The Sign of the Added Mass Coefficients for 2-D Structures

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Highlights:

- The diagonal coefficients in the added mass matrix, for a single two-dimensional structure which satisfies the John condition in water of infinite depth, are proven to be non-negative.
- The heave added mass coefficient for a symmetric pair of structures, which individually satisfy the John condition but move as a single structure, is shown to be non-negative in the range $(2n-1)\pi \leq Ks \leq 2n\pi$. The corresponding sway coefficient is shown to be non-negative when $2n\pi \leq Ks \leq (2n+1)\pi$. Here *n* is an integer, *K* is the infinite-depth wave number and *s* is the length of the free surface between the structures.

1 Introduction

A structure is forced to make small oscillations in water of infinite depth in a single mode of motion. The coefficient of proportionality in the part of the complex-valued hydrodynamic force on the structure that is proportional to minus its acceleration, is a diagonal term in the added mass matrix. Numerical calculations show that this coefficient is positive at all frequencies for many structures. However, if the structure is shallowly-submerged, it may be negative in some frequency ranges [1, 2]. Negative added mass also occurs when one or more elements of the structure enclose a portion of the free surface, for example a pair of surface-piercing cylinders in two dimensions [3, 4, 5].

Falnes [6] showed that the diagonal coefficients in the added mass matrix are proportional to the difference in the time averaged kinetic and potential energies of the relevant fluid motion. A similar argument to that employed in [7] is used here to show that the potential energy is less than or equal to the kinetic energy for a single two-dimensional structure which satisfies the John condition and oscillates in a single mode of motion. So the diagonal coefficients in the added mass matrix are non-negative for such a structure at all frequencies. The work is extended to find frequency ranges in which the added mass coefficients for a symmetric pair of such structures are non-negative, using the method in [8]. The ranges for non-negative symmetric and antisymmetric added mass coefficients are shown to be complementary. Numerical calculations for two semi-circular cylinders show that the frequencies at which negative added mass occurs are consistent with these results.

2 Formulation



Figure 1: A single structure which satisfies the John condition

A two-dimensional, surface-piercing structure makes small amplitude oscillations at angular frequency ω , in water of infinite depth. In its equilibrium position, the structure is assumed to intersect the mean free surface at two points only $(\pm a, 0)$, as illustrated in Figure 1, and to be such that a vertical line drawn down through the fluid from each point on the mean free surface does not intersect the structure. John [9] showed that the linear, unforced frequency domain potential for such a body is zero and so the structure is said to satisfy the John condition.

If the component of the velocity of the structure in the *p*th mode of motion is given by $Re[v_p e^{-i\omega t}]$ then the corresponding velocity potential is $Re[v_p \phi_p(x, z) e^{-i\omega t}]$, where ϕ_p is harmonic and satisfies

$$\frac{\partial \phi_p}{\partial n} = n_p$$
 on the structure, (1)

where n_p is the *p*th component of the unit inward normal. The linearised free surface condition is

$$K\phi_p - \frac{\partial\phi_p}{\partial z} = 0$$
 on $z = 0, \ x < -a, \ x > a, \ K = \omega^2/g.$ (2)

There is no motion at large depths and only outward propagating waves as $x \to \pm \infty$.

3 The sign of the added mass coefficient for a single structure

An application of the divergence theorem to $\phi_p \nabla \overline{\phi}_p + \overline{\phi}_p \nabla \phi_p$ in the fluid yields the relationship between the non-dimensional diagonal coefficient in the added mass matrix μ_{pp} and the difference between the time-averaged kinetic and potential energies derived in [6], namely

$$\mu_{pp} = Re\left[\frac{1}{A_0} \int_B \phi_p \, n_p \, \mathrm{d}S\right] = \frac{1}{A_0} \lim_{M \to \infty} \left[\int_{D_- \cup D_+ \cup D_B} |\nabla \phi_p|^2 \, \mathrm{d}V - K \int_{F_- \cup F_+} |\phi_p|^2 \, \mathrm{d}x\right],\tag{3}$$

where A_0 is the cross-sectional area of the structure. Green's theorem is applied to ϕ_p and $e^{iK(x-s)+Kz}$ in the region $x \ge s > a$. Both functions represent outgoing waves at infinity and so the only contribution comes from the line x = s and yields

$$\int_{-\infty}^{0} \left[\frac{\partial \phi_p}{\partial x} e^{Kz} - i K \phi_p e^{Kz} \right]_{x=s} dz = 0.$$
(4)

The second term in (4) is integrated by parts and then the equation is rearranged to give

$$\phi_p(s,0) = \int_{-\infty}^0 \left[\frac{\partial \phi_p}{\partial z} - i \frac{\partial \phi_p}{\partial x} \right]_{x=s} e^{Kz} dz.$$
(5)

