Total transmission through narrow gaps in channels

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Highlights:

• A small gap approximation is used to show total transmission through multiple narrow gaps in barriers across a channel.

• Coupled integral equations are solved exactly on the basis of the small gap approximation.

1. Introduction

The phenomenon of 'extraordinary transmission' was first discovered in the field of optics. A recent review is given by Garcia de Abajo (2007).

Total transmission of a plane wave at a certain frequency is very familiar in the theory of linear water waves involving transmission over a long obstacle or past a pair of obstacles. See for example Newman (1965) and Porter & Evans (1995).

The solution to the problem of the transmission of plane waves through a gap in a barrier spanning a narrow channel is well-known. See for example Jones (1986) for an electromagnetic context. In this paper we show that 100% transmission can occur at a given frequency for an infinite sequence of spacings, when one or more extra identical barriers with small gaps are introduced. Specifically we consider the transmission of long waves down a channel of width $2d$ containing N equally-spaced thin rigid barriers spanning the channel each containing a central narrow gap of width $2a$. It is shown by comparison with a full numerical treatment that a small-gap approximation to the single barrier is remarkably accurate for a range of gap sizes. The technique is extended to obtain closed-form expressions for the reflection and transmission coefficients through the periodic array which are shown, as the spacing between the barriers increases, to reduce to the simple expressions based on a wide-spacing aproximation (WSA) given, for example, by Martin (2014). A key feature of the problem is the derivation of the condition under which we are in a pass or stop-band for the periodic structure. Also obtained is the condition for total reflection which disappears as the barrier spacing increases which is consistent with its non-appearance under the WSA.

2. The N-barrier problem

Since the channel walls and the barriers extend throughout the depth it is possible to factor out the depth dependence and we also assume a time harmonic dependence $e^{-i\omega t}$. Thus we seek a two-dimensional potential $\phi(x,y)$ satisfying

$$(\nabla^2 + k^2)\phi = 0 \tag{1}$$

in the fluid, where the wavenumber k is the real positive root of

$$\omega^2 = gk\tanh kh \tag{2}$$

where h is the depth of the channel. The barriers occupy $x = b_n = nb$, $n = 0,1,2,\ldots,N-1$, and the gaps occupy $L = \{|y| < a\}$. The no-flow condition on the walls demands

$$\phi_y(x,\pm d) = 0, \quad -\infty < x < \infty \tag{3}$$

whilst on the nth barrier ($n = 0,1,2,\ldots,N-1$)

$$\phi_x(b_n^{\pm}, y) = 0, \quad a < |y| < d. \tag{4}$$

For $x < 0$, general solutions of (1) satisfying (3) are

$$\phi(x,y) = e^{ikx} + R_N e^{-ikx} + \sum_{r=1}^{\infty} \frac{F_r^{(0)} e^{\alpha_r x}\cos p_r y}{\alpha_r} \tag{5}$$

where $p_r = r\pi/d$, $\alpha_r = (p_r^2 - k^2)^{1/2}$, $\alpha_0 = -ik$, $k < \pi/d$. For $x > (N-1)b$, we write

$$\phi(x,y) = T_N e^{ik(x-b_{N-1})} - \sum_{r=1}^{\infty} \frac{F_r^{(N-1)} e^{-\alpha_r(x-b_{N-1})}\cos p_r y}{\alpha_r}. \tag{6}$$

Finally, for $(n-1)b < x < nb$, $n = 1,2,\ldots,N-1$, we write

$$\phi(x,y) = \sum_{r=0}^{\infty} \frac{(F_r^{(n)}\cosh\alpha_r(x-b_{n-1}) - \ldots}{\alpha_r\sinh\alpha_r b} \\ \frac{\ldots F_r^{(n-1)}\cosh\alpha_r(b_n - x))\cos p_r y}{}. \tag{7}$$

In the above R_N and T_N are the reflection and transmission coefficients for N barriers and $F_r^{(n)}$ are undetermined coefficients. These definitions ensure that $\phi_x(b_n^+, y) = \phi_x(b_n^-, y)$, $|y| < d$, $n = 0,1,2,\ldots,N-1$ and we write

$$\phi_x(b_n, y) \equiv F^{(n)}(y) = \sum_{r=0}^{\infty} F_r^{(n)}\cos p_r y \tag{8}$$

whence (4) is used to give

$$F_r^{(n)} = \frac{1}{\epsilon_r}\int_L F^{(n)}(t)\cos p_r t\,dt \tag{9}$$

with $\epsilon_0 = 2d$, $\epsilon_r = d$, $r > 0$. In particular,

$$\mathrm{i}k(1 - R_N) = F_0^{(0)}, \quad \mathrm{i}kT_N = F_0^{(N-1)}. \tag{10}$$

