}))}$$

is the complex velocity $u - iv$ at $z = z_0$ Assume $z_0(t) = x_0(t) + iy_0(t)$ represents the Lagrangian path of the centre of the vortex at time t. Then

$$u - iv \equiv \frac{dx_0}{dt} - i\frac{dy_0}{dt}\left(\equiv \frac{d\overline{z}_0}{dt}\right) = \frac{\Gamma}{4\pi y_0}.$$

That is to say,

$$\frac{dx_0}{dt} = \frac{\Gamma}{4\pi y_0}, \qquad \frac{dy_0}{dt} = 0.$$

These are coupled ODEs but the latter integrates to give $y_0(t) = y_0(0)$ and using this in the former and integrating gives

$$x_0(t) = x_0(0) + \frac{\Gamma t}{4\pi y_0(0)}$$

So the vortex moves with a constant speed $u = \Gamma/4\pi y_0$ parallel to the wall. The image vortex also moves like this and so the pair of vortices travel in tandem parallel to the wall.

6.7 Blasius' Theorem

Theorem: Suppose a fixed rigid body, boundary C (a closed loop) is in a steady flow, potential $W(z)$. Then the force $\mathbf{F} = (F_x, F_y)$ is determined from

$$\boxed{F_x - iF_y = i\tfrac{1}{2}\rho\int_C (W'(z))^2\, dz} \tag{44}$$

Proof: We have

$$W'(z) = u - iv = q\mathrm{e}^{-i\chi}$$

so $|\mathbf{u}| = q$. So, Bernoulli (no g, no t) says

$$p/\rho + \tfrac{1}{2}q^2 = p_0/\rho$$

where p_0 is a constant background pressure which exists in the absence of the flow. Hence

$$p = p_0 - \tfrac{1}{2}\rho q^2.$$

and the force on C is given by

$$\mathbf{F} = -\int_C p\hat{\mathbf{n}}\,\mathrm{d}s = \int_C \rho q^2 \hat{\mathbf{n}}\,\mathrm{d}s$$

since $\int_C p_0 \hat{\mathbf{n}}\,\mathrm{d}s = \int_C \nabla p_0 dxdy = 0$ by the divergence theorem and the fact that p_0 is a constant.

By definition, the flow velocity is everywhere parallel to the boundary C and letting $\chi(s)$ denote the angle that C makes to the positive x-axis as a function of arclength gives

$$\frac{dx}{ds} = \cos\chi, \quad \frac{dy}{ds} = \sin\chi, \qquad \Rightarrow \qquad dz = dx + idy = \mathrm{e}^{i\chi}ds.$$

Also, by geometrical considerations $\hat{\mathbf{n}} = (y_s, -x_s) = (\sin\chi, -\cos\chi)$. So if we write $\mathbf{F} = (F_x, F_y)$ in terms of its components and define a **complex force** $F = F_x - iF_y$ then the force from above can be written out as

$$\begin{aligned} F = F_x - iF_y = -\int_C p(\sin\chi + i\cos\chi)\,\mathrm{d}s &= -i\int_C p\mathrm{e}^{-i\chi}\,\mathrm{d}s \\ &= i\tfrac{1}{2}\rho\int_C (q^2\mathrm{e}^{-2i\chi})\,\mathrm{e}^{i\chi}\,\mathrm{d}s \\ &= i\tfrac{1}{2}\rho\int_C (q\mathrm{e}^{-i\chi})^2\,\mathrm{d}z = i\tfrac{1}{2}\rho\int_S (W'(z))^2\,\mathrm{d}z \end{aligned}$$

6.8 Method of Images: Flows next to cylinders

Theorem: Suppose $f(z)$ is a complex potential in the absence of a cylinder with no singularities in $|z| < a$. Then

$$\boxed{W(z) = f(z) + \overline{f}(a^2/z)} \tag{45}$$

is the complex potential representing a flow in the presence of a cylinder on $|z| = a$.

This is called the **Milne-Thompson circle theorem**.

Proof: On $|z| = a$, $z\overline{z} = a^2$ and so $a^2/z = \overline{z}$. Hence

$$\overline{f}(a^2/z) = \overline{f}(\overline{z}) = \overline{f(z)}$$

Thus, on $|z| = a$, $W(z) = f(z) + \overline{f(z)}$ and $\Im\{W\} = \psi = 0$. The streamline may be replaced by rigid boundary and no new singularities have emerged in $|z| > a$.

E.g. 6.5: Choose $f(z) = Uz$ (§6.2, uniform flow). Then (45) gives us

$$W(z) = Uz + U\frac{a^2}{z}$$

I.e. stream plus horizontal dipole of strength $\mu = -2\pi U a^2$.

Note: Exactly the flow found in §5.3.1 for flow past a cylinder.

Using Blasius, the complex force is

$$F_x - iF_y = \frac{1}{2}i\rho\int_C U^2\left(1 - \frac{a^2}{z^2}\right)\,\mathrm{d}z = \frac{1}{2}i\rho U^2\int_C\left(1 - 2\frac{a^2}{z^2} + \frac{a^4}{z^4}\right)dz = 0$$

since there are no simple poles inside C. We already had this result from §5.8 when U is constant.

6.8.1 Problem: Vortex outside a cylinder

Find the complex potential for a point vortex outside a cylinder and determine its motion.

In absence of cylinder, point vortex at z_0 is $f(z) = \dfrac{-i\Gamma}{2\pi}\log(z - z_0)$. With a cylinder, radius $a < |z_0|$ (45) gives

$$\begin{aligned}
W(z) &= -\frac{i\Gamma}{2\pi}\log(z - z_0) + \frac{i\Gamma}{2\pi}\log\left(\frac{a^2}{z} - \overline{z_0}\right)\\
&= -\frac{i\Gamma}{2\pi}\left\{\log(z - z_0) - \log\left(\frac{1}{z}(-\overline{z_0})\left(z - \frac{a^2}{\overline{z_0}}\right)\right)\right\}\\
&= -\frac{i\Gamma}{2\pi}\left\{\log(z - z_0) + \log(z) - \log\left(z - \frac{a^2}{\overline{z_0}}\right) - \log(-\overline{z_0})\right\}
\end{aligned}$$

The 2nd and 3rd terms are images at the origin and an inverse point to z_0 and the last term is a constant and can be ignored, because constants do not affect the flow velocities which are determined by derivatives.

(i) Motion of vortex

The velocity field at $z = z_0$ is due to the image vortices, or

$$u - iv = W'(z_0) - f'(z_0) = -\frac{i\Gamma}{2\pi}\left\{\frac{1}{z_0} - \frac{1}{z_0 - a^2/\overline{z_0}}\right\}$$

Better to work in polar coordinates, so let $z_0 = r_0(t)\mathrm{e}^{i\theta_0(t)}$ track the position of the vortex whence

$$\begin{aligned}
q\mathrm{e}^{-i\chi} = -\frac{i\Gamma}{2\pi}\left\{\frac{1}{r_0\mathrm{e}^{i\theta_0}} - \frac{r_0\mathrm{e}^{-i\theta_0}}{r_0^2 - a^2}\right\} &= \frac{i\Gamma}{2\pi}\mathrm{e}^{-i\theta_0}\left(\frac{a^2}{r_0(r_0^2 - a^2)}\right)\\
&= \frac{\Gamma a^2}{2\pi r_0(r_0^2 - a^2)}\mathrm{e}^{-i(\theta_0 - \pi/2)}.
\end{aligned}$$

Thus, the speed of the point vortex is $\Gamma a^2/2\pi r_0(r_0^2 - a^2)$ and its direction is at right angles to its position. Remembering the representation for velocity in polars:

$$\mathbf{u} = \dot{r_0}\hat{\mathbf{r}} + r_0\dot{\theta}_0\hat{\boldsymbol{\theta}} = -\frac{\Gamma a^2}{2\pi r_0(r_0^2 - a^2)}\hat{\boldsymbol{\theta}}$$

(Mech 1) means

$$\frac{dr_0}{dt} = 0, \qquad \frac{d\theta_0}{dt} = -\frac{\Gamma a^2}{2\pi r_0^2(r_0^2 - a^2)}$$

and the first equation integrates to $r_0(t) = r_0(0)$, a constant (initial radial distance to the vortex). The second integrates to

$$\theta_0(t) = \theta_0(0) - \frac{\Gamma a^2 t}{2\pi r_0^2(0)(r_0^2(0) - a^2)}.$$

Thus, the vortex moves at constant angular velocity in a circle around the cylinder.

(ii) Force on cylinder

From the Blasius formula and our definition of $W(z)$ we have

$$\begin{aligned} F_x - iF_y &= \tfrac{1}{2}i\rho \int_C \left(-\frac{i\Gamma}{2\pi}\right)^2 \left(\frac{1}{z - z_0} + \frac{1}{z} - \frac{1}{z - a^2/\bar{z}_0}\right)^2 dz \\ &= -\frac{i\rho\Gamma^2}{8\pi^2} \int_{|z|=a} \left(\frac{1}{(z - z_0)^2} + \frac{1}{z^2} + \frac{1}{(z - a^2/\bar{z}_0)^2}\right. \\ &\qquad\qquad \left. + \frac{2}{z(z - z_0)} - \frac{2}{(z - z_0)(z - a^2/\bar{z}_0)} - \frac{2}{z(z - a^2/\bar{z}_0)}\right) dz \end{aligned}$$

We can use Cauchy's Residue Theorem to evaluate the integral. The first 3 terms in the integral are poles of order 2 and don't contribute. Also, z_0 is outside $|z| = a$ but $a^2/\bar{z}_0$ is inside $|z| = a$ and only simple poles inside will count. So we get

$$F_x - iF_y = -\frac{i\rho\Gamma^2}{8\pi^2}(2\pi i)\left(\frac{2}{-z_0} - \frac{2}{(a^2/\bar{z}_0 - z_0)} - \frac{2}{a^2/\bar{z}_0} - \frac{2}{-a^2/\bar{z}_0}\right).$$

The last two terms cancel and the others combine as

$$F_x - iF_y = \frac{\rho\Gamma^2}{2\pi}\left(-\frac{1}{z_0} - \frac{\bar{z}_0}{a^2 - |z_0|^2}\right) = \frac{\rho\Gamma^2 a^2}{2\pi z_0(|z_0|^2 - a^2)}.$$

Writing $z_0 = r_0 e^{i\theta_0}$ shows that the force is of magnitude

$$\frac{\rho\Gamma^2 a^2}{2\pi r_0(r_0^2 - a^2)}$$

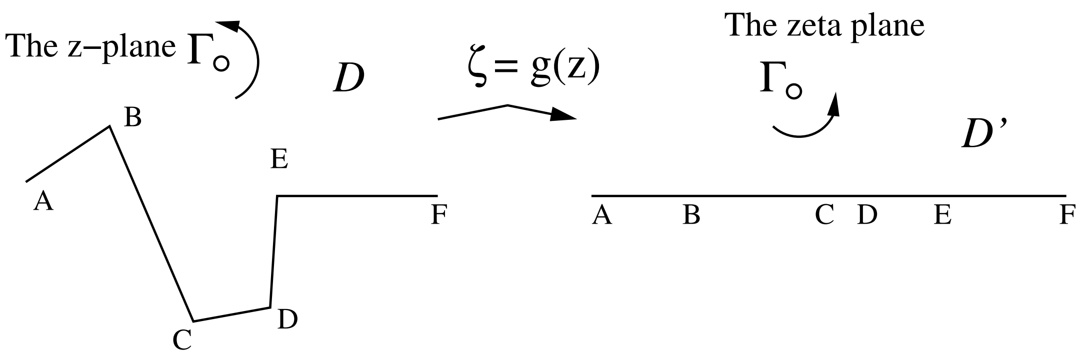
and is in the direction of θ_0.

