UNIVERSITY OF BRISTOL

Examination for the Degree of B.Sc. and M.Sci. (Level 5)

MULTIVARIABLE CALCULUS

MATH 20901

(Paper Code MATH-20901)

January 2016, 1 hours 30 minutes

This paper contains **two** *questions. Both answers are used for assessment.*

Calculators are **not** *permitted in this examination.*

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

Do not turn over until instructed.

1. (a) (6 marks)
A function $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^n$ maps $\mathbf{x} = (x_1, \ldots, x_m)$ into $\mathbf{F} = (F_1, \ldots, F_n)$. Define the derivative of the map, $\mathbf{F}'(\mathbf{x})$. What is special about $\mathbf{F}'(\mathbf{x})$ if $\mathbf{F}$ is a linear map ?

(b) (6 marks)
If $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $\mathbf{F}(\mathbf{x}) = (x_1 x_2^2 / x_3, \cos(x_1 x_3))$, calculate $\mathbf{F}'(\mathbf{x})$.

(c) Two functions $\mathbf{u}(\mathbf{x})$ and $\mathbf{v}(\mathbf{x})$ map $\mathbb{R}^3 \to \mathbb{R}^3$, where $\mathbf{x} = (x_1, x_2, x_3)$.

(i) (8 marks)
Define $\nabla(\mathbf{u} \cdot \mathbf{v})$ in terms of $\mathbf{u}'(\mathbf{x})$ and $\mathbf{v}'(\mathbf{x})$.

(ii) (10 marks)
By considering $\mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{v})$ and $\mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{u})$ show that

$$\nabla(\mathbf{u} \cdot \mathbf{v}) = (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{v} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{u} + \mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{v}) + \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{u}).$$

(d) Let $\mathbf{r} : \mathbb{R}^2 \to \mathbb{R}^2$ be a map defining the transformation $x = r\cos\theta$, $y = r\sin\theta$.

(i) (6 marks)
Compute the Jacobian determinant of the transformation, $\dfrac{\partial(x,y)}{\partial(r,\theta)}$. When is the map invertible ?

(ii) (6 marks)
Construct a local basis $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ in the new coordinate system.

(iii) (8 marks)
For a scalar function $f(x,y)$ derive the expression for $\boldsymbol{\nabla} f$ in the local basis $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$. Hence, or otherwise show that

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2$$

Continued...

2. (a) (8 marks)
   Calculate the gradient of the function $f(\mathbf{r}) = \sin(xz) - \sin(yx)$ where $\mathbf{r} = (x, y, z)$ and confirm in this case that $\boldsymbol{\nabla} \times \boldsymbol{\nabla} f = 0$.

   (b) (6 marks)
   For any scalar functions f, g, prove that $\boldsymbol{\nabla} \times (f\boldsymbol{\nabla} g) = \boldsymbol{\nabla} f \times \boldsymbol{\nabla} g$.

   (c) (8 marks)
   You are given that a vector field $\mathbf{v}(\mathbf{r})$ is everywhere parallel to the normals of a family of surfaces $g(\mathbf{r}) = \text{constant}$. Show that

$$\mathbf{v} \cdot (\boldsymbol{\nabla} \times \mathbf{v}) = 0$$

   (d) In this part of the question, consider the vector field $\mathbf{f} = (2z, x, y^2)$ and the surface S described by the paraboloid $z = 4 - x^2 - y^2$ in $z \geq 0$.

   (i) (6 marks)
   Calculate $\boldsymbol{\nabla} \times \mathbf{f}$.

   (ii) (12 marks)
   Calculate the surface integral

$$\int_S \boldsymbol{\nabla} \times \mathbf{f} \cdot d\mathbf{S}$$

   in which $d\mathbf{S}$ is directed outwards from the surface of the paraboloid.

   (iii) (10 marks)
   State Stokes's theorem and use it to make an independent calculation of the integral.

End of examination.