Continuity of pressure across the gap in the nth barrier demands $\phi(b_n^+, y) = \phi(b_n^-, y)$, $y \in L$ and gives, after using (5), (6), (7) and (9), $n = 1, 2, \ldots, N-2$,

$$\int_L \big(F^{(n+1)}(t)K_1(y,t) - 2F^{(n)}(t)K_2(y,t) + F^{(n-1)}(t)K_1(y,t)\big)\,dt = 0, \tag{11}$$

and

$$\int_L \big(F^{(1)}(t)K_1(y,t) - F^{(0)}(t)K_3(y,t)\big)\,dt = 2, \tag{12}$$

$$\int_L \big(F^{(N-1)}(t)K_3(y,t) - F^{(N-2)}(t)K_1(y,t)\big)\,dt = 0 \tag{13}$$

all for $y \in L$, in which

$$K_1(y,t) = \sum_{r=0}^{\infty}\frac{\cos p_r y \cos p_r t}{\epsilon_r \alpha_r \sinh \alpha_r b}, \tag{14}$$

$$K_2(y,t) = \sum_{r=0}^{\infty}\frac{\coth\alpha_r b\,\cos p_r y\cos p_r t}{\epsilon_r\alpha_r}, \tag{15}$$

and

$$K_3(y,t) = \sum_{r=0}^{\infty}\frac{(1+\coth\alpha_r b)\cos p_r y\cos p_r t}{\epsilon_r\alpha_r}. \tag{16}$$

3. The single barrier approximation

This generic problem forms the basis of the approach to the N-barrier problem. It is straightforward to show that $R_1 + T_1 = 1$ and that the horizontal velocity across L, $F^{(0)}(y)$ satisfies

$$\int_L F^{(0)}(t)K(y,t)\,dt = -R_1, \quad y \in L, \tag{17}$$

with

$$\int_L F^{(0)}(t)\,dt = 2\mathrm{i}kd\,T_1 \tag{18}$$

where

$$K(y,t) = \sum_{r=1}^{\infty}\frac{\cos p_r y\cos p_r t}{\alpha_r d}. \tag{19}$$

Note that the term $r = 0$ does not appear here. At this point we exploit the assumption that $a/d \ll 1$ so that the kernel $K(y,t)$ is dominated by a logarithmic term. Thus, we can write (e.g. Jones (1986) equn. (16.1))

$$K(y,t) = -\frac{1}{2\pi}\ln 2|\cos(\pi y/d) - \cos(\pi t/d)| + \sum_{r=1}^{\infty}\Big(\frac{1}{\alpha_r d} - \frac{1}{r\pi}\Big)\cos p_r y\cos p_r t \tag{20}$$

so that for $y, t \to 0$

$$K(y,t) \sim -\frac{1}{2\pi}\ln|y^2 - t^2| + \frac{1}{\pi}(S - \ln(\pi/d)) \tag{21}$$

where

$$S = \pi\sum_{r=1}^{\infty}\Big(\frac{1}{\alpha_r d} - \frac{1}{r\pi}\Big). \tag{22}$$

Substituting (21) into (17) and using the fact that $F^{(0)}(y)$ is even in y, gives

$$\int_L F^{(0)}(t)\ln|y - t|\,dt = A, \quad y \in L, \tag{23}$$

where $A = \pi R_1 + (S - \ln(\pi/d))\int_L F^{(0)}(t)\,dt$. The singular integral equation (23), where $F^{(0)}(y)(a^2 - y^2)^{1/2}$ is bounded, has an explicit solution, but all we require is the result

$$\int_L F^{(0)}(t)\,dt = A/\ln(a/2). \tag{24}$$

For a proof see Cooke (1970), and, for applications in water waves, Evans (1975), and Packham & Williams (1972) who also describe a three-dimensional version. It follows from (18), (23) and (24) that