E.g. if $z_0 = b > a$, a real number, then $F_y = 0$ and $F_x > 0$ and the cylinder feels a force towards the vortex.

6.9 Conformal mappings

For more complicated geometries, it can be hard to identify a complex potential representing the flow.

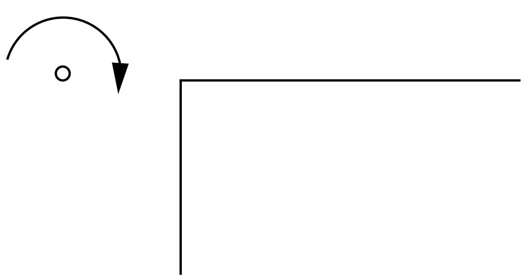
A very powerful tool in complex analysis involves using a coordinate transformation in complex coordinates or **conformal mapping** to map a complicated domain into a simpler one in which the flow can be found. The inverse mapping then provides the solution to the original complicated problem.



Note: One has to formally check a number of technicalities of this approach which we do not have time for here. Thus, it can be shown that an analytic function g, say, defining the mapping from a region $\mathcal{D}$ in the z-plane to a region $\mathcal{D}_1$ in the ζ-plane with $\zeta = g(z)$ preserves boundary conditions, the strength of singularities and harmonicity of functions. For example, a source of strength m located at $z = z_0$ in $\mathcal{D}$ satisfying Laplace's equation next to a wall with a no-flow condition is transformed under the mapping g to a source of strength m at $\zeta = \zeta_0 = g(z_0)$ next to a (geometrically transformed) wall with a no-flow condition in $\mathcal{D}_1$.

6.9.1 Problem: Vortex next to a corner

Find the flow generated by a point vortex in the presence of a right-angled boundary.



Consider the mapping $\zeta = g(z) \equiv z^{2/3}$ such that $z = \zeta^{3/2} \equiv g^{-1}(\zeta)$ the inverse.

With $z = re^{i\theta}$, when $\theta = 0$ so that $z = r$, $\zeta = r^{2/3}$. That is, the positive real axis in the z-plane is mapped to the positive real axis in the ζ-plane.

When $\theta = \frac{3}{2}\pi$, $z = -ir$ and so $\zeta = r^{2/3}e^{i\pi} = -r^{3/2}$. That is, the line from $z = 0$ to $z = -i\infty$ is mapped to the negative real axis in the ζ-plane.

So a point vortex in the z-plane at $z = z_0$ of strength Γ in the presence of a rigid corner from $z = -i\infty$ to $z = 0$ to $z = \infty$ is mapped to a point vortex of strength Γ at $\zeta = \zeta_0$ where $\zeta_0 = g(z_0) = z_0^{2/3}$ with a rigid wall along the real ζ-axis. I.e. the problem in the ζ-plane is a point vortex in the upper half plane bounded by a wall.

The problem in the mapped ζ-plane is one whose solution we have already found by elementary methods (see §6.6.1): it has the solution

$$W_1(\zeta) = -\frac{i\Gamma}{2\pi}\log(\zeta - \zeta_0) + \frac{i\Gamma}{2\pi}\log(\zeta - \overline{\zeta}_0)$$

where $\zeta_0 = z_0^{2/3}$ and $\zeta = z^{2/3}$. The use of the subscript 1 refers to the fact that this is the complex potential in the domain $\mathcal{D}_1$.

The complex potential describing the flow in the z-plane is therefore

$$W(z) = -\frac{i\Gamma}{2\pi}\log(z^{2/3} - z_0^{2/3}) + \frac{i\Gamma}{2\pi}\log(z^{2/3} - \overline{z}_0^{2/3})$$

To find the path of the vortex, subtract the vortex at z_0 from $W(z)$ as before:

$$(u - iv)_{z=z_0} = \frac{d}{dz}\left(-\frac{i\Gamma}{2\pi}\log(z^{2/3} - z_0^{2/3}) + \frac{i\Gamma}{2\pi}\log(z^{2/3} - \overline{z}_0^{2/3}) + \frac{i\Gamma}{2\pi}\log(z - z_0)\right)_{z=z_0}.$$

A bit tricky... need to Taylor expand the function $z^{2/3} - z_0^{2/3}$ about $z = z_0$ and get

$$z^{2/3} - z_0^{2/3}(= f(z_0) + (z - z_0)f'(z_0) + ...) = 0 + (z - z_0)(2/3)z_0^{-1/3} + (1/2)(z - z_0)^2(-2/9)z_0^{-4/3} + \ldots$$

So

$$(u - iv)_{z=z_0} = \frac{d}{dz}\left(-\frac{i\Gamma}{2\pi}\log\left((2/3)z_0^{-1/3} - (1/9)(z - z_0)z_0^{-4/3}\right) + \frac{i\Gamma}{2\pi}\log(z^{2/3} - \overline{z}_0^{2/3})\right)_{z=z_0}$$

and this gives

$$\frac{d\bar{z}_0}{dt} = -\frac{i\Gamma}{2\pi}\left(\frac{1}{6z_0} - \frac{(2/3)z_0^{-1/3}}{(z_0^{2/3} - \bar{z}_0^{2/3})}\right).$$

Pretty damned awkward to work out the path the vortex takes from this equation.

6.9.2 Velocities under maps

Under a conformal mapping $\zeta = g(z)$, the relationship between velocity fields in $\mathcal{D}$ and $\mathcal{D}_1$ is most easily established using the chain rule:

$$W'(z) = u - iv = \frac{\mathrm{d}W_1}{\mathrm{d}\zeta}\frac{\mathrm{d}\zeta}{\mathrm{d}z} \equiv W_1'(\zeta)g'(z)$$

E.g. 6.6: Consider the map $\zeta = z\mathrm{e}^{-i\alpha} \equiv g(z)$. The line $z = s\mathrm{e}^{i\alpha}$, $-\infty < s < \infty$ is mapped to $\zeta = s$ (in fact easy to see this is a rotation of axes through an angle α.)

A uniform flow $W_1(\zeta) = U\zeta$ is transformed into $W(z) = Uz\mathrm{e}^{-i\alpha}$ whilst $g'(z) = \mathrm{e}^{i\alpha}$ so

$$u - iv = W_1'(\zeta)g'(z) = U\mathrm{e}^{-i\alpha} = U\cos\alpha - iU\sin\alpha$$

so $\mathbf{u} = (u, v) = (U\cos\alpha, U\sin\alpha)$ is a flow rotated through α, as expected.

Key points: For 2D flows, complex variables can be used to describe the flow domain. Any analytic function of complex variable z represents a complex potential $W(z)$ with real part velocity potential and imaginary part streamfunction. This is a powerful tool for generating complex flows.

The derivative $W'(z)$ provides the velocity components, the integral of $W'(z)$ around a closed loop provides information on sources of mass flux and circulation, the integral of $W'(z)^2$ provides us with information about the forces.

The method of images allows us to consider flows next to planar and circular boundaries. Complex mappings are introduced as a method of solving flows in complex domains by mapping complex variables into a domain in which the flow can easily be solved.

7 Flow past cylinders and aerofoils

7.1 Flow past a circular cylinder with circulation

Start with e.g. in §6.8 in which we formulate the flow past a cylinder and add circulation of strength Γ to the flow by placing an "image" point vortex at the origin. The complex potential is

$$W(z) = Uz + U\frac{a^2}{z} - \frac{i\Gamma}{2\pi}\log(z) \tag{46}$$

If we write $z = r\mathrm{e}^{i\theta}$ then

$$\begin{aligned} W(z) &= Ur\mathrm{e}^{i\theta} + U\frac{a^2}{r\mathrm{e}^{i\theta}} - \frac{i\Gamma}{2\pi}\log(r\mathrm{e}^{i\theta}) \\ &= Ur\mathrm{e}^{i\theta} + U\frac{a^2}{r}\mathrm{e}^{-i\theta} - \frac{i\Gamma}{2\pi}\log(r) + \frac{\Gamma\theta}{2\pi} \\ &= U\cos\theta\left(r + \frac{a^2}{r}\right) + \frac{\Gamma\theta}{2\pi} + iU\sin\theta\left(r - \frac{a^2}{r}\right) - \frac{i\Gamma}{2\pi}\log(r) \end{aligned}$$

Since $W = \phi + i\psi$ we have

$$\phi = U\cos\theta\left(r + \frac{a^2}{r}\right) + \frac{\Gamma\theta}{2\pi}$$

(the same as (28) when $\Gamma = 0$) and

$$\psi = U\sin\theta\left(r - \frac{a^2}{r}\right) - \frac{\Gamma}{2\pi}\log(r)$$

7.1.1 Analysis of flow

The flow components are

$$u_r = \frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = U\cos\theta\left(1 - \frac{a^2}{r^2}\right)$$

which is zero on $r = a$ (confirming that the addition of circulation has retained the no-flow condition on the cylinder) and

$$u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{\partial\psi}{\partial\theta} = -U\sin\theta\left(1 + \frac{a^2}{r^2}\right) + \frac{\Gamma}{2\pi r}$$

