MATH 20901 Multivariable Calc. MODEL ANSWERS JAN '16

from notes

Q1) (a) $\underline{F}'(x)$ is the $n\times m$ matrix with i,jth entry $\dfrac{\partial F_i}{\partial x_j}$. if $\underline{F}$ is linear, $\underline{F}'(x)$ is a constant matrix.

simple example

(b) $\underline{F} = \left(\dfrac{x_1 x_2^2}{x_3}, \cos(x_1 x_3) \right)$

So

$$\underline{F}'(\underline{x}) = \begin{pmatrix} x_2^2/x_3 & \dfrac{2x_1x_2}{x_3} & -\dfrac{x_1x_2^2}{x_3^2} \\ -\sin(x_1x_3).x_3 & 0 & -x_1\sin(x_1x_3) \end{pmatrix}$$

unseen

(c) (i)

$$[\nabla(\underline{u}.\underline{v})]_i = \frac{\partial}{\partial x_i} u_j v_j$$

$$= \frac{\partial u_j}{\partial x_i}.v_j + \frac{\partial v_j}{\partial x_i} u_j \quad (*)$$

$$= \left[(\underline{u}')^T\underline{v} + (\underline{v}')^T\underline{u}\right]_i$$

$$\therefore \nabla(\underline{u}.\underline{v}) = (\underline{u}')^T\underline{v} + (\underline{v}')^T\underline{u}.$$

unseen, but we do $\underline{v}\times\nabla\times\underline{v}$!

(ii)

$$[\underline{u}\times\nabla\times\underline{v}]_i = \varepsilon_{ijk} u_j \quad \varepsilon_{klm}\frac{\partial}{\partial x_l}v_m$$

$$= \varepsilon_{kij}\varepsilon_{klm} u_j \frac{\partial v_m}{\partial x_l}$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})u_j \frac{\partial v_m}{\partial x_l}$$

$$= u_j\frac{\partial v_j}{\partial x_i} - u_j\frac{\partial v_i}{\partial x_j} = u_j\frac{\partial v_j}{\partial x_i} - (\underline{u}.\nabla)v_i$$

Similarly

$$[\underline{v} \times \nabla \times \underline{u}]_i = \frac{\partial u_j}{\partial x_i} v_j - (\underline{v}.\nabla) u_i$$

if we add the two results together & substitute from (*) we have

$$[\underline{u} \times \nabla \times \underline{v}) + (\underline{v} \times \nabla \times \underline{u}) = \nabla(\underline{u}.\underline{v}) - (\underline{u}.\nabla)\underline{v} - (\underline{v}.\nabla)\underline{u}$$

Hence result.

(A) $(x, y) = \underline{r}(r, \theta) = (r\cos\theta, r\sin\theta)$. So

(i)

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

map is invertible if $r \neq 0$.

(ii) $\hat{\underline{r}} = \frac{\partial \underline{r}}{\partial x} = (\cos\theta, \sin\theta)$ & $h_r = \left|\frac{\partial \underline{r}}{\partial x}\right| = 1$

$\hat{\underline{\theta}} = \frac{1}{h_\theta}\frac{\partial \underline{r}}{\partial \theta} = \frac{1}{h_\theta}(-r\sin\theta, r\cos\theta)$ & $h_\theta = \left|\frac{\partial \underline{r}}{\partial \theta}\right| = r$

$= (-\sin\theta, \cos\theta)$

(iii) from notes $\nabla f = \frac{\hat{\underline{r}}}{h_r}\frac{\partial f}{\partial r} + \frac{\hat{\underline{\theta}}}{h_\theta}\frac{\partial f}{\partial \theta}$

$= \hat{\underline{r}}\frac{\partial f}{\partial r} + \frac{\hat{\underline{\theta}}}{r}\frac{\partial f}{\partial \theta}$

following lecture notes

Now $f_x^2 + f_y^2 = |\nabla f|^2 = \nabla f . \nabla f = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2$ since $\hat{\underline{r}}.\hat{\underline{\theta}} = 0$

unseen.

Q2a)

Standard calculation / similar to examples

$$\underline{f} = \sin(xz) - \sin(yx)$$

$$\nabla f = \left(z\cos(xz) - y\cos(yx),\ -x\cos(yx),\ x\cos(xz)\right)$$

$$\nabla \times \nabla f = \left(\frac{\partial}{\partial y}(x\cos xz) + \frac{\partial}{\partial z}(-x\cos yx)\right)\underline{\hat{x}}$$

$$+\left(\frac{\partial}{\partial z}(z\cos(xz)) - \frac{\partial}{\partial x}(x\cos xz)\right)\underline{\hat{y}}$$

$$+\left(\frac{\partial}{\partial x}(-x\cos yx) - \frac{\partial}{\partial y}(-y\cos yx)\right)\underline{\hat{z}}$$

$$= 0\,\underline{\hat{x}} + \left(\cos xz - \cos xz - xz\sin(xz) + xz\sin xz\right)\underline{\hat{y}}$$

$$+\left(-\cos yx + \cos yx + xy\sin xy + xy\sin xy\right)\underline{\hat{z}}$$

$$= \underline{0}$$

unseen

~~(b) $\nabla \times \underline{v} = \underline{v}$.~~

~~Take divergence of both sides & use $\nabla\cdot(\nabla\times\underline{v}) = 0$~~

~~for any $\underline{v}$ to get $\nabla\cdot\underline{v} = 0$.~~

on problems sheet / prob class example

(c) $$[\nabla\times(f\nabla g)]_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}\left(f\frac{\partial g}{\partial x_k}\right)$$

$$= \varepsilon_{ijk}\frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_k} + f\,\varepsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial g}{\partial x_k}$$

$$= [\nabla f\times\nabla g]_i + f\,\underbrace{[\nabla\times\nabla g]_i}_{0 \text{ by an identity}}$$

Hence result

(c) if $\underline{v}$ is parallel to normal of surfaces g = constant then $\underline{v}$ points in the dir^{n} of ∇g.

However $\underline{v}$ is <u>not</u> ∇g, it's only proportional to ∇g so

$$\underline{v} = f(\underline{r})\nabla g.$$

unseen

Now $\nabla \times \underline{v} = \nabla f \times \nabla g$ by part (c)

& $\underline{v} \cdot \nabla \times \underline{v} = f \nabla g \cdot (\nabla f \times \nabla g) = 0$

by vector identity.

(d) (i). $\underline{f} = (2z, x, y^2)$

standard

$$\nabla \times \underline{f} = \begin{vmatrix} \hat{\underline{x}} & \hat{\underline{y}} & \hat{\underline{z}} \\ \partial_x & \partial_y & \partial_z \\ 2z & x & y^2 \end{vmatrix} = 2y\hat{\underline{x}} + 2\hat{\underline{y}} + \hat{\underline{z}}$$

$$= (2y, 2, 1)$$

standard method, unseen example

(ii) Introduce mapping $\underline{s}(u,v) : D \to \mathbb{R}^3$ where $D = \{(u,v) \mid u^2 + v^2 < 2\}$ & $\underline{s}(u,v) = (u, v, 4 - u^2 - v^2)$

Then $\underline{s}_u = (1, 0, -2u)$ $\underline{s}_v = (0, 1, -2v)$

& $\underline{N} = \underline{s}_u \times \underline{s}_v = \begin{vmatrix} \hat{\underline{x}} & \hat{\underline{y}} & \hat{\underline{z}} \\ 1 & 0 & -2u \\ 0 & 1 & -2v \end{vmatrix} = (2u, 2v, 1)$

points <u>outwards</u>.

$$\therefore \int_S \nabla\times\underline{F}\cdot d\underline{S} = \int_D \underline{F}(\underline{s})\cdot\underline{N}\,du\,dv$$

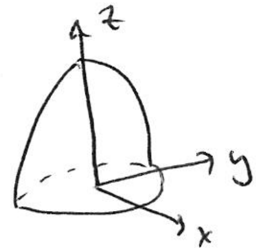
$$= \int_D (2v,2,1)\cdot(2u,2v,1)\,du\,dv$$

$$= \int_D \underbrace{4uv + 4v}_{\text{odd in } u,v \text{ so } \int = 0} + 1\,du\,dv$$

$$= \pi(2^2) = \underline{4\pi}$$

[or use mapping $\underline{s}(r,\theta) = (r\cos\theta, r\sin\theta, 4-r^2)$]

(iii) Stokes' theorem:



$$\int_S \nabla\times\underline{F}\cdot d\underline{S} = \oint_C \underline{F}\cdot d\underline{r}$$

Standard method

where here C is the circle $x^2+y^2=4$ & by RH rule we go anticlockwise. Let C be parