MODEL ANSWERS (Jan '15)

unseen examples similar to probclass

Q1 (a) A linear map is one which can be expressed as $\underline{f}(\underline{x}) = A\underline{x}$. This implies $f(\lambda\underline{x}) = \lambda A\underline{x}$
$= \lambda f(\underline{x})$

(i) is clearly linear & $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

(ii) satisfies $f(\lambda\underline{x}) = \lambda^2 f(\underline{x})$ & is non-linear.

(b) unit vector $\underline{v} = \frac{1}{\sqrt{3}}(1, -1, 1)$

unseen example, standard method as in lectures

Method (i): $D_{\underline{v}}\underline{f} = \frac{d}{dt}\left(\underline{f}(\underline{x} + \underline{v}t)\right)\Big|_{t=0}$

$$= \frac{d}{dt}\left(\left(x_2 - \frac{t}{\sqrt{3}}\right)\left(x_3 + \frac{t}{\sqrt{3}}\right), \left(x_1 + \frac{t}{\sqrt{3}}\right)\left(x_2 - \frac{t}{\sqrt{3}}\right)\right)\Big|_{t=0}$$

$$= \left(\frac{-x_3}{\sqrt{3}} + \frac{x_2}{\sqrt{3}}, \frac{-x_1}{\sqrt{3}} + \frac{x_2}{\sqrt{3}}\right)$$

Method (ii) $\underline{f}'(\underline{x}) = \begin{pmatrix} 0 & x_3 & x_2 \\ x_2 & x_1 & 0 \end{pmatrix}$ is Jacobian matrix.

&

$$D_{\underline{v}}\underline{f} = \underline{f}'(\underline{x}).\underline{v} = \begin{pmatrix} 0 & x_3 & x_2 \\ x_2 & x_1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}/\sqrt{3}$$ ... same answer.

(c) if $\underline{H}(\underline{x}) = \underline{G}(\underline{x}) \circ \underline{F}(\underline{x})$

then $\underline{H}'(\underline{x}) = \underline{G}'(\underline{F}(\underline{x}))\,\underline{F}'(\underline{x})$

— bookwork ... from lectures —

if $\underline{H}(\underline{x}) = \underline{x}$ then $\underline{H}'(\underline{x}) = I$, the identity.

$\therefore \quad I = \underline{G}'(\underline{F}(\underline{x}))\,\underline{F}'(\underline{x})$

In this case, G must represent the inverse of $\underline{F}$, call it $\underline{F}^{-1}$ & so

$$I = (F^{-1})'(F(\underline{x}))\,F'(\underline{x})$$

$$\Rightarrow (F'(\underline{x}))^{-1} = (F^{-1})'(\underline{F}(\underline{x}))$$

i.e. the inverse of the derivative is equal to the derivative of the inverse.

(d) (i) $$\frac{\partial(x,y)}{\partial(\mu,\nu)} = \begin{pmatrix} \sinh\mu\cos\nu & -\cosh\mu\sin\nu \\ \cosh\mu\sin\nu & \sinh\mu\cos\nu \end{pmatrix}$$

is the Jacobian matrix.

similar to e.g. on homeworks

The Jacobian determinant is $\left|\frac{\partial(x,y)}{\partial(\mu,\nu)}\right| =$

$$\sinh^2\mu\cos^2\nu + \cosh^2\mu\sin^2\nu = \sinh^2\mu + \sin^2\nu$$

Map is invertible apart from when this is zero, when $\mu = 0$, $\nu = n\pi$.

unseen, similar to q's on homeworks

(ii) Using part (c)

$$\frac{\partial(\mu,\nu)}{\partial(x,y)} = \frac{1}{\sinh^2\mu + \sin^2\nu}\begin{pmatrix} \sinh\mu\cos\nu & +\cosh\mu\sin\nu \\ -\cosh\mu\sin\nu & \sinh\mu\cos\nu \end{pmatrix}$$

so $$\frac{\partial\mu}{\partial y} = \frac{\cosh\mu\sin\nu}{\sinh^2\mu + \sin^2\nu}$$

at $\mu = 0$, $\nu = \pi/2$ $\quad \frac{\partial\mu}{\partial y} = 1.$

unseen

(e) $\Delta f = 0$. $\quad \Delta f^2 = \nabla\cdot\nabla(f^2) = 0$

$$\nabla(f^2) = 2f\nabla f \quad \text{so} \quad 0 = \nabla\cdot f\nabla f$$
$$= \underbrace{f\nabla^2 f}_{=0} + \nabla f\cdot\nabla f.$$

so $\|\nabla f\|^2 = 0 \Rightarrow \nabla f = 0 \Rightarrow f$ is a constant.

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Standard but unseen e.g.

Q2) (a) (i) $\delta_{jm}\delta_{jm} = \delta_{jj} = 3$

(ii) $\varepsilon_{ijk}\varepsilon_{ikj} = \delta_{jk}\delta_{kj} - \delta_{jj}\delta_{kk} = 3 - 9 = -6.$

in class / in homework

(b) (i) $(\underline{a}\times\underline{r})_i = \varepsilon_{ijk}a_j x_k$

$$\nabla\cdot(\underline{a}\times\underline{r}) = \frac{\partial}{\partial x_i}\varepsilon_{ijk}a_j x_k = \varepsilon_{ijk}a_j\delta_{ik}$$
$$= \varepsilon_{kjk}a_j = 0$$

(ii)
$$(\nabla\times(\underline{a}\times\underline{r}))_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}\varepsilon_{klm}a_l x_m$$
$$= \varepsilon_{kij}\varepsilon_{klm}a_l\delta_{mj}$$
$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})a_l\delta_{mj}$$
$$= 3a_i - a_i = 2a_i$$

$$\therefore\ \nabla\times(\underline{a}\times\underline{r}) = 2\underline{a}$$

Standard calculation

(c) (i)
$$\nabla\times\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ -y & x & z \end{vmatrix} = 2\underline{k}$$

(ii)
$$\underline{N}(u,v) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = (-u\cos v, -u\sin v, u)$$

$\underline{N}$ is normal to the surface S, routine evaluation similar to e.g.s

(iii) $$\int_S (\nabla \times \underline{f}) \cdot d\underline{S} = \int_0^{2\pi} \int_0^1 (2\underline{k}) \cdot (-u\cos v, -u\sin v, u)\, du\, dv$$

$$= 2\pi \cdot 2 \cdot \tfrac{1}{2} = 2\pi$$

(iv) By Stokes, $\int_S (\nabla \times \underline{f}) \cdot d\underline{S} = \oint_C \underline{f} \cdot d\underline{r}$

$$= \int_0^{2\pi} (-\sin\theta, \cos\theta, 1) \cdot \frac{df}{dt}\, dt.$$

where $f(t) = (\cos\theta, \sin\theta, 1)$ parametrises C

$$\therefore \int_S (\nabla \times f) \cdot d\underline{S} = \int_0^{2\pi} 1 \, dt = 2\pi.$$

unseen

(d) consider.

$$\int_S (\nabla \times (\underline{a} \times \underline{r})) \cdot d\underline{S} = \oint_C (\underline{a} \times \underline{r}) \cdot d\underline{r}$$

$$\shortparallel$$

$$2\underline{a} \cdot \int_S d\underline{S} = \underline{a} \cdot \oint_C \underline{r} \times d\underline{r}$$ using vector identity.

$$\therefore \underline{a} \cdot \left(\int_S d\underline{S} - \tfrac{1}{2} \oint_C \underline{r} \times d\underline{r} \right) = 0$$

but $\underline{a}$ is arbitrary, so must be that $(\quad) = 0$.