On $r = a$ this is

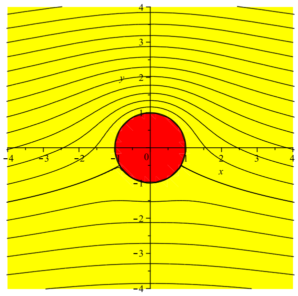
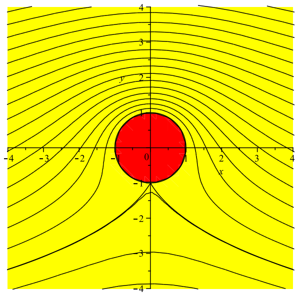
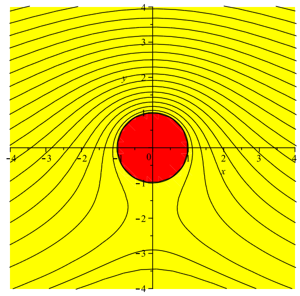
$$u_\theta = -2U\sin\theta + \frac{\Gamma}{2\pi a}$$

The **stagnation points** on the cylinder are where $u_r = 0$ and $u_\theta = 0$ and occur where

$$\sin\theta = \frac{\Gamma}{4\pi Ua} \tag{47}$$

If $\Gamma = 0$, these are at $\theta = 0, \pi$ (front and back of cylinder). If $|\Gamma| < 4\pi U a$ two stagnation points sit symmetrically about $x = 0$ on cylinder. If $|\Gamma| > 4\pi U a$ no real roots of (47) exist (the stagnation point moves into the body of the fluid).

Exercise: Repeat the analysis using $W'(z)$ to access the fluid velocity components to come to the same conclusion as above.

7.1.2 Forces on a circular cylinder

From (46)

$$W'(z) = U - U\frac{a^2}{z^2} - \frac{i\Gamma}{2\pi z}$$

Using Blasius around the closed circle C of radius a centred at the origin to compute the force gives

$$\begin{aligned} F_x - iF_y &= i\tfrac{1}{2}\rho \int_C \left(U - U\frac{a^2}{z^2} - \frac{i\Gamma}{2\pi z} \right)^2 \mathrm{d}z \\ &= i\tfrac{1}{2}\rho \int_C \left(U^2 - \frac{iU\Gamma}{\pi z} + C_2 z^{-2} + C_3 z^{-3} + C_4 z^{-4} \} \right) \mathrm{d}z \end{aligned}$$

where C_i are constants we don't need since the only contribution by Cauchy's Residue Theorem is from the simple pole, so

$$F_x - iF_y = i\tfrac{1}{2}\rho.(2\pi i). \left(-\frac{iU\Gamma}{\pi} \right) = i\rho U \Gamma$$

So $F_x = 0$ meaning there is no drag force ![15]

If $\Gamma = 0$ then $F_y = 0$ (expected since the flow is symmetric). In general $F_y = -\rho U \Gamma$ so if $\Gamma < 0$ the cylinder experiences a lift force. This is called the **Magnus effect**. [16]

[15] Reality is more complicated and this result holds for an ideal fluid; viscous effects lead to a much more complicated flow.

[16] Practically how can one introduce circulation in a flow involving a cylinder (mathematically we have placed an image vortex inside the cylinder) ? Viscous forces on a spinning cylinder tend to drag the fluid with the rotating surface creating a circulating component of the flow and this is familiar to us as swerve on a football for example

7.2 Forces on arbitrary cylinders

You might be tempted to argue that the previous result, namely that the force in the direction of the uniform stream is zero, to be a consequence of the symmetry of the cylinder. As we shall show now, this is not the case; a cylinder of *any* cross section experiences no force in a uniform stream.

Consider a cylinder of arbitrary cross-section C, in a uniform stream, U, which generates circulation of strength Γ. Far away from the cylinder, it has to be that

$$W(z) \to Uz - \frac{i\Gamma}{2\pi} \log z$$

(origin inside C) since all other contributions to the flow on account of the local perbutative effects of the cylinder must decay (and faster than that for a source). So

$$W'(z) \to U - \frac{i\Gamma}{2\pi z}, \qquad \text{as } z \to \infty$$

Also, by definition, $W(z)$ is analytic outside C in the fluid and so

$$\int_C \left(W'(z)\right)^2 \, \mathrm{d}z = \lim_{R\to\infty} \int_{|z|=R} \left(W'(z)\right)^2 \, \mathrm{d}z$$

is a consequence of Cauchy's Residue Theorem, since there are no singularities in between C and $|z| = R$.

So

$$\int \left(W'(z)\right)^2 \, \mathrm{d}z = \int_{|z|=R} \left(U - \frac{i\Gamma}{2\pi z}\right)^2 \, \mathrm{d}z$$

Referring to §7.1.2 for the circle, easy to see that we get the same result of

$$F_x - iF_y = iU\rho\Gamma$$

I.e. in the absence of circulation there is no force on a cylinder of arbitrary cross section. This is known as **D'Alembert's paradox**.

However, *with* circulation there is a lift force is $-U\rho\Gamma$ (this is called the **Kutta-Joukowski lift theorem**).

7.3 Problem: Lift on aerofoils

We use the previous theory to demonstrate the fundamental ideas that underpin flight: how to aerofoils generate lift.

7.3.1 Mapping plates to circles

Consider the (inverse) mapping[17] called the **Joukowski map**

$$z = \zeta + \frac{a^2}{\zeta} \tag{48}$$

[17] Mappings are often easiest to construct by consideration of the inverse for actually, fairly obvious reasons

Now $\zeta = a\mathrm{e}^{i\sigma}$ for $0 < \sigma < 2\pi$ describes a circle of radius a in the ζ-plane. In the z-plane

$$z = a\mathrm{e}^{i\sigma} + \frac{a^2}{a\mathrm{e}^{i\sigma}} = 2a\cos\sigma$$

which describes the line $-2a < x < 2a$, $y = 0$ in the z-plane. The mapping squishes the circle flat like you'd squash a rubber tube between two hands.

Note: We can invert the mapping (48) by writing it as $\zeta^2 - z\zeta + a^2 = 0$ and solving as

$$\zeta = \tfrac{1}{2}(z + \sqrt{z^2 - 4a^2})$$

but the square root is awkward so we will avoid this.

Note: We are going to represent our aerofoil as a flat plate of length $4a$ in the z-plane and now have a mapping from a circle, whose solutions we already understand.

E.g. 7.1: In the ζ-plane if we choose a solution representing a streaming flow, speed U, past a cylinder of radius a (E.g. 6.5)

$$W_1(\zeta) = U\left(\zeta + \frac{a^2}{\zeta}\right)$$

Then (48) gives us

$$W(z) = Uz$$

which is a uniform flow in the positive x-direction. This is expected as the 'plate' is transparent to the flow. Obviously this is not a lift-generating solution and we need to do more.

7.3.2 Inclined flow past a plate with circulation

Consider the complex potential from §7.1 for horizontal flow past a cylinder with circulation Γ, but in a complex ξ-plane. We call it

$$W_2(\xi) = U\left(\xi + \frac{a^2}{\xi}\right) - \frac{i\Gamma}{2\pi}\log(\xi)$$

We know from §7.1.2, §7.2, that this flow generates a lift force of $-\rho U\Gamma$.

The mapping (see E.g. 6.6) $\xi = \zeta\mathrm{e}^{-i\alpha}$ rotates axes through α and results in the complex potential in the ζ-plane of

$$W_1(\zeta) = U\left(\zeta\mathrm{e}^{-i\alpha} + \frac{a^2\mathrm{e}^{i\alpha}}{\zeta}\right) - \frac{i\Gamma}{2\pi}\log(\zeta)$$

plus a constant which we can ignore (because $W'(\zeta)$ determines the flow). I.e. the complex potential $W_1(\zeta)$ represents the flow angled at α to the horizontal past a circular cylinder with circulation.

Finally we map using (48) into the z-plane which determines a complex potential $W(z)$ representing the inclined flow upon a horizontal plate of length $4a$ with circulation.

Note: What has happens to the dipole & vortex embedded in the original complex potential W_2 under these rotation and squishing maps ? It is sitting on the plate[18]

Note: We could write down $W_1(z)$ explicitly as we have determined $\zeta = g(z)$, but this is not necessary...

The velocity in the z-plane is determined (see §6.9.2) by

$$u - iv = W_1'(\zeta)\frac{\mathrm{d}\zeta}{\mathrm{d}z} = W_1'(\zeta)\frac{1}{\dfrac{\mathrm{d}\zeta}{\mathrm{d}z}}$$

(from MVC: the derivative of the inverse map is the inverse of the derivative of the map) which is more convenient since this allows us to write

$$u - iv = \left[U\left(\mathrm{e}^{-i\alpha} - \frac{a^2\mathrm{e}^{i\alpha}}{\zeta^2}\right) - \frac{i\Gamma}{2\pi\zeta}\right] \Big/ \left(1 - \frac{a^2}{\zeta^2}\right)$$

We are interested in velocity on the plate, and if we let $\zeta = a\mathrm{e}^{i\sigma}$ which we've already established maps to $z = 2a\cos\sigma$ then

$$\begin{aligned} u - iv &= \frac{U(\mathrm{e}^{-i\alpha} - \mathrm{e}^{i\alpha}\mathrm{e}^{-2i\sigma}) - i(\Gamma/2\pi a)\mathrm{e}^{-i\sigma}}{(1-\mathrm{e}^{-2i\sigma})} \\ &= \frac{U(\mathrm{e}^{i(\sigma-\alpha)} - \mathrm{e}^{-i(\sigma-\alpha)}) - i(\Gamma/2\pi a)}{(\mathrm{e}^{i\sigma} - \mathrm{e}^{-i\sigma})} \\ &= \frac{U\sin(\sigma-\alpha) - \Gamma/4\pi a}{\sin\sigma} \end{aligned} \tag{49}$$

Note: The right-hand side is real, so $v = 0$ as we expect.

7.3.3 The Kutta condition

In the absence of circulation (49) gives us

$$u = \frac{U\sin(\sigma-\alpha)}{\sin\sigma}$$

and $u \to \infty$ when $\sigma \to 0, \pi$. These correspond to the sharp leading and trailing edges of the plate. This is to be expected – in §6.3 we showed that the flow speed goes to infinity around sharp edges in the boundary.