$$\left.\begin{aligned} T_1 &= \cos\delta\,\mathrm{e}^{\mathrm{i}\delta} \\ R_1 &= -\mathrm{i}\sin\delta\,\mathrm{e}^{\mathrm{i}\delta}\end{aligned}\right\} \quad \tan\delta = 2\kappa(S(\kappa) - \ln\nu) > 0 \tag{25}$$

where

$$\kappa = kd/\pi < 1, \quad \nu = \pi a/2d. \tag{26}$$

The phase of T_1 will play a key role in the general theory and we make use of the result which follows from the above that if

$$\int_L F^{(n)}(t)K(y,t)\,dt = C, \quad y \in L, \tag{27}$$

then for $y, t \to 0$

$$\frac{1}{2d}\int_L F^{(n)}(y)dy = F_0^{(n)} = kC\cot\delta. \tag{28}$$

4. Solution for N barriers

Returning to the general case we have, from (14) to (16), as $y, t \to 0$,

$$K_1(y,t) \sim \frac{(-\operatorname{cosec} kb + E_1)}{2kd}, \tag{29}$$

$$K_2(y,t) - K(y,t) \sim \frac{(-\cot kb + E_2)}{2kd}, \tag{30}$$

and

$$K_3(y,t) - 2K(y,t) \sim \frac{(-\cot kb + E_2 + \mathrm{i})}{2kd}, \tag{31}$$

where

$$E_1 = \sum_{r=1}^{\infty}\frac{2kd}{\alpha_r d\sinh\alpha_r b}, \quad E_2 = \sum_{r=1}^{\infty}\frac{2kd\mathrm{e}^{-\alpha_r b}}{\alpha_r d\sinh\alpha_r b} \tag{32}$$

where the $E_i \to 0$ as $b/d \to \infty$, $i = 1, 2$. Substituting in (11)–(13), using (28) and defining $F_0^{(n)} = k\mu_n$ gives

$$\mu_{n+1} - 2\cos\alpha\mu_n + \mu_{n-1} = 0, \quad n = 1, 2, \ldots, N-2 \tag{33}$$

and

$$(p + \mathrm{i})\mu_0 + 2 = q\mu_1, \tag{34}$$

$$(p + \mathrm{i})\mu_{N-1} = q\mu_{N-2}. \tag{35}$$

In the above

$$p = 2\tan\delta - (\cot\kappa\lambda - E_2), \quad q = (-\mathrm{cosec}\,\kappa\lambda + E_1) \tag{36}$$

with $\kappa = kd/\pi$, $\lambda = \pi b/d$ and

$$\cos\alpha = \frac{(\cos\alpha_0 - E_2\sin kb)}{(1 - E_1\sin kb)} \tag{37}$$

where

$$\cos\alpha_0 = \frac{\cos(\delta + kb)}{\cos\delta} \equiv \frac{\cos(\delta + kb)}{|T_1|}. \tag{38}$$

Clearly $\alpha \to \alpha_0$ as $E_i \to 0$ which corresponds to $b/d \to 0$ or a wide-spacing approximation (WSA). For parameter values such that $|\cos\alpha| < 1$ it turns out we are in a pass-band which ensures wave transmission through the periodic array. For other values we are in a stop-band and transmission is not possible. See for example Linton & McIver (2001) equn. (6.52).
Finally,

$$R_N = 1 + \mathrm{i}\mu_0, \quad T_N = -\mathrm{i}\mu_{N-1}. \tag{39}$$

The solution of (33) satisfying (35) is, for $n = 0, 1, 2, \ldots, N-1$,

$$\mu_n = \mu_{N-1}\big((p + \mathrm{i})U_{N-n-2} - qU_{N-n-3}\big)/q \tag{40}$$

where $U_n(\alpha) = \sin(n+1)\alpha/\sin\alpha$, so that $U_{-2} = -1$, $U_{-1} = 0$, $U_0 = 1$. Thus from (39)

$$R_N = 1 + \mathrm{i}\mu_0 = \frac{q\mu_1 - (p - \mathrm{i})\mu_0}{q\mu_1 - (p + \mathrm{i})\mu_0} \tag{41}$$

after using (35). Now (40) can be used to show

$$R_N = \frac{(p^2 + 1)U_{N-2} - 2pqU_{N-3} + q^2U_{N-4}}{(p + \mathrm{i})^2U_{N-2} - 2q(p + \mathrm{i})U_{N-3} + q^2U_{N-4}}. \tag{42}$$

Similarly

$$T_N = \frac{2\mathrm{i}q}{(p + \mathrm{i})^2U_{N-2} - 2q(p + \mathrm{i})U_{N-3} + q^2U_{N-4}}. \tag{43}$$

The above expressions turn out to hold for $N = 2$ also where only (32) and (33) are required. The energy condition $|R_N|^2 + |T_N|^2 = 1$ can be shown to be satisfied exactly after some algebra.
It is clear from (43) that $T_N = 0$ if $q = 0$ for all $N > 1$ which is obvious on physical grounds. Less obvious is the fact that the condition is independent of the smallness of the gap. Now from (36) and (32), $q = 0$ implies

$$\sin\kappa\lambda = \left[\sum_{r=1}^{\infty} \frac{2\kappa}{(r^2 - \kappa^2)^{1/2}\sinh\lambda(r^2 - \kappa^2)^{1/2}}\right]^{-1}. \tag{44}$$

It is clear that provided the right-hand-side of (44) is less than unity, solutions of the form $\lambda(\kappa)$ exist, but that there will be a cut-off at, say, $\lambda = \lambda_c(\kappa)$ above which no solution is possible. This is consistent with the WSA valid for large λ which predicts no solution.
The vanishing of $|T_N|$ for any $N > 1$ is an unusual phenomenon, rare in water wave problems which was first shown by Evans & Morris (1972) in considering the scattering of waves by a pair of partially-immersed vertical barriers. For detailed and accurate computations see Porter & Evans (1995). However, it is the phenomenon of total transmission which is the main interest in this paper, particularly in the light of the assumption of small gaps. Thus the numerator in (42) is real so it can be expected that $R_N = 0$ for certain values of p, q, and α. As a special case we consider wide barrier spacing when the E_i in (32) tend to zero. After considerable algebra it can be shown that (42) and (43) become

$$R_N = \frac{U_{N-1}R_1}{U_{N-1} - T_1\mathrm{e}^{\mathrm{i}kb}U_{N-2}} \tag{45}$$

$$T_N = \frac{T_1}{U_{N-1} - T_1\mathrm{e}^{\mathrm{i}kb}U_{N-2}} \tag{46}$$

in agreement with Martin (2014) equn. (21). The condition $R_N = 0$ for total transmission is now simply $U_{N-1}(\alpha_0) = 0$ or $\alpha_0 = m\pi/N$, $m = 1, 2, \ldots, N-1$ so that from (38)

$$\cos\frac{m\pi}{N} = \frac{\cos(\delta + kb)}{\cos\delta}, \quad m = 1, 2, \ldots, N-1. \tag{47}$$

Thus for example, for $N = 2$ we have $\cos(\delta + kb) = 0$ or $\delta + kb = (2p-1)\pi/2$, p an integer.

5. Results

In Fig.1 the solid lines describe the variation of $|R_1|$ with kd for gap sizes $a/d = 0.1, 0.2, 0.3, 0.4$ using an accurate numerical method described in Porter & Evans (1995). The crosses are computed using the small-gap result given in (25). The agreement for $a/d = 0.1$ is excellent over the whole range of $kd < \pi$ and this value together with $kd = 1$ will be used in the further computations for multiple barriers.
The result (47) shows that a WSA approximation provides $N - 1$ equations to determine when $R_N = 0$ and $|T_N| = 1$ for each region in which $\cos\alpha_0 < 1$ and we are in a pass-band, and we might expect that to be the case generally when the numerator of (42) vanishes. This is confirmed in Fig. 2 where $N = 4$ and $a/d = 0.1$ throughout. The solid lines show, in $(kd, b/d)-$space,

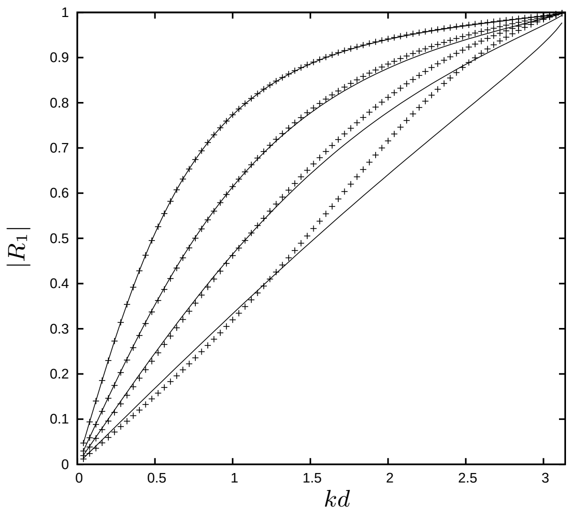

Figure 1: Comparison of small gap approximation against exact results for a single barrier.

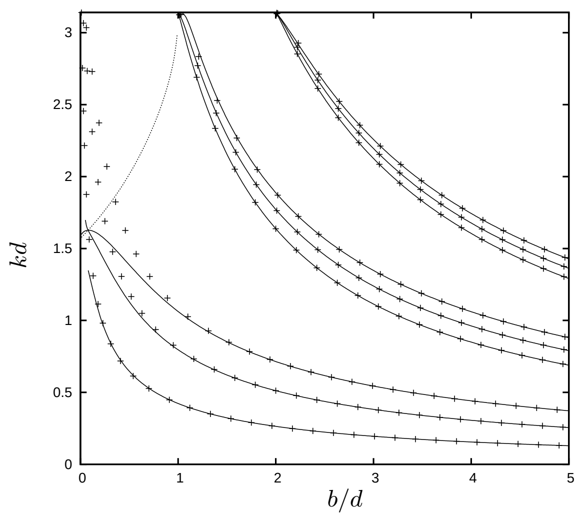

Figure 2: Solid curves/points show $R_4 = 0$ for $N = 4$ barriers with $a/d = 0.1$ using exact/WSA theory. The dotted line is $T_4 = 0$.

where $R_4 = 0$ and the crosses are based on the WSA. Thus for example for $kd = 1$ as b/d increases there are three different spacings at which $R_4 = 0$ followed by a gap before a further three cut in and so on. Alternatively, Fig. 2 shows that at a given spacing there are three distinct wavenumbers for which total transmission occurs with further groups of three at higher frequencies occurring for larger spacings. It is also clear that for most purposes the WSA is entirely adequate in predicting the results. The solution for $b/d \lesssim a/d$ is less clear as it conflicts with the small-gap approximation. Also shown in Fig. 2 is a dotted line on which $T_4 = 0$ derived from (44) and which has no counterpart in a WSA. It is possible to consider a semi-infinite array of barriers by ignoring condition (35) and assuming $\mu_n = A\mathrm{e}^{\pm in\alpha}$ as a solution of (33) whence (34) gives

$$R_\infty = \frac{q\mathrm{e}^{\pm \mathrm{i}\alpha} - (p - \mathrm{i})}{q\mathrm{e}^{\pm \mathrm{i}\alpha} - (p + \mathrm{i})} \tag{48}$$

the sign chosen so that $|R_\infty|$ does not exceed unity. The WSA counterpart of (48) requires replacing α by α_0. Fig. 3 shows a plot of $|R_4|$ and $|R_\infty|$ against b/d based on the exact small-gap theory with the WSA results overlaid. It shows clearly the triplets of zeros of R_4 separated by stop bands.

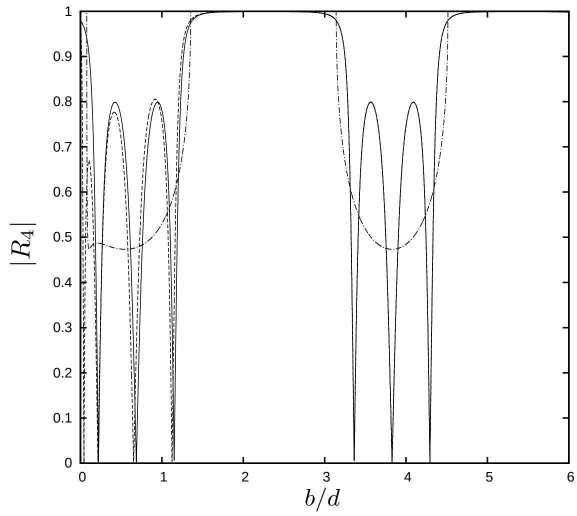

Figure 3: Solid/dashed lines are exact/WSA results for $N = 4$ barriers with $a/d = 0.1$ and $kd = 1$. The chained line is $|R_\infty|$.

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