UNIVERSITY OF BRISTOL

School of Mathematics

MULTIVARIABLE CALCULUS
MATH 20901
(Paper code MATH–20901J)

January 2018 1 hour 30 minutes

This paper contains **TWO** *questions.*
Both answers are used for assessment.

Calculators are **not** *permitted in this examination.*

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

Do not turn over until instructed.

1. (a) (10 marks)
   What condition must a general function $\mathbf{F}(\mathbf{x})$ satisfy to be linear ?
   Let $\mathbf{x} = (x_1, x_2)$. Which of the following is a linear map ? In each case find the derivative.

$$\text{(i) } \mathbf{G}(\mathbf{x}) = (x_1/x_2, x_2^2 + x_1^2, \mathrm{e}^{x_1}), \qquad \text{(ii) } \mathbf{H}(\mathbf{x}) = (x_1 + x_2, x_1 - x_2)$$

   (b) (10 marks)
   Find the directional derivative of $\mathbf{G}$ in the direction $(1, -2)$ using two different methods.

   (c) Let $\mathbf{r} : \mathbb{R}^2 \to \mathbb{R}^2$ be the mapping $\mathbf{r}(r, \theta) = (r\cos\theta, r\sin\theta) \equiv (x, y)$.

   (i) (6 marks)
   Calculate the Jacobian of the mapping, $\partial(x, y)/\partial(r, \theta)$. When is the mapping invertible ?

   (ii) (8 marks)
   Construct a local basis $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ as functions of (r, θ) and define the gradient operator in terms of the local basis.

   (iii) (10 marks)
   Use the definition of the gradient operator to show that the divergence of $\mathbf{F} = F_r(r, \theta)\hat{\mathbf{r}}$ is

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \frac{\partial F_r}{\partial r} + \frac{F_r}{r}.$$

   (iv) (6 marks)
   Find the most general function $\phi(r)$ satisfying $\Delta\phi = r$.

Continued...

2. (a) (10 marks)
For two general vector fields $\mathbf{u}(\mathbf{r})$, $\mathbf{v}(\mathbf{r})$ prove that $\boldsymbol{\nabla}\cdot(\mathbf{u}\times\mathbf{v}) = \mathbf{v}\cdot(\boldsymbol{\nabla}\times\mathbf{u}) - \mathbf{u}\cdot(\boldsymbol{\nabla}\times\mathbf{v})$.

(b) For this part let $\mathbf{F}(\mathbf{r}) = (z^2 - y^2, x^2 - z^2, y^2 - x^2)$.

(i) (6 marks)
Calculate $\boldsymbol{\nabla}\cdot\mathbf{F}$ and $\boldsymbol{\nabla}\times\mathbf{F}$.

(ii) (10 marks)
Let C be the closed curve that is the boundary of the triangle with vertices at the points $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. Specify a direction along C and consider the closed-loop integral

$$\int_C \mathbf{F}\cdot d\mathbf{r}.$$

Explain why the contribution to the integral is the same from each edge of C, and evaluate the integral above.

(iii) (6 marks)
State Stokes' theorem and use it to evaluate the surface integral

$$\int_S (\boldsymbol{\nabla}\times\mathbf{F})\cdot d\mathbf{S}$$

where S is the triangle formed by the plane $x+y+z=1$ with C as its boundary. The normal to the surface points towards the origin.

(iv) (10 marks)
Verify your result by a direct calculation of the surface integral in part (b)(iii).

(c) (8 marks)
Let $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ be vector fields in $\mathbb{R}^3$ which also depend on time, t, and let them satisfy the relations

$$\frac{\partial\mathbf{E}}{\partial t} = \boldsymbol{\nabla}\times\mathbf{B}, \qquad \frac{\partial\mathbf{B}}{\partial t} = -\boldsymbol{\nabla}\times\mathbf{E}.$$

Show that, for any fixed volume $V\subset\mathbb{R}^3$ enclosed by a surface S,

$$\frac{d}{dt}\int_V \tfrac{1}{2}\left(|\mathbf{E}|^2 + |\mathbf{B}|^2\right) dV = -\int_S (\mathbf{E}\times\mathbf{B})\cdot d\mathbf{S},$$

stating any results used.

End of examination.