We also note that, there are stagnation points in the flow at $\sigma = \alpha$ and $\sigma = \alpha + \pi$. In the physical z-plane these are points (for small α) under the plate backwards of the leading egde and on the top of the plate forwards from the trailing edge.

[18] one has to be a bit cleverer than we've been and invent a mapping which has some girth to it so that the singularities in the original potential are still removed from the physical domain. Mathematically, we are OK though.

An aerofoil has a rounded leading edge which suppresses the singularity there. There are subtle physical reasons why a fluid will avoid sharp turns elsewhere, and our focus turns to the trailing edge.

The Kutta condition resolves the singularity at the trailing edge by incorporating the right amount of circulation such that the stagnation point found forward of the trailing edge on top of the plate when there is no circulation moves to the trailing edge to suppress the singularity.

Thus, if we choose

$$\Gamma = -4U\pi a \sin\alpha \tag{50}$$

from (49) with

$$u = \frac{U(\sin(\sigma-\alpha)+\sin(\alpha))}{\sin\sigma} \to U\cos\alpha$$

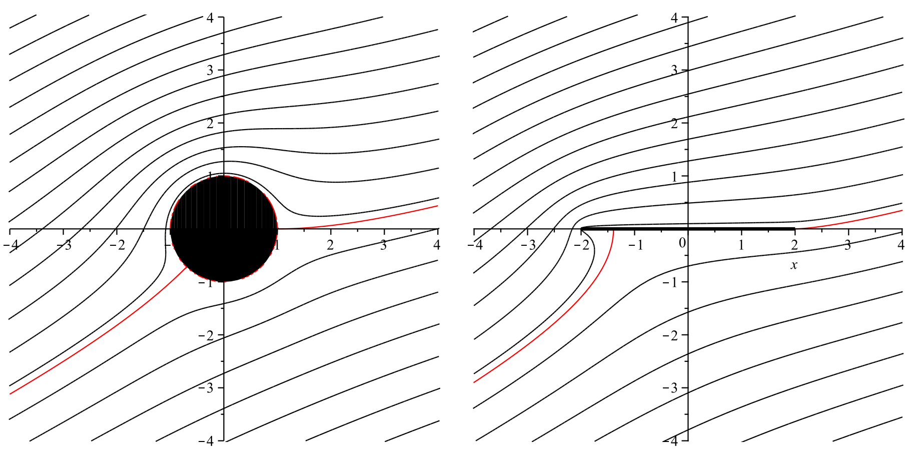
as $\sigma \to 0$ (e.g. using L'Hopital's rule) but, most importantly, this is a finite value !

This means that the lift force on our plate provided by the flow round the cylinder and mapped into the flow past the plate is

$$F_y = -\rho U\Gamma = \pi U^2 \rho L \sin\alpha$$

where $L = 4a$ is the length of the plate (the *chord*). This is actually a pretty good approximation to a real wing under ideal flying conditions (e.g. not under high angles of attack.)

Below is a graphical representation of the aerofoil flow in the mapped plane around the circle and the flow in the physical plane satisfying the Kutta condition at the rear of the plate.



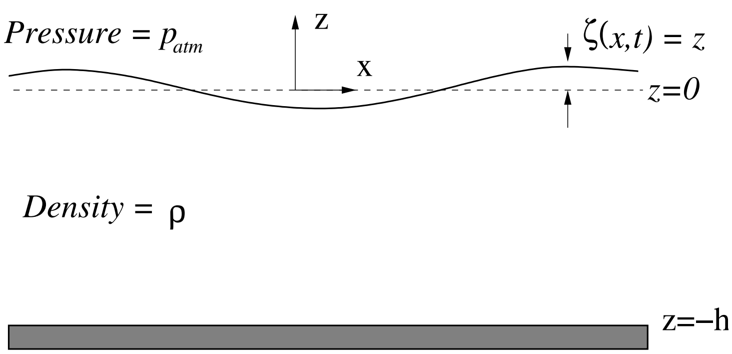
Remarks: Finally, we comment on the presence of circulation around the wing, which is the crucial ingredient needed for flying. In the first instance, there is nothing wrong with that from the point of view of a potential flow description. However, where is this circulation coming from when one imagines starting up the flow, with zero circulation ? Since the net circulation in a large circle around the wing must vanish initially, and the Kutta condition requires a circulation $\Gamma < 0$ around the wing, this means that in the process of the point of separation moving to the trailing edge, vortices of positive circulation (rotating counterclockwise) are shed from the wing. Eventually the vortices are convected downstream, and no longer matter for the problem.

8 Free surfaces and waves

Free surface flows are defined by a fluid motion in which a portion of the boundary of the fluid does not move in a prescribed way. That is, the fluid surface evolves in time as part of the solution. Perhaps the easiest example to understand is that of the surface of a body of heavy fluid such as water which we observe will support wave-like motion (e.g. surface of cup of tea).

We are not going to need any new physics to describe such a surface. These principles are already in place. What we know from observations of the surface of water (on the ocean for example) is that the solutions can be very complicated (e.g. breaking waves !)

We shall avoid such complications in due course by making certain approximations. For now, we assume that the flow is irrotational and the fluid is incompressible and inviscid[19]. We shall also assume 2D flows but this is only to keep things simple.



8.1 Governing equations (non-linear)

We can use potential theory, so $\mathbf{u} = \boldsymbol{\nabla}\phi$ and

$$\nabla^2\phi = 0 \tag{51}$$

Note: $\mathbf{u} = (u, 0, w)$ and $\phi = \phi(x, z, t)$ is still 2D but the motion is in the (x, z) plane.

Choose $z = 0$ to coincide with the undisturbed free surface and set the base of the fluid at $z = -h$, a constant.

Let the free surface of the water in motion be described by

$$\boxed{z = \zeta(x, t).}$$

Below $z = \zeta(x, t)$ we have our fluid, density ρ and above we have air of negligible density and constant atmospheric pressure p_{atm}.

On $z = -h$ the kinematic boundary condition says that $\mathbf{u} \cdot \hat{\mathbf{z}} = 0$ or $w = 0$ or

$$\frac{\partial \phi}{\partial z} = 0, \qquad \text{on } z = -h \tag{52}$$

[19]all good approximations unless the surface is in extreme conditions

So far so good. Now the interesting bit... the free surface. Two condition apply here: (i) the kinematic boundary condition says that a fluid particle on the surface has to move with the surface; (ii) the pressure on the surface of the fluid must equal p_{atm} (this is called the **dynamic boundary condition**[20])

(i) The surface can be described by $S(x, z, t) = z - \zeta(x, t) = 0$. Then according to (29), $DS/Dt = 0$. In other words

$$\frac{D}{Dt}(z - \zeta(x,t)) = \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + w\frac{\partial}{\partial z}\right)(z - \zeta(x,t))$$
$$= -\frac{\partial \zeta}{\partial t} - \frac{\partial \phi}{\partial x}\frac{\partial \zeta}{\partial x} + \frac{\partial \phi}{\partial z} = 0, \qquad \text{on } z = \zeta(x,t) \tag{53}$$

(ii) Unsteady Bernoulli on surface gives

$$p_{atm}/\rho + \tfrac{1}{2}\left(\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right) + \frac{\partial \phi}{\partial t} + g\zeta = \mathcal{C}(t), \qquad \text{on } z = \zeta(x,t) \tag{54}$$

Note: Laplace's equation, (51) looks OK and (52) looks OK. But the two free surface conditions (53), (54) look like a real headache. For a start they are **non-linear**. But worse still, they must be applied on a surface which is part of the solution ! But let's not get upset; afterall, we know that this free surface problem gives rise to all sorts of complex solutions like breaking waves. So it is not a surprise that things look complicated... and they really are !

8.2 Linearised equations

The way forward is to make the following approximation

$$\zeta(x,t) \ll h, \qquad \left|\frac{\partial \zeta}{\partial x}\right| \ll 1$$

which limits the free surface to **small amplitudes**. It follows that $|\phi| \ll 1$ also.

This allows us to make two further approximations:

(A) products of small terms can be neglected. This enables us to remove the non-linear terms from our boundary conditions.

E.g. Since ϕ is small and so is ζ then

$$\left|\frac{\partial \phi}{\partial x}\frac{\partial \zeta}{\partial x}\right| \ll \left|\frac{\partial \zeta}{\partial t}\right|$$

(B) Since ζ is small, we can use a Taylor expansion around the mean level $z = 0$ and subsequently use (A) to neglect small terms. This enables us to overcome the difficulty of not knowing where the boundary conditions should be imposed.

[20] we've used this before, as early as the lock gate problem but without giving it a special name.

E.g.:

$$\left.\frac{\partial\phi}{\partial t}\right|_{z=\zeta} = \left.\frac{\partial\phi}{\partial t}\right|_{z=0} + \zeta\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial t}\right)_{z=0} + \zeta^2\ldots$$

But according to (A) the second (and later) term(s) on the RHS is much smaller than the first so we neglect it.

Applying these principles reduces (53) to

$$\boxed{\frac{\partial\zeta}{\partial t} = \frac{\partial\phi}{\partial z}, \qquad \text{on } z=0} \tag{55}$$

and (54) to

$$\frac{\partial\phi}{\partial t} + g\zeta = \mathcal{C}(t) - p_{atm}/\rho, \qquad \text{on } z=0 \tag{56}$$

Finally, the terms on the RHS are not space-dependent and so do not affect the flow velocities. Formally we can redefine

$$\phi \to \phi - p_{atm}t/\rho + \int^t \mathcal{C}(t')dt'$$

and this does not affect any of the other conditions of the problem. It's only effect is that (56) is now

$$\boxed{\frac{\partial\phi}{\partial t} + g\zeta = 0, \qquad \text{on } z=0} \tag{57}$$

This completes the process known as **linearisation**.

Note: Linearised pressure *in* the fluid is

$$\frac{p(x,z,t)}{\rho} + \frac{\partial\phi}{\partial t} + gz = \mathcal{C}(t)$$

and because of $\phi \to \phi'$ this transforms to

$$\frac{p(x,z,t) - p_{atm}}{\rho} = -\frac{\partial\phi}{\partial t} - gz \tag{58}$$

the LHS representing the pressure in excess of atmospheric and the RHS has a hydrostatic component ($-gz$) which exists in the absence of any motion and a dynamic component $-\partial\phi/\partial t$ which therefore accounts for the pressure variations due to fluid motion.

