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On Stokes' Coefficients and the Wave Resistance of a Towed Body

by Dmitri V. Maklakov[1] and Alexander G. Petrov[2]
[1]Kazan (Volga region) Federal University, Kazan, Russia
[2]Moscow Institute of Physics and Technology (State University),
Institute for Problems in Mechanics of RAS, Moscow, Russia
dmaklak@kpfu.ru, petrovipmech@gmail.com

Highlights:

In this work we have deduced an infinite system of quadratic equations with respect to the Stokes coefficients which define the periodic progressive waves in water of finite depth. The system has a compact form and has been derived by means of a new variational equation for steady periodic flows above a level bottom. By solving the system we have constructed an analytical ten-termed expansion in the amplitude for the wave resistance of a two-dimensional body that creates the waves. The coefficients of the expansion depend only on the mean depth of the waves. The obtained expansion has been compared with Kelvin's one-termed formula and with an accurate numerical solution.

1 Introduction

In the scientific literature there is a considerable number of works devoted to finding the Stokes coefficient which define the periodic progressive waves in water of finite depth. The first computer algorithm has been developed by Schwartz (1974). As in the initial work by Stokes (1880), Schwartz has used the boundary condition of constant pressure at the free surface and obtained a cubic system of equations with respect to the Stokes coefficients. Longuet-Higgins (1978) has demonstrated that after some transformations the cubic system can be reduced to a quadratic one. In this work we have derives a simple variational equation for steady periodic flows above a level bottom which is especially convenient for studying steady periodic gravity waves in water of finite depth. The equation leads to a very compact system of quadratic equations with respect to the Stokes coefficients. We have developed an effective algorithm of computing the coefficients in the form of a series in powers of the wave amplitude. The exact formula for the wave resistance R_w derived in Maklakov & Petrov (2014) allows us to deduce new analytical formulae for R_w.

2 Variational equations for steady periodic flows and gravity waves

Consider a steady flow of an ideal fluid bounded by a λ-periodic line l_z from above and by a horizontal bottom $y = 0$ from below. Under the λ-periodicity we understand the property $z = x + iy \in l_z \Rightarrow z + \lambda \in l_z$. The complex potential of the flow $w = \phi + i\psi$ satisfies the following boundary conditions

$$\begin{aligned} \operatorname{Im} w = \psi = Q \quad &\text{for} \quad z \in l_z, \\ \operatorname{Im} w = \psi = 0 \quad &\text{for} \quad y = 0, \end{aligned} \tag{1}$$

where Q is the volume flux. In the flow domain the complex conjugate velocity dw/dz is a λ-periodic function, but the complex potential $w(z)$ increases by an increment C (by a circulation) on every period:

$$w(z + \lambda) - w(z) = C.$$

For a fixed upper boundary l_z the flow is defined uniquely by specifying either the volume flux Q, in this case the increment of the potential C should be found in solving, or by specifying C to determine Q.

In the flow domain consider one period, shown in Fig. 1(a). Let s be the arc abscissa of the line l_z, reckoned from a certain point L.

Proposition. *Consider a steady λ-periodic flow bounded from above by the line l_z with the velocity distribution $v(s)$ on l_z. Assume that the flow flux is Q and the circulation is C. Let the boundary l_z be varied by shifting each of its points on the distance $\delta n(s)$ in the direction of the normal in so manner that the new line l_z^* be also λ-periodic. Let for the new λ-periodic flow, bounded from above by the line l_z^*, the flux and circulation be $Q + \delta Q$ and $C + \delta C$, respectively (fig. 1a). Then the following variational equation holds*

$$\int_{LS} v^2(s)\delta n(s) ds = C\delta Q - Q\delta C, \tag{2}$$

where the curve LS is any period of the line l_z.

The prove of the proposition is based on the Cauchy theorem for analytic functions.

Consider a system of periodic gravity waves in the wave-fixed reference frame in which the flow is steady. Then the parameter of periodicity λ is the wavelength. Let the density of the fluid be ρ. As in Longuet-Higgins (1975) we non-dimensionalize all wave parameters by choosing $\lambda/(2\pi)$, $\sqrt{g\lambda/(2\pi)}$ and ρ as the scales for length, velocity and density, respectively. In what follows all physical quantities will be dimensionless with accordance to the chosen scales. Now the dimensionless wavelength is 2π, in the boundary conditions (1) and in the variational equation (2) the parameters Q and C are scaled by $\sqrt{g\lambda^3/(2\pi)^3}$.

For the periodic wave problem we should add to the boundary conditions (1) the Bernoulli equation on the unknown free surface l_z: $\frac{v^2}{2} + y = R$, where R is the dimensionless Bernoulli constant (total head).

