UNIVERSITY OF BRISTOL

School of Mathematics

MULTIVARIABLE CALCULUS
MATH 20901
(Paper code MATH–20901J)

January 2017   1 hour 30 minutes

This paper contains **TWO** *questions.*
Both answers are used for assessment.

Calculators are **not** *permitted in this examination.*

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

Do not turn over until instructed.

1. (a) The map $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $\mathbf{F}(\mathbf{x}) = (x^2 + y^3, \cos x + \sin y)$, where $\mathbf{x} = (x, y)$.

   (i) (6 marks)
   Calculate $\mathbf{F}'(\mathbf{x})$.

   (ii) (10 marks)
   Show that close to the point $\mathbf{x} = (\pi/2, 0)$,

$$\mathbf{F}(\mathbf{x}) \approx (-\tfrac{1}{4}\pi^2 + \pi x, \tfrac{1}{2}\pi - x + y).$$

   (iii) (8 marks)
   Giving reasons, state whether the pair of simultaneous equations $u = x^2 + y^3$, $v = \cos x + \sin y$ can be inverted to give (x, y) uniquely in terms of (u, v) in the neighbourhood of points (x_0, y_0) lying, firstly, on the line $x = 0$ and, secondly, on the line $x = \frac{1}{2}\pi$.

   (b) Now let $\mathbf{G}(\mathbf{r}) = \left(\dfrac{-y}{x^2 + y^2}, \dfrac{x}{x^2 + y^2}, 0 \right)$ with $\mathbf{r} = (x, y, z)$.

   (i) (10 marks)
   Calculate $\displaystyle\int_C \mathbf{G} \cdot \mathrm{d}\mathbf{r}$ as a path integral where C is the circle of radius a lying in the (x, y)-plane, centred on the origin and oriented anti-clockwise.

   (ii) (10 marks)
   State Green's theorem in the plane and use it to calculate $\displaystyle\int_C \mathbf{G} \cdot \mathrm{d}\mathbf{r}$ as an integral over the area of the circle with boundary C.

   (iii) (6 marks)
   Do your two answers to parts (b)(i) and (b)(ii) coincide ? If not, provide an explanation as to why they don't.

Continued...

2. Throughout this question, $\mathbf{r} = (x, y, z)$ and $r = |\mathbf{r}|$.

   (a) (6 marks)
   (i) Evaluate $\boldsymbol{\nabla} \cdot \mathbf{r}$; (ii) show that $\boldsymbol{\nabla} r = \mathbf{r}/r$.

   (b) (i) (6 marks)
   For two arbitrary scalar functions $f(\mathbf{r})$ and $g(\mathbf{r})$, show that $\boldsymbol{\nabla} \cdot (f \boldsymbol{\nabla} g) = f \Delta g + \boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g$.

   (ii) (10 marks)
   Hence show that

$$\int_V (f\Delta g - g\Delta f)\mathrm{d}V = \int_{\partial V} (f\partial_n g - g\partial_n f)\mathrm{d}S$$

   where $\partial_n \equiv \hat{\mathbf{n}} \cdot \boldsymbol{\nabla}$ and $\hat{\mathbf{n}}$ is the outward unit normal to the closed surface ∂V bounding the volume V. Clearly state any results or theorems used.

   (c) (18 marks)
   A function $f(r)$ is defined by

$$f(r) = \begin{cases} \dfrac{1}{2a} - \dfrac{r^2}{2a^3}, & r < a, \\ \dfrac{1}{r}, & r > a. \end{cases}$$

   Calculate $\mathbf{F} = \boldsymbol{\nabla} f$ in $r < a$ and $r > a$ and confirm that $\mathbf{F}$ is continuous across $r = a$. Hence show that $\Delta f = 0$ in $r > a$ whilst in $r < a$, $\Delta f = C$, where C is a constant to be determined. You may find results from part (a) useful.

   (d) (10 marks)
   Using results from parts (b) and (c) (or otherwise) show, for any function g satisfying $\Delta g = 0$ inside the sphere $V = \{r \mid r < a\}$, that

$$\int_V g \,\mathrm{d}V = \frac{a}{3} \int_{\partial V} g \,\mathrm{d}S$$

   where $\partial V = \{r \mid r = a\}$ is the surface of the sphere.
   Give an example of a function g (not identically zero) which confirms this result.

End of examination.