8.3 Travelling waves

Let us assume the surface is sinusoidal[21] and takes the following form:

$$\zeta(x,t) = H\sin(kx - \omega t) \tag{59}$$

Then

[21] why not ?

- **Wave height** is H; ζ oscillates between $\pm H$.
- Motion is periodic with **period** $T = 2\pi/\omega$ (ω is **angular frequency**).
- Surface repeats in space every $\lambda = 2\pi/k$ (λ is **wavelength**)
- Moves to the *right* with speed $c = \omega/k$ (c is called **phase speed**, see APDE2.)

At the moment (59) is not a solution, it's an ansatz (a guess). We need to see if it satisfies our governing equations (the linearised versions) and this means we need $\phi(x, z, t)$.

Since from (57) $\dfrac{\partial \phi}{\partial t} = -g\zeta$ reasonable to write

$$\phi(x, z, t) = \cos(kx - \omega t)Z(z)$$

for some $Z(z)$. Really this is a separation of variables solution (e.g. APDE2) and we've written $\phi = X(x, T)Z(z)$.

Then from (51)

$$-k^2 \cos(kx - \omega t)Z(z) + \cos(kx - \omega t)Z''(z) = 0$$

and so $Z''(z) - k^2 Z(z) = 0$. A general solution[22] is

$$Z(z) = A \cosh k(z + h) + B \sinh k(z + h)$$

for constants A, B. Because of (52), need $B = 0$ and so

$$Z(z) = A \cosh k(z + h)$$

At this point we still have two conditions (55) and (57) to apply, but only one constant, A, remains unknown[23]

From (57) first

$$gH \sin(kx - \omega t) = -A\omega \sin(kx - \omega t) \cosh kh$$

and then

$$A = \frac{-gH}{\omega \cosh kh}$$

determines the constant A, meaning that ϕ is now fully known. Finally, applying (55) gives

$$-\omega H \cos(kx - \omega t) = kA \cos(kx - \omega t) \sinh kh$$

and this therefore implies

$$-\omega H = k \left(\frac{-gH}{\omega \cosh kh} \right) \sinh kh$$

or

$$\boxed{\frac{\omega^2}{g} = k \tanh kh} \qquad (60)$$

[22]can write other forms of the general solution here like $Z(z) = C\mathrm{e}^{-kz} + D\mathrm{e}^{kz}$ or $Z(z) = E \cosh kz + F \sinh kz$

[23]Awkward

This is called the **dispersion relation** (for gravity waves on finite depth water).

Note: In the derivation of the solution we chose (59) as a sine function. We could have also chosen a cosine with equal success. In general we could combine both sine and cosine and work with **complex exponentials**.

8.4 The character of waves

With reference to (60) we make the following notes:

- Changing $k \to -k$, (60) still holds. Then

$$\zeta(x,t) = H \sin(kx + \omega t)$$

  is a wave travelling to the left, speed $c = -\omega/k$.

- $k = 2\pi/\lambda$ is called the **wavenumber** (the number of wavelengths that fit into 2π).
- (60) implies complicated relations between period, wavelength and speed. Very qualitatively, large wavelengths have long periods and high phase speed (and vice versa).

8.4.1 Long waves

If $kh \ll 1$ (so that $\lambda \gg h$) then waves are long compared to the depth.

Then $\tanh kh \approx kh$ and (60) tells us $\omega^2/g \approx k^2 h$ and the wave speed is

$$c = \omega/k = \sqrt{gh}$$

Note: In the long wavelength limit, wave speed does not depend on wavelength of wave period. Waves of all frequencies travel at the same speed and they are called **non-dispersive**.

8.4.2 Short waves

If $kh \gg 1$ (so that $\lambda/h \ll 1$) then waves are short compared to depth

Then $\tanh kh \approx 1$ and (60) tells us $\omega^2/g \approx k$ or

$$c = \omega/k = \sqrt{g/k} = \sqrt{g\lambda/2\pi}$$

In the short wavelength limit, the wave speed doesn't depend on depth (because it's relatively far away) but does depend on wavelength (and period).

E.g. 8.1: (2004 Indonesian Boxing Day Tsunami)

Indian ocean is deep $\sim$ 4km, but an earthquake moves a lot of water and consequently waves happen to be very long ($\sim$ 10km) so $\lambda/h \gg 1$ and "shallow water" approximately applies. So wave speed is

$$c = \sqrt{gh} = \sqrt{10 \times 4000} = 200\text{ms}^{-1} = 400\text{mph}$$

(which matches observations made at that time). The energy flux or power (per length of wave) transported by a wave in the shallow limit happens to be given (no proof) by $E = \frac{1}{2}\rho g c H^2$, where $c = \sqrt{gh}$ here. Energy flux must be constant for all x, so

$$E = \tfrac{1}{2}\rho g H^2 \sqrt{gh} = const$$

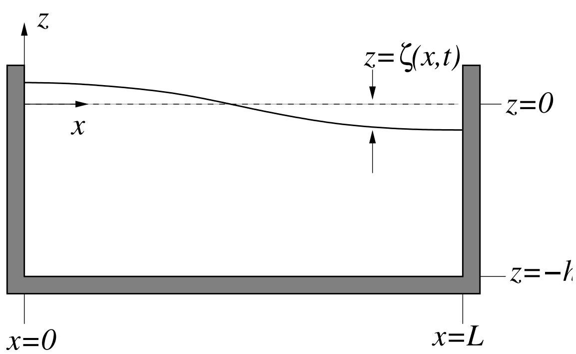
and this implies $H^4 h = const$. Therefore $H \sim 1/h^{1/4}$.

If depth goes from 4000m to 4m the wave height increases by a factor of $1000^{1/4} \approx 6$. So a 50cm tsunami wave (typically they are this small out to sea – boats will not notice them) in the ocean is a 3m wave at the coast.

A 3m wave might not seem *that* big (you'll see waves this big around the UK), but it's the power of the wave that causes the disruption. Putting in the numbers above gives $E = 250,000$W/m (watts per metre) or 250kW/m. That's the same power as 10 electric cars generate for every metre of coastline.

8.5 Problem: Oscillations in a closed container

Liquids readily slosh back and forth in a closed container (e.g. tea in a tea-cup). There are many associated important practical problems: sloshing in road tankers, water on decks of ships, resonance in harbours, etc...



Example: consider a two-dimensional rectangular box with rigid walls at $x = 0, L$ and a bottom on $z = -h$, filled with fluid to $z = 0$.

We adopt the small-amplitude theory from before but require additional kinematic boundary conditions at the two lateral sides of the container. These are that $\mathbf{u} \cdot \hat{\mathbf{n}} \equiv \mathbf{u} \cdot \hat{\mathbf{x}} \equiv u = 0$ or

$$\frac{\partial \phi}{\partial x} = 0, \qquad \text{on } x = 0, L \tag{61}$$

8.5.1 Normal modes

This is different to before as we do not seek travelling waves. However, we still expect solutions which are periodic in time. So we write

$$\zeta(x,t) = \eta(x) \sin \omega t$$

for some unknown $\eta(x)$ (c.f. normal mode solutions in APDE2), and assume a time-compatible separation solution for ϕ

$$\phi(x, z, t) = X(x)Z(z)\cos\omega t$$

Then Laplace's equation gives

$$X''(x)Z(z)\cos\omega t + Z''(z)X(x)\cos\omega t = 0$$

which implies

$$X''(x)Z(z) + Z''(z)X(x) = 0$$

This separates (APDE2):

$$\frac{X''(x)}{X(x)} = -\frac{Z''(z)}{Z(z)} = -k^2 \tag{62}$$

where $-k^2$ is the separation constant.

Solving (62) for $Z(z)$, with (52) gives $Z(z) = A\cosh k(z+h)$ as before.

We also note that we can combine (55) and (57) to eliminate ζ by adding g times the first equation to $\partial/\partial t$ of the second to give

$$\frac{\partial^2\phi}{\partial t^2} = -g\frac{\partial\phi}{\partial z}, \qquad \text{on } z = 0$$

which implies

$$-\omega^2 X(x)Z(0)\cos\omega t = -gX(x)Z'(0)\cos\omega t$$

and it follows that $\omega^2/g = k\tanh kh$ as before in (60)[24]

Now solving (62) for $X(x)$ gives

$$X(x) = B\cos(kx) + C\sin(kx)$$

and this solution, when subjected to (61) has to satisfy $X'(0) = X'(L) = 0$.

Then easy to show (APDE2) that must have $C = 0$ whilst the separation constants are $\boxed{k = n\pi/L}$ and so

$$X(x) = B\cos(n\pi x/L)$$

So pulling everything together we have

$$\phi(x, z, t) = C\cos(n\pi x/L)\cosh k(z+h)\cos\omega t$$

for some constant $C = A.B$. This is our normal mode solution in which the frequency is determined from using $k = n\pi/L$ in (60) to give

$$\omega^2/g = (n\pi/L)\tanh(n\pi h/L) \equiv \omega_n^2/g, \qquad n = 0, 1, 2, \ldots \tag{63}$$

Also, we can now use the solution in either linearised condition applying on the surface to determine $\eta(x)$ and it follows that

$$\zeta(x, t) = \frac{\omega C\cosh kh}{g}\cos(n\pi x/L)\sin\omega t$$

[24]Not unsurprising as applying conditions at the bottom and surface of the fluid are independent of the lateral boundary conditions.

In constrast to the travelling wave solutions where waves of any frequency could propagate on the surface of a fluid, here the solution exists only at a set of discrete wave frequencies given by (63). The value of n (the **mode number**) tells you how many oscillations are occurring across the box.

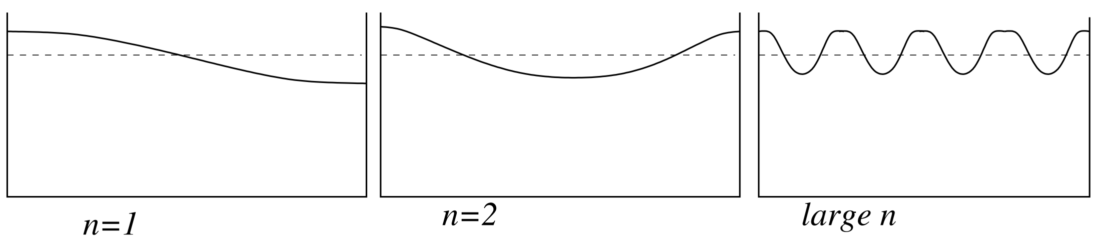
Note: $n = 0$ is not a valid mode number since (63) gives $\omega = 0$, $\phi = 1$, and $\zeta = 0$ and there is no motion in the fluid.[25]

The **fundamental frequency** is the lowest (gravest) frequency, given here by $n = 1$.

E.g. Let $L = 1$m, $h = 20$cm. Then the fundamental frequency ($n = 1$) is

$$\omega \equiv \omega_1 = \sqrt{9.81 \times \pi \times \tanh(\pi \times 0.2)} = 4.14\text{s}^{-1}$$

given a period of $\tau = 2\pi/4.14 = 1.51$ seconds. The shape of the surface is given by the variation of x in $\zeta(x, t)$, which is $\cos(\pi x/L)$ and it is modulated by the signal $\sin \omega t$.



The $n = 2$ mode gives a period of $\tau = 0.86$s a shape function of $\cos(2\pi x/L)$ etc...

As $n \to \infty$, $\tanh(n\pi h/L) \to 1$ and so $\omega_n \to \sqrt{gn\pi/L}$.

[25] In fact, you can discount a sloshing mode with no x-dependence – a flat surface oscillating up and down – as it would violate mass conservation in the container.

Appendix: Revision

A summary of key results from previous courses: MVC, MCF and APDE2.

A.6 Suffix notation and summation convention

Suppose that we have two vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$. Then the **dot product** is defined to be

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{3} u_i v_i \qquad \text{or, more simply, write} \qquad \mathbf{u} \cdot \mathbf{v} = u_i v_i$$

(drop the summation symbol on the understanding that **repeated suffices** imply summation.)

Definition A.1 (Kronecker delta)

The **Kronecker delta** *is defined by*

$$\delta_{ij} = \left\{ \begin{array}{ll} 1, & i = j \\ 0, & i \neq j \end{array} \right\}$$

So in summation convention, $\delta_{ij} a_j = a_i$ since

$$\delta_{ij} a_j \equiv \sum_{j=1}^{3} \delta_{ij} a_j = a_i$$

Example A.2 (Uses of the Kronecker delta)

1. $\delta_{ii} = 3$
2. $\delta_{ij} u_i v_j = u_j v_j \equiv \mathbf{u} \cdot \mathbf{v}$.

Definition A.3 (Levi-cevita tensor)

The **antisymmetric symbol** ϵ_{ijk} *is defined by*

- $\epsilon_{123} = 1$
- ϵ_{ijk} *is zero if there are any repeated suffices. E.g.* $\epsilon_{113} = 0$.
- *Interchanging any two suffices reverses the sign: e.g.* $\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{kji}$
- *Above implies invariant under cyclic rotation of suffices:* $\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}$

With this definition all 27 permutations are defined. There are only 6 non-zero components,

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1, \qquad \epsilon_{213} = \epsilon_{132} = \epsilon_{321} = -1,$$

Definition A.4 (cross product)

The **cross product** *is defined by*

$$\mathbf{w} = \mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2)\hat{\mathbf{x}} + (u_3 v_1 - u_1 v_3)\hat{\mathbf{y}} + (u_1 v_2 - u_2 v_1)\hat{\mathbf{z}}$$

But can now be written in component form as

$$w_i = \epsilon_{ijk} u_j v_k$$

where summation over j and k occurs.

Example A.5 (triple product)

Consider the **triple product**,

$$\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = w_i \epsilon_{ijk} u_j v_k$$

where summation is over i, j, k so result is scalar. It follows that

$$\begin{aligned}
\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) &= \epsilon_{jki} w_i u_j v_k = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \\
&= \epsilon_{kij} w_i u_j v_k = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) \\
&= -\epsilon_{jik} w_i u_j v_k = -\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v}) \\
&= -\epsilon_{ikj} w_i u_j v_k = -\mathbf{w} \cdot (\mathbf{v} \times \mathbf{u})
\end{aligned}$$

Example A.6 (double product)

The **double product** *of ϵ_{ijk} is*

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

Example A.7 (vector triple product)

The **vector triple product** *is defined by the result*

$$\mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = (\mathbf{w} \cdot \mathbf{v})\mathbf{u} - (\mathbf{w} \cdot \mathbf{u})\mathbf{v}$$

Proof:

$$\begin{aligned}
[\mathbf{w} \times (\mathbf{u} \times \mathbf{v})]_i &= \epsilon_{ijk} w_j [\mathbf{u} \times \mathbf{v}]_k \\
&= \epsilon_{ijk} w_j \epsilon_{klm} u_l v_m \\
&= \epsilon_{kij} \epsilon_{klm} w_j u_l v_m \\
&= (\delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}) w_j u_l v_m \\
&= w_j u_i v_j - w_j u_j v_i = (\mathbf{w} \cdot \mathbf{v}) u_i - (\mathbf{w} \cdot \mathbf{u}) v_i
\end{aligned}$$

True for $i = 1, 2, 3$, hence result.

A.7 Differential operators

Here we consider operations on a function $\phi(\mathbf{r})$ and a vector field $\mathbf{f}(\mathbf{r})$ where $\mathbf{r} = (x_1, x_2, x_3)$. One can regard $\boldsymbol{\nabla}$ as the vector operator $\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$. Without using any information (but always remembering the true meaning of the symbol!), one can also use the much sleaker notation $\boldsymbol{\nabla} \equiv (\partial_1, \partial_2, \partial_3)$. I will usually do that whenever using summation convention. Then

- The **gradient** is $\boldsymbol{\nabla}\phi$. So $[\boldsymbol{\nabla}\phi]_i = \frac{\partial \phi}{\partial x_i} \equiv \partial_i \phi$
- The **divergence** is $\boldsymbol{\nabla} \cdot \mathbf{f} = \frac{\partial f_i}{\partial x_i} \equiv \partial_i f_i$ (in summation convention)
- The **curl** is $\boldsymbol{\nabla} \times \mathbf{f}$ where $[\boldsymbol{\nabla} \times \mathbf{f}]_i = \epsilon_{ijk} \frac{\partial f_k}{\partial x_j} \equiv \epsilon_{ijk} \partial_j f_k$.
- The **Laplacian** is $\nabla^2 \phi = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\phi = \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2}$
- $(\mathbf{f} \cdot \boldsymbol{\nabla})\mathbf{f} = \left(f_1 \frac{\partial f_1}{\partial x_1} + f_2 \frac{\partial f_1}{\partial x_2} + f_3 \frac{\partial f_1}{\partial x_3} \right) \hat{\mathbf{x}} + \left(f_1 \frac{\partial f_2}{\partial x_1} + f_2 \frac{\partial f_2}{\partial x_2} + f_3 \frac{\partial f_2}{\partial x_3} \right) \hat{\mathbf{y}}$
  $+ \left(f_1 \frac{\partial f_3}{\partial x_1} + f_2 \frac{\partial f_3}{\partial x_2} + f_3 \frac{\partial f_3}{\partial x_3} \right) \hat{\mathbf{z}}$

Example A.8

1. $[\boldsymbol{\nabla}(x_j)]_i = \partial_j x_i = \delta_{ij}$
2. $\boldsymbol{\nabla} \cdot \mathbf{r} = \partial_i x_i = \delta_{ii} = 3$
3. $[\boldsymbol{\nabla} \times \mathbf{r}]_i = \epsilon_{ijk} \partial_k x_j = \epsilon_{ijk} \delta_{kj} = \epsilon_{ijj} = 0$
4. $\boldsymbol{\nabla} r = \mathbf{r}/r$, *where* $r^2 \equiv \mathbf{r} \cdot \mathbf{r}$.

   Proof: $[\boldsymbol{\nabla} r]_i = \partial_i \sqrt{x_j x_j} = \frac{\partial_i x_j^2}{2\sqrt{x_j^2}} = \frac{x_j \partial_i x_j}{r} = \frac{x_j \delta_{ij}}{r} = \frac{x_i}{r}$.

A.8 Three important vector identities

1. $\boldsymbol{\nabla} \cdot (\phi \mathbf{f}) = \partial_i (\phi f_i) = \phi \partial_i f_i + f_i \partial_i \phi = \phi \boldsymbol{\nabla} \cdot \mathbf{f} + \mathbf{f} \cdot \boldsymbol{\nabla}\phi$
2. $\boldsymbol{\nabla} \times (\phi \mathbf{f}) = \phi(\boldsymbol{\nabla} \times \mathbf{f}) + (\boldsymbol{\nabla}\phi \times \mathbf{f})$

   Proof: $[\boldsymbol{\nabla} \times (\phi \mathbf{f})]_i = \epsilon_{ijk} \partial_j (\phi f_k) = \phi \epsilon_{ijk} \partial_j f_k + f_k \epsilon_{ijk} \partial_j \phi = \phi [\boldsymbol{\nabla} \times \mathbf{f}]_i + [\boldsymbol{\nabla}\phi \times \mathbf{f}]_i$.

3. $\mathbf{f} \times (\boldsymbol{\nabla} \times \mathbf{f}) = \boldsymbol{\nabla}(\frac{1}{2}\mathbf{f} \cdot \mathbf{f}) - (\mathbf{f} \cdot \boldsymbol{\nabla})\mathbf{f}$

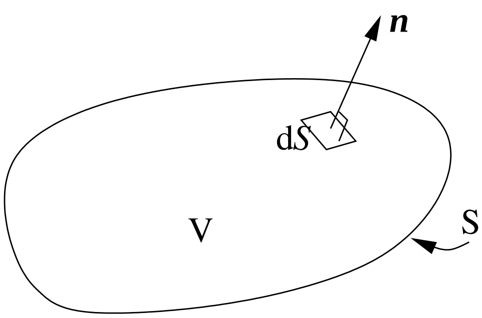
Proof:

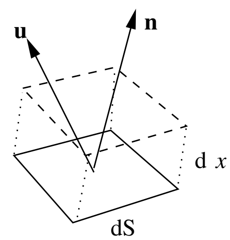
$$\begin{aligned}[\mathbf{f} \times (\boldsymbol{\nabla} \times \mathbf{f})]_i &= \epsilon_{ijk} f_j [\boldsymbol{\nabla} \times \mathbf{f}]_k = \epsilon_{ijk} f_j \epsilon_{klm} \partial_l f_m = \epsilon_{kij}\epsilon_{klm} f_j \partial_l f_m \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) f_j \partial_l f_m = f_j \partial_i f_j - f_j \partial_j f_i = \partial_i \frac{1}{2} f_j f_j - (\mathbf{f} \cdot \boldsymbol{\nabla}) f_i\end{aligned}$$

The result follows.

A.9 Integral results

A.9.1 The divergence theorem





Consider a volume V bounded by a closed surface S with outward unit normal $\hat{\mathbf{n}}$. Then for a vector field $\mathbf{f}(\mathbf{r})$

$$\boxed{\int_V \boldsymbol{\nabla} \cdot \mathbf{f} dV = \int_S \mathbf{f} \cdot \hat{\mathbf{n}} dS}$$

(and note $\hat{\mathbf{n}} dS \equiv d\mathbf{S}$)

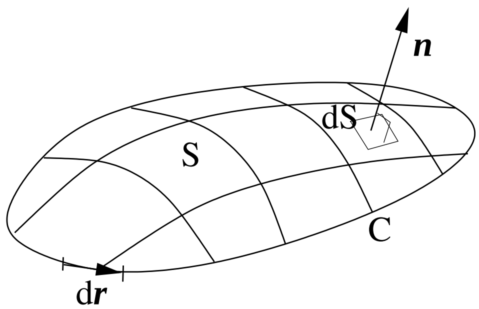
Corollary: Let $\mathbf{f}(\mathbf{r}) = \mathbf{a}\phi(\mathbf{r})$ where $\mathbf{a}$ is an arbitrary constant vector, and $\phi(\mathbf{r})$ a scalar function. Since $\boldsymbol{\nabla} \cdot (\mathbf{a}\phi(\mathbf{r})) = \mathbf{a} \cdot \boldsymbol{\nabla}\phi(\mathbf{r})$, so the divergence theorem reduces to

$$\mathbf{a} \cdot \int_V \boldsymbol{\nabla}\phi dV = \mathbf{a} \cdot \int_S \phi \hat{\mathbf{n}} dS$$

True for any $\mathbf{a}$, so

$$\boxed{\int_V \boldsymbol{\nabla}\phi dV = \int_S \phi \hat{\mathbf{n}} dS}$$

A.9.2 Stokes' theorem



Let C be a closed curve bounding a surface S with unit normal $\hat{\mathbf{n}}$.

Then for a vector field $\mathbf{f}(\mathbf{r})$,

$$\boxed{\int_C \mathbf{f}\cdot d\mathbf{r} = \int_S (\boldsymbol{\nabla}\times\mathbf{f})\cdot\hat{\mathbf{n}}dS}$$

where $d\mathbf{r}$ is a line element on C.

Corollary: Let $\mathbf{f}(\mathbf{r}) = \mathbf{a}\phi(\mathbf{r})$ where $\mathbf{a}$ is an arbitrary constant vector. Then

$$\int_C \mathbf{a}\phi\cdot d\mathbf{r} = \int_S (\boldsymbol{\nabla}\times\mathbf{a}\phi)\cdot\hat{\mathbf{n}}dS$$

and from an earlier result $(\boldsymbol{\nabla}\times\mathbf{a}\phi) = \phi(\boldsymbol{\nabla}\times\mathbf{a}) + (\boldsymbol{\nabla}\phi\times\mathbf{a}) = (\boldsymbol{\nabla}\phi\times\mathbf{a})$. Using the vector triple product result, $(\boldsymbol{\nabla}\phi\times\mathbf{a})\cdot\hat{\mathbf{n}} = -\mathbf{a}\cdot(\boldsymbol{\nabla}\phi\times\hat{\mathbf{n}})$ and then

$$\mathbf{a}\cdot\int_C \phi d\mathbf{r} = -\mathbf{a}\cdot\int_S \boldsymbol{\nabla}\phi\times\hat{\mathbf{n}}dS$$

Therefore

$$\int_C \phi d\mathbf{r} = -\int_S \boldsymbol{\nabla}\phi\times\hat{\mathbf{n}}dS$$

A.10 Curvilinear coordinate systems

Many problems can be approached more simply by choosing a coordinate system that fits a given geometry. Instead of writing the position vector $\mathbf{r}$ as a function of Cartesian coordinates (x, y, z), $\mathbf{r}$ is now written as function of three new coordinates: $\mathbf{r}(q_1, q_2, q_3)$. The coordinate lines are swept out by varying one of the coordinates, keeping the other two constant. We will deal only with the by far most important case of *orthogonal* coordinate systems, in which the coordinate lines always intersect one another at right angles.

Evidently, $\dfrac{\partial\mathbf{r}}{\partial q_i}$ is a vector which points in the direction of the i-th coordinate line.

If each of these vectors are normalized to unity, we obtain the local basis system:

$$\hat{\mathbf{q}}_i = \frac{1}{h_i}\frac{\partial\mathbf{r}}{\partial q_i}, \quad h_i \equiv \left|\frac{\partial\mathbf{r}}{\partial q_i}\right|. \tag{64}$$

The quantities $h_i(q_1, q_2, q_3)$ are called *scale factors* or *metric coefficients*. The fact that the coordinate system is orthogonal means that all $\hat{\mathbf{q}}_i$, computed at a point (q_1, q_2, q_3), are orthogonal. However, the direction of $\hat{\mathbf{q}}_i$ of course changes as one goes along the coordinate lines.

Example A.9 (Cylindrical polar coordinate system)

The position vector $(x, y, z) = \mathbf{r}$ *is*

$$\mathbf{r} = (r\cos\theta, r\sin\theta, z)\,,$$

and the coordinates (q_1, q_2, q_3) *are* (r, θ, z). *Thus the scale factors become* $h_1 = 1$, $h_2 = r$, *and* $h_3 = 1$, *and the local basis is*

$$\begin{aligned}\hat{\mathbf{r}} &= \frac{\partial \mathbf{r}}{\partial r} = (\cos\theta, \sin\theta, 0)\,,\\ \hat{\boldsymbol{\theta}} &= \frac{1}{r}\frac{\partial \mathbf{r}}{\partial \theta} = (-\sin\theta, \cos\theta, 0)\,,\\ \hat{\mathbf{z}} &= \frac{\partial \mathbf{r}}{\partial z} = (0, 0, 1).\end{aligned}$$

It is simple to confirm that $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\mathbf{z}}$ *are indeed mutially orthogonal.*

Remark A.10 (Integration)

As the coordinates are varied by δq_1, δq_2, *and* δq_3, *respectively,* $\mathbf{r}$ *describes a volume whose sides are orthogonal. The length of each side is* $h_i \delta q_i$, *and thus the volume of the cuboid is* $dxdydz = h_1 h_2 h_3 dq_1 dq_2 dq_3$. *Thus integration in a curvilinear coordinate system is achieved by the formula*

$$\int_V f(\mathbf{r})dxdydz = \int_V f(q_1, q_2, q_3) h_1 h_2 h_3 dq_1 dq_2 dq_3. \tag{65}$$

E.g. In cylindrical polars, the volume element is $r dr d\theta dz$.

In the curvilinear coordinate system, $\boldsymbol{\nabla}$ has the representation (see MVC for derivation)

$$\boldsymbol{\nabla} = \frac{\hat{\mathbf{q}}_1}{h_1}\frac{\partial}{\partial q_1} + \frac{\hat{\mathbf{q}}_2}{h_2}\frac{\partial}{\partial q_2} + \frac{\hat{\mathbf{q}}_3}{h_3}\frac{\partial}{\partial q_3}. \tag{66}$$

This is sufficient to derive the vector identities which follow (again, see MVC notes for details).

A.11 Cylindrical polar coordinates

Coordinate system is $\mathbf{r} = (r, \theta, z)$ where the relationship to Cartesians is $x = r\cos\theta$, $y = r\sin\theta$. The unit vectors are $\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\theta + \hat{\mathbf{y}}\sin\theta$, $\hat{\boldsymbol{\theta}} = -\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{y}}\cos\theta$ and $\hat{\mathbf{z}}$. In the following, $\mathbf{f} = (f_r, f_\theta, f_z) \equiv f_r\hat{\mathbf{r}} + f_\theta\hat{\boldsymbol{\theta}} + f_z\hat{\mathbf{z}}$.

- The gradient is $\boldsymbol{\nabla}\phi = \hat{\mathbf{r}}\frac{\partial\phi}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial\phi}{\partial\theta} + \hat{\mathbf{z}}\frac{\partial\phi}{\partial z}$
- The divergence is $\boldsymbol{\nabla}\cdot\mathbf{f} = \frac{1}{r}\frac{\partial(rf_r)}{\partial r} + \frac{1}{r}\frac{\partial f_\theta}{\partial\theta} + \frac{\partial f_z}{\partial z}$
- The curl is $\boldsymbol{\nabla}\times\mathbf{f} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & \hat{\mathbf{z}} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial z \\ f_r & rf_\theta & f_z \end{vmatrix}.$
- The Laplacian is $\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}$
- $(\mathbf{f}\cdot\boldsymbol{\nabla})\mathbf{f} = \left(f_r\frac{\partial f_r}{\partial r} + \frac{f_\theta}{r}\frac{\partial f_r}{\partial\theta} + f_z\frac{\partial f_r}{\partial z} - \frac{f_\theta^2}{r}\right)\hat{\mathbf{r}} + \left(f_r\frac{\partial f_\theta}{\partial r} + \frac{f_\theta}{r}\frac{\partial f_\theta}{\partial\theta} + f_z\frac{\partial f_\theta}{\partial z} + \frac{f_r f_\theta}{r}\right)\hat{\boldsymbol{\theta}} + \left(f_r\frac{\partial f_z}{\partial r} + \frac{f_\theta}{r}\frac{\partial f_z}{\partial\theta} + f_z\frac{\partial f_z}{\partial z}\right)\hat{\mathbf{z}}$

A.12 Spherical polar coordinates

Coordinate system is $\mathbf{r} = (r, \varphi, \theta)$ where the relationship to Cartesians is $x = r\sin\varphi\cos\theta$, $y = r\sin\varphi\sin\theta$, $z = r\cos\varphi$.[26]

The unit vectors are $\hat{\mathbf{r}} = \hat{\mathbf{x}}\sin\varphi\cos\theta + \hat{\mathbf{y}}\sin\varphi\sin\theta + \hat{\mathbf{z}}\cos\varphi$, $\hat{\boldsymbol{\varphi}} = \hat{\mathbf{x}}\cos\varphi\cos\theta + \hat{\mathbf{y}}\cos\varphi\sin\theta - \hat{\mathbf{z}}\sin\varphi$ and $\hat{\boldsymbol{\theta}} = -\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{y}}\cos\theta$.