It is easy to demonstrate that for any λ–periodic line and any differentiable function $P(y)$ the following variational equation holds

$$\delta \int_{LS} P(y)dx = \int_{LS} P'(y)\delta n(s)ds, \qquad (3)$$

where LS is one period of the line.

Consider now one wave period located between neighboring wave crests (Fig. 1 a). On the free surface the boundary condition $v^2 = 2R - 2y$ is satisfied. Let us vary the free-surface by a function $\delta n(s)$. Then by virtue of (3)

$$\begin{aligned}\int_{LS} v^2(s)\delta n(s)ds &= \int_{LS} (2R-2y)\delta n(s)ds \\ &= \delta \int_{LS} (2Ry - y^2)dx = 2\pi\delta M[\eta],\end{aligned} \qquad (4)$$

where the functional

$$M[\eta] = 2R\, d - D^2$$

depends only on the shape of the free surface with the equation $y = \eta(x)$, d and D are the mean and root-mean-square depths, respectively:

$$d = \frac{1}{2\pi}\int_x^{x+2\pi} \eta(x)dx, \quad D^2 = \frac{1}{2\pi}\int_x^{x+2\pi} \eta^2(x)dx.$$

It follows from (2) that

$$2\pi\, \delta M[\eta] = C\delta Q - Q\delta C = C^2 \delta\left(\frac{Q}{C}\right). \qquad (5)$$

For the wave period the domain of the complex potential $w = \varphi + i\psi$ is a rectangle, shown in Fig. 1 b. We map conformally this rectangle onto an annulus (see Fig. 1 c) with an outer radius of unity and inner radius of

$$r_0 = \exp\left(-\frac{2\pi Q}{C}\right). \qquad (6)$$

A usual assumption (see e.g. Longuet-Higgins, 1975; Cokelet, 1977) in the theory of nonlinear periodic waves is that in the bottom-fixed reference frame the waves propagate with the velocity c_a, equal to the average fluid velocity at any horizontal level completely within the fluid in the wave-fixed reference frame (in steady flow). That is $c_a = \frac{C}{2\pi}$. Now the equation (5) can be rewritten as

$$\delta M[\eta] = -c_a^2 \frac{\delta r_0}{r_0}, \qquad (7)$$

which is just the variational equation for steady periodic waves.

3 System of quadratic equations for the Stokes coefficients

We shall seek the conformal mapping of the annulus (Fig. 1 c) in the parametric ζ-plain onto the flow domain of the one wave period in the form

$$z(\zeta) = 2\pi + i\log\zeta + iy_0 + i\sum_{n=1}^{\infty} y_n\left(\zeta^n - \frac{r_0^{2n}}{\zeta^n}\right). \qquad (8)$$

The representation (8), being a variant of the Stokes method (see Stokes, 1880), is often used in the nonlinear wave theory (see e.g. Schwartz, 1974). By virtue of symmetry the Stokes coefficients y_n ($n = 0, 1, 2\ldots$) are real. Because on the bottom $\operatorname{Im} z = 0$, we find from (8) that $y_0 = -\log r_0$. The parametric equations of the free surface are

$$\begin{aligned} x_s(\gamma) &= 2\pi - \gamma - \sum_{n=1}^{\infty} \alpha_n \sin n\gamma, \\ y_s(\gamma) &= -\log r_0 + \sum_{n=1}^{\infty} \beta_n \cos n\gamma, \end{aligned} \qquad (9)$$

where $\alpha_n = 1 + r_0^{2n}$, $\beta_n = 1 - r_0^{2n}$, γ is a polar angle in the ζ-plane. After some algebra it is possible to demonstrate that

$$M = (R + \log r_0)\,\Lambda - \frac{1}{2}(\Lambda_1 + \Lambda_2) - \log^2 r_0 - 2R\,\log r_0,$$

where

$$\begin{aligned} \Lambda &= \sum_{n=1}^{\infty} n\beta_{2n} y_n^2, \quad \Lambda_1 = \sum_{n=1}^{\infty} \beta_n^2 y_n^2, \\ \Lambda_2 &= \sum_{k=2}^{\infty} y_k \sum_{n=1}^{k-1} \gamma_{k-n,n}\, y_n\, y_{k-n}, \end{aligned} \qquad (10)$$

$$\gamma_{m,n} = m\beta_n\beta_{2m+n} + n\beta_m\beta_{m+2n}.$$

Now in the variational equation (7) the left-hand side is a function of the Stokes coefficients y_n ($n = 1, 2, 3, \ldots$) and the parameter r_0. Differentiating M with respect to y_n at fixed r_0, we come to the following infinite system of quadratic equations:

$$\begin{aligned} (n\beta_{2n}K - \beta_n^2)\, y_n &= \frac{1}{2}\sum_{n=1}^{n-1} \gamma_{n-m,m}\, y_{n-m}\, y_m \\ &+ \sum_{m=1}^{\infty} \gamma_{m,n}\, y_{m+n} y_m, \quad n = 1, 2, 3, \ldots, \end{aligned} \qquad (11)$$

where

$$K = 2(R + \log r_0). \qquad (12)$$

After finding y_n at a fixed r_0 we are able to calculate

$$c_a^2 = -r_0\frac{\partial M}{\partial r_0} = K - \Lambda + \frac{r}{2}\left(\frac{\partial \Lambda_1}{\partial r_0} + \frac{\partial \Lambda_2}{\partial r_0} - K\frac{\partial \Lambda}{\partial r_0}\right). \qquad (13)$$

We shall seek the coefficients y_n and the parameter K in the form of expansions in powers of the wave amplitude $a = (h_c - h_t)/2$, where h_c and h_t are the heights of the wave crests and troughs, respectively. The coefficients of these expansions are functions of the parameter $\varepsilon = \exp(-2d)$. Let the highest power of a in these expansions be N, where N is an odd number. Then

$$K = \sum_{k=0}^{(N-1)/2} \varkappa_{2n} a^{2n} + \mathrm{O}(a^{N+1}), \tag{14}$$

$$y_i = \sum_{n=0}^{[(N-i)/2]} a_{i,2n} a^{i+2n} + \mathrm{o}(a^N), \quad i = 1, 2, \ldots, N, \tag{15}$$

and because the parameters a and ε are given we should add to the system (11) two more equations

$$a = \sum_{n=1}^{(N+1)/2} \beta_{2n-1} y_{2n-1}, \quad r_0^2 = \varepsilon \exp\left(\sum_{n=1}^{N} n\beta_{2n} y_n^2\right).$$

The first equation follows from the second equation in (9), the latter one follows from the identity

$$d = \Lambda/2 - \log r_0. \tag{16}$$

We have developed an effective algorithm of finding the coefficients $\varkappa_{2n}$, $a_{i,2n}$. The algorithm can be easily programmed (for example, by the MATHEMATICA package) and allows one to use exact arithmetics and symbolic computations. After determination the coefficients by making use of (13) one can calculate

$$c_a^2 = \sum_{n=0}^{(N-1)/2} c_{2n} a^{2n} + \mathrm{O}(a^{N+1}), \tag{17}$$

where c_{2n} are functions of the parameter ε.

4 Analytical formulae for the wave resistance

Consider a two-dimensional body that moves horizontally from right to left at constant speed c in a channel of finite depth h. Assume that in the body frame of reference the flow is steady. Then the wave train generated by the body also moves from right to left with the same velocity c. In the body frame of reference we have far upstream a uniform stream with velocity c and far downstream the train of steady periodic waves (Fig. 2).

Due to the generation of waves the body experiences a resistance, which we denote by R_w. Maklakov & Petrov (2014) have deduced an exact analytical formula for R_w:

$$R = 3V + \frac{1}{2}(\Delta d)^2 + (c^2 - d)\Delta d, \tag{18}$$

where V is the mean potential energy of the wave:

$$V = \frac{1}{4\pi}\int_x^{x+2\pi} [\eta(x) - d]^2 dx = \frac{1}{4}\left(-\frac{\Lambda^2}{2} + \Lambda_1 + \Lambda_2\right), \tag{19}$$

and $\Delta d = h - d$ is the defect of levels (the difference between the undisturbed level far upstream and the mean level far downstream).

Let us assume that the mean depth d and amplitude a of the waves are known. By means of equations (10), (15), (19) it is easy to derive an expansion for the mean potential energy V. Now to determine R_w by equation (18) one needs to find the defect of levels Δd and the speed of the body c. Because the flux Q and the Bernoulli constant R far upstream and far downstream of the body are equal we can write

$$c(d + \Delta d) = Q, \quad c^2 + 2\Delta d = 2(R - d).$$

From equations (6), (12), (16) we deduce that $Q = c_a(d - \Lambda/2)$, $2(R-d) = K - \Lambda$. This allows us to derive a cubic equation with respect to Δd:

$$(K - \Lambda - 2\Delta d)(\Delta d + d)^2 = c_a^2 (d - \Lambda/2)^2.$$

In this equation the parameters Λ, c_a^2 and K are expressed by ε and a by formulae (10), (14), (15) and (17). The solution is represented as an expansion in even powers of a up to the terms of order a^{N-1}. After finding Δd we determine $c^2 = K - \Lambda - 2\Delta d$. Inserting the found Δd and c^2 in (18), we get the wave resistance in the form

$$R_w = R_2 a^2 + R_4 a^4 + R_6 a^6 + \cdots + R_{N-1} a^{N-1} + \mathrm{O}(a^{N+1}), \tag{20}$$

where the coefficients R_{2k} depends on $\varepsilon = \exp(-2d)$. The first coefficient $R_2 = \frac{1}{4}\left(1 - \frac{2d}{\sinh 2d}\right)$ coincides with that at a^2 obtained by Kelvin (1887).

The computations of the wave resistance by the Stokes method have been carried out at $N = 21$, i. e with the asymptotic accuracy up to the terms of a^{20}, and all ten coefficients R_{2k} have been found analytically. But an analytical representation of the coefficient R_6 is already very cumbersome. For the coefficient R_4 we have

$$R_4 = R_{40} + R_{41} d + R_{42} d^2,$$

$$R_{40} = \frac{11\cosh 2d + \cosh 4d + 2\cosh 6d - 5}{128 \sinh^5 d\,(d\cosh d - \sinh d)},$$

$$R_{41} = \frac{10\cosh 2d - 118\cosh 4d - 14\cosh 6d - 3\cosh 8d - 19}{1024\sinh^6 d\cosh d\,(d\cosh d - \sinh d)},$$

$$R_{42} = \frac{-7\cosh 2d + 10\cosh 4d + \cosh 6d + 5}{128\sinh^7 d\,(d\cosh d - \sinh d)}.$$

Comparison of the wave resistance obtained by analytical formulae and by the accurate numerical method of the paper by Maklakov (2002) is presented on the graphs of Fig. 3.

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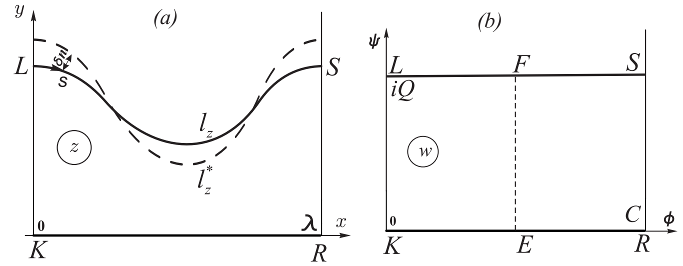

Figure 1: (a) One period. (b) Domain of the complex potential $w = \varphi + i\psi$. (c) Parametric ζ-plane.

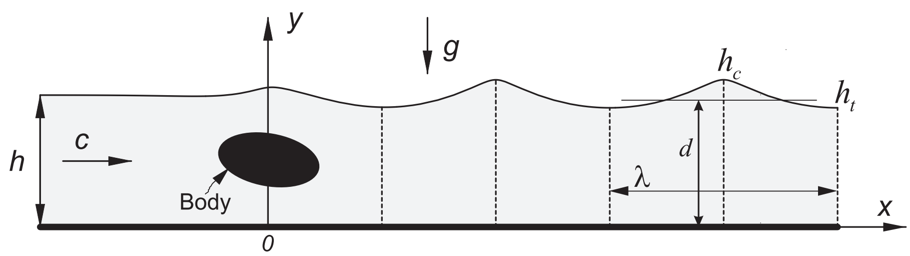

Figure 2: Scheme of the steady flow

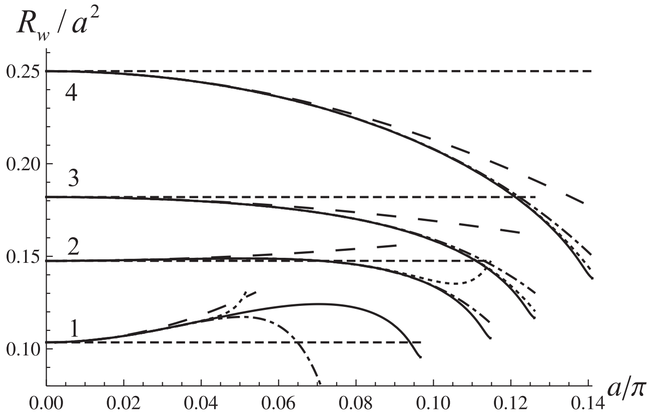

Figure 3: Comparison of accurate numerical results with analytical formulae at the depths $\frac{d}{2\pi} = 0.15$ (1), 0.2 (2), 0.25 (3) ∞ (4): solid lines, the waves have been computed by the method of the paper by Maklakov (2002); dashed lines, $n = 1$ (Kelvin's one-termed formula), dashed lines with long dashes, $n = 2$; dot-and-dash lines, $n = 5$; dotted lines, $n = 10$, where n is the number of terms in (20).