UNIVERSITY OF BRISTOL

Examination for the Degree of B.Sc. and M.Sci. (Level 5)

MULTIVARIABLE CALCULUS
MATH 20901
(Paper Code MATH-20901)

January 2015, 1 hours 30 minutes

This paper contains **two** *questions. Both answers are used for assessment.*

Calculators are **not** *permitted in this examination.*

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

Do not turn over until instructed.

1. (a) (6 marks)

   Explain what is meant by a linear map. With $\mathbf{x} = (x_1, x_2, x_3)$ which of these are linear maps from $\mathbb{R}^3 \to \mathbb{R}^2$:

   $$\text{(i) } \mathbf{f}(\mathbf{x}) = (x_2 + x_3, x_1 + x_2), \qquad \text{and} \qquad \text{(ii) } \mathbf{f}(\mathbf{x}) = (x_2 x_3, x_1 x_2).$$

   Explain why.

   (b) (14 marks)

   Let $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $\mathbf{f}(\mathbf{x}) = (x_2 x_3, x_1 x_2)$. Find the derivative of $\mathbf{f}$ in the direction $(1, -1, 1)$ using two independent methods.

   (c) (8 marks)

   Let $\mathbf{F}, \mathbf{G} \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and define $\mathbf{H}(\mathbf{x}) = \mathbf{G}(\mathbf{x}) \circ \mathbf{F}(\mathbf{x})$.

   State how the chain rule is applied to the derivative $\mathbf{H}'(\mathbf{x})$ in terms of $\mathbf{G}$ and $\mathbf{F}$.

   Now using $\mathbf{H}(\mathbf{x}) = \mathbf{x}$, show that the inverse of the derivative of $\mathbf{F}$ is identical to the derivative of the inverse of $\mathbf{F}$.

   (d) Consider the mapping $(x, y) = (\cosh\mu \cos\nu, \sinh\mu \sin\nu)$.

   (i) (8 marks)

   Show that the Jacobian determinant is given by $\sinh^2\mu + \sin^2\nu$ and provide conditions under which map is invertible.

   (ii) (6 marks)

   Use the previous results to compute $\dfrac{\partial\mu}{\partial y}$ at $\mu = 0$, $\nu = \frac{1}{2}\pi$.

   (e) (8 marks)

   If f is a function such that both $\triangle f = 0$ and $\triangle(f^2) = 0$, show that f must be a constant.

Continued...

2. (a) (6 marks)
   Evaluate $\delta_{jm}\delta_{jm}$ and $\epsilon_{ijk}\epsilon_{ikj}$.

   (b) (10 marks)
   With $\mathbf{r} = (x, y, z)$ and $\mathbf{a}$ a constant vector, calculate

   $$\text{(i) } \nabla \cdot (\mathbf{a} \times \mathbf{r}) \qquad \text{and} \qquad \text{(ii) } \nabla \times (\mathbf{a} \times \mathbf{r}).$$

   (c) The surface, S, of an open cone of unit height is parametrised by by $\mathbf{s}(u, v) = (u \cos v, u \sin v, u)$, for $0 < v \leq 2\pi$, $0 < u < 1$. A vector field $\mathbf{f}$ is given by $\mathbf{f}(x, y, z) = (-y, x, z)$.

   (i) (4 marks)
   Calculate $\nabla \times \mathbf{f}$.

   (ii) (6 marks)
   Calculate the vector $\mathbf{N}(u, v) = \mathbf{s}_u \times \mathbf{s}_v$. What geometric property does this vector have ?

   (iii) (8 marks)
   Hence find

   $$\int_S \nabla \times \mathbf{f} \cdot d\mathbf{S}.$$

   (iv) (8 marks)
   Use Stokes' theorem to calculate (c)(iii) by an alternative means.

   (d) (8 marks)
   For a general surface S connecting a closed curve C show that

   $$\int_S d\mathbf{S} = \frac{1}{2} \oint_C \mathbf{r} \times d\mathbf{r}.$$

   [Hint: use $\mathbf{f} = \mathbf{a} \times \mathbf{r}$ in Stokes' theorem.]

End of examination.