In the following, $\mathbf{f} = f_r\hat{\mathbf{r}} + f_\varphi\hat{\boldsymbol{\varphi}} + f_\theta\hat{\boldsymbol{\theta}}$.

- The gradient is $\boldsymbol{\nabla}\phi = \hat{\mathbf{r}}\frac{\partial\phi}{\partial r} + \hat{\boldsymbol{\varphi}}\frac{1}{r}\frac{\partial\phi}{\partial\varphi} + \hat{\boldsymbol{\theta}}\frac{1}{r\sin\varphi}\frac{\partial\phi}{\partial\theta}$
- The divergence is $\boldsymbol{\nabla}\cdot\mathbf{f} = \frac{1}{r^2}\frac{\partial(r^2 f_r)}{\partial r} + \frac{1}{r\sin\varphi}\frac{\partial(\sin\varphi f_\varphi)}{\partial\varphi} + \frac{1}{r\sin\varphi}\frac{\partial f_\theta}{\partial\theta}$
- The curl is $\boldsymbol{\nabla}\times\mathbf{f} = \frac{1}{r^2\sin\varphi}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\varphi}} & r\sin\varphi\hat{\boldsymbol{\theta}} \\ \partial/\partial r & \partial/\partial\varphi & \partial/\partial\theta \\ f_r & rf_\varphi & r\sin\varphi f_\theta \end{vmatrix}.$
- The Laplacian is $\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial\phi}{\partial\varphi}\right) + \frac{1}{r^2\sin^2\varphi}\frac{\partial^2\phi}{\partial\theta^2}$
- Gulp ! $(\mathbf{f}\cdot\boldsymbol{\nabla})\mathbf{f} = \left(f_r\frac{\partial f_r}{\partial r} + \frac{f_\varphi}{r}\frac{\partial f_r}{\partial\varphi} + \frac{f_\theta}{r\sin\varphi}\frac{\partial f_r}{\partial\theta} - \frac{f_\varphi^2 + f_\theta^2}{r}\right)\hat{\mathbf{r}} + \left(f_r\frac{\partial f_\varphi}{\partial r} + \frac{f_\varphi}{r}\frac{\partial f_\theta}{\partial\varphi} + \frac{f_\theta}{r\sin\varphi}\frac{\partial f_\varphi}{\partial\theta} + \frac{f_r f_\varphi}{r} - \frac{f_\theta^2\cot\varphi}{r}\right)\hat{\boldsymbol{\varphi}} + \left(f_r\frac{\partial f_\theta}{\partial r} + \frac{f_\varphi}{r}\frac{\partial f_\theta}{\partial\varphi} + \frac{f_\theta}{r\sin\varphi}\frac{\partial f_\theta}{\partial\theta} + \frac{f_r f_\theta}{r} + \frac{f_\varphi f_\theta\cot\varphi}{r}\right)\hat{\boldsymbol{\theta}}$

[26] This definition and ordering of variables is often called the "Mathematician's version" and is aligned with the MVC course, but beware that it is common to see the "Physicist's version" in which φ is used to measure the azimuthal angle and θ the polar inclination

A.13 Line and surface integrals

When the curve C is parametrised by $\mathbf{r} = \mathbf{p}(t)$ for $t_1 < t < t_2$.

$$\int_C \mathbf{u}(\mathbf{r}) \cdot d\mathbf{r} = \int_{t_1}^{t_2} \mathbf{u}(\mathbf{p}(t)) \cdot \mathbf{p}'(t) dt$$

E.g.: in 2D plane polars, $\mathbf{p}(\theta) = r(\cos\theta, \sin\theta) = r\hat{\mathbf{r}}$ parametrised a circle and $\mathbf{p}'(\theta) = r\hat{\boldsymbol{\theta}} = r(-\sin\theta, \cos\theta)$ so $d\mathbf{r} = \hat{\boldsymbol{\theta}} r d\theta$.

Next, $ds = |d\mathbf{r}|$ in

$$\int_C f(\mathbf{r}) ds = \int_{t_1}^{t_2} f(\mathbf{p}(t)) |\mathbf{p}'(t)| dt$$

E.g.: for a circle, $ds = |d\mathbf{r}| = r d\theta$

For 2D integrals given a mapping $\mathbf{s}(u, v)$ from $(u, v) \in D$ to S

$$\int_S \mathbf{u}(\mathbf{r}) \cdot \hat{\mathbf{n}} dS = \iint_D \mathbf{u}(\mathbf{s}(u, v)) \cdot \mathbf{N} du dv$$

($\hat{\mathbf{n}} dS \equiv d\mathbf{S}$) where $\mathbf{N} = \mathbf{s}_u \times \mathbf{s}_v$ (check directions of normal are correct).

E.g.: For a sphere of radius r, $\mathbf{s}(\varphi, \theta) = r(\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta)$ and $\mathbf{N} = r^2 \sin\varphi \hat{\mathbf{r}}$ so $\hat{\mathbf{n}} dS = \hat{\mathbf{r}} r^2 \sin\varphi d\varphi d\theta$

Next, $dS = |d\mathbf{S}|$

$$\int_S f(\mathbf{r}) dS = \iint_D f(\mathbf{s}(u, v)) |\mathbf{N}| du dv$$

E.g.: For a sphere of radius r, $dS = r^2 \sin\varphi d\varphi d\theta$

A.14 Cauchy-Riemann equations

A complex function $f(z) = u(x, y) + iv(x, y)$ where $z = x + iy$ and u and v are real function is differentiable if the Cauchy-Riemann equations

$$u_x(x, y) = v_y(x, y), \qquad u_y(x, y) = -v_x(x, y),$$

hold. A consequence of this is that the functions u are v are said to be *harmonic*, that is to say they satisfy $\nabla^2 u = \nabla^2 v = 0$.

A function which is differentiable at every point in an open set V is said to be holomorphic (or analytic). It follows that a holomorphic function satisfies the Cauchy-Riemann equations on V.

A.15 Cauchy's Integral Theorems

A function f which is holomorphic on V and the path C lies within V then

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - a} dz$$

where a lies inside C and the integral around C is counter-clockwise. If a lies outside C then the integral evaluates to zero.

It follows further that

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz.$$

Cauchy's residue theorem states that

$$\boxed{\oint_C f(z) dz = 2\pi i \sum \text{Res}(f, a_k)}$$

where $f(z)$ is holomorphic on $V \backslash \{a_1, \ldots, a_n\}$ where a_i are isolated singularites – that is to say $f(z)$ is meromorphic on V.

In the above, if a is a pole of order n then

$$\text{Res}(f; a) = \lim_{z \to a} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} (z-a)^n f(z)$$

If the pole is order 1 then

$$\text{Res}(f; a) = \lim_{z \to a} (z-a) f(z)$$

and if $f(z) = g(z)/h(z)$ where $h(z)$ has a zero of order 1 then

$$\text{Res}(f; a) = g(a)/h'(a)$$

A.16 Separation of variables

A.16.1 Cartesians

In Cartesian coordinates, separation solutions of Laplace's equation are of the form

$$\cosh \nu x \cos \nu y, \quad \cosh \nu x \sin \nu y, \quad \sinh \nu x \cos \nu y, \quad \sinh \nu x \sin \nu y$$

for any complex ν. E.g. if ν is replaced by $i\nu$ with ν real, the positions of x and y in the above are interchanged. The values of ν are normally determined by imposing lateral boundary conditions.

Separation solutions of the wave equation $(\nabla^2 + k^2)\phi = 0$ are of the form

$$\cos \nu x \cos \mu y$$

(with sines interchangeable with cosines as above) where $\nu^2 + \mu^2 = k^2$ and ν and μ determined by boundary conditions.

A.16.2 Plane polars

Separation solutions of Laplace's equation in 2D plane polar coordinates are

$$r^\nu \cos \nu\theta, \quad r^\nu \sin \nu\theta$$

for all real ν (positive and negative). For solutions which are continuous for all θ, ν must be an integer.

Separation solutions of the 2D wave equation $(\nabla^2 + k^2)\phi = 0$ in plane polars are (assuming continuous solutions for all θ) are

$$J_n(kr)\cos n\theta, \quad J_n(kr)\sin n\theta, \quad Y_n(kr)\cos n\theta, \quad Y_n(kr)\sin n\theta,$$

where J_n and Y_n are Bessel functions. They both oscillate and decay at infinity but $Y_n(kr)$ are singular at the origin and $J_n(kr)$ are bounded at the origin.

A.16.3 3D cylindrical polars

By separating $\mathrm{e}^{\pm\mu z}$ Laplace's equation in 3D is transformed in to the 2D wave equation in plane polars, so solutions are

$$\mathrm{e}^{\pm\mu z} J_n(\mu r)\cos n\theta$$

as the first of the four combinations indicated in the previous section. If μ is replaced by $i\mu$ to turn exponentials in z into oscillatory terms, the Bessel functions J_n and Y_n are replaced by *modified Bessel functions* I_n and K_n.

For the wave equation in 3D cylindrical polars we separate in the same way but the argument in the exponential becomes $\sqrt{\mu^2 + k^2}$.

A.16.4 3D spherical polars

In 3D spherical polars, solutions of Laplace's equation which are axisymmetric about the polar axis (independent of θ) are

$$r^n P_n(\cos\varphi), \quad \text{and} \quad P_n(\cos\varphi)/r^{n+1}$$

for $n = 0, 1, 2, \ldots$ where P_n are Legendre polynomials ($P_0(x) = 1$, $P_1(x) = x$ and $P_2(x) = \frac{1}{2}(3x^2 - 1)$, and so on.)

The first set are bounded at the origin, unbounded at infinity and for the second these are reversed. Singular solutions at the origin are therefore:

$$\frac{1}{r}, \quad \frac{\cos\varphi}{r^2}, \quad \frac{3\cos^2\varphi - 1}{2r^3}, \ldots$$

and those bounded at the origin are

$$1, \quad r\cos\varphi, \quad \tfrac{1}{2}r^2(3\cos^2\varphi - 1), \ldots$$