Now

$$\left|\frac{\partial\phi_p}{\partial z} - i\frac{\partial\phi_p}{\partial x}\right|^2 = 2\left(\left|\frac{\partial\phi_p}{\partial z}\right|^2 + \left|\frac{\partial\phi_p}{\partial x}\right|^2\right) - \left|\frac{\partial\phi_p}{\partial z} + i\frac{\partial\phi_p}{\partial x}\right|^2 \le 2\left|\nabla\phi_p\right|^2,\tag{6}$$

so an application of the Cauchy-Schwarz inequality to (5) and then integration over F_+ yields

$$K \int_{F_{+}} |\phi_{p}(x,0)|^{2} \,\mathrm{d}x \le \int_{D_{+}} |\nabla \phi_{p}|^{2} \,\mathrm{d}V.$$
(7)

A similar analysis in x < -a produces the same inequality but with F_+ and D_+ replaced by F_- and D_- . Both inequalities are substituted into (3) to give

$$\mu_{pp} = Re\left[\frac{1}{A_0}\int_B \phi_p \, n_p \,\mathrm{d}S\right] \ge \frac{1}{A_0}\int_{D_B} |\nabla\phi_p|^2 \,\mathrm{d}V \ge 0. \tag{8}$$

Thus the diagonal terms in the added mass matrix are non-negative for a single structure which satisfies the John condition.

4 The sign of the added mass coefficient for two structures



Figure 2: Two structures which individually satisfy the John condition

The analysis used in $\S3$ is performed for two structures which individually satisfy the John condition but move as a single structure and yields

$$\mu_{pp} = Re\left[\frac{1}{A_0}\int_B \phi_p n_p \,\mathrm{d}S\right] \ge \frac{1}{A_0}\left[\int_{D_i} |\nabla \phi_p|^2 \,\mathrm{d}V - K\int_{F_i} |\phi_p|^2 \,\mathrm{d}x\right],\tag{9}$$

where F_i represents the free surface between the structures and D_i the fluid region below. It remains to determine where the right-hand side of (9) is non-negative. On F_i the function w_p is defined as in [8] by

$$w_p(x) = \int_{-\infty}^0 \phi_p(x, z) e^{Kz} \, dz.$$
 (10)

The operator d^2/dx^2 is applied to (10) and yields $d^2w_p/dx^2 + K^2w_p = 0$. If the system of structures is symmetric, the heave potential ϕ_3 is symmetric in x, and so

$$w_3(x) = B_3 \cos Kx = \int_{-\infty}^0 \phi_3(x, z) e^{Kz} \, \mathrm{d}z, \quad x \in F_i,$$
(11)

where B_3 is a complex constant. Integration by parts in (11) followed by applications of the Cauchy and Schwarz inequalities yields an inequality which is integrated over F_i to give

$$K \int_{F_i} |\phi_3(x,0)|^2 \, \mathrm{d}x \le K^2 |B_3|^2 \left[2K(b-a) + \sin 2K(b-a) \right] + \int_{D_i} \left| \frac{\partial \phi_3}{\partial z} \right|^2 \, \mathrm{d}V. \tag{12}$$

Differentiation of (11) with respect to x and an application of the Cauchy-Schwarz inequality followed by integration over F_i yields

$$K^{2}|B_{3}|^{2} \left[2K(b-a) - \sin 2K(b-a)\right] \leq \int_{D_{i}} \left|\frac{\partial\phi_{3}}{\partial x}\right|^{2} dV.$$
(13)

A combination of (12) and (13) gives

$$K \int_{F_i} |\phi_3(x,0)|^2 \, \mathrm{d}x \le 2K^2 |B_3|^2 \sin 2K(b-a) + \int_{D_i} |\nabla\phi_3|^2 \, \mathrm{d}V \le \int_{D_i} |\nabla\phi_3|^2 \, \mathrm{d}V \tag{14}$$

if $\sin 2K(b-a) \leq 0$, that is if $(2n-1)\pi \leq 2K(b-a) \leq 2n\pi$, where n is a positive integer. Substitution of (14) into (9) shows that the heave added mass is non-negative in this range. The sway potential ϕ_1 is antisymmetric in x and so $w_1(x) = B_1 \sin Kx$ and a similar analysis gives

$$K \int_{F_i} |\phi_1(x,0)|^2 \, \mathrm{d}x \le -2K^2 |B_1|^2 \sin 2K(b-a) + \int_{D_i} |\nabla \phi_1|^2 \, \mathrm{d}V \le \int_{D_i} |\nabla \phi_1|^2 \, \mathrm{d}V \tag{15}$$

if $\sin 2K(b-a) \ge 0$, that is if $2n\pi \le 2K(b-a) \le (2n+1)\pi$, where n is a non-negative integer. Substitution of (15) into (9) shows that the sway added mass is non-negative in this range.

Numerical calculations of the heave and sway added mass coefficients, μ_{33} and μ_{11} , are presented in Figure 3 for a pair of surface-piercing, semi-circular cylinders for which b = 2a. The shaded area represents the frequency ranges in which μ_{11} must be non-negative and the complementary frequency ranges are where μ_{33} must be non-negative. It should be noted that μ_{11} and μ_{33} are not negative everywhere outside these intervals and work is currently in progress to use the wide-spacing approximation to find ranges of frequencies at which negative values of μ_{11} and μ_{33} occur.



Figure 3: The heave — and sway - - added mass for 2 semi-circular cylinders, b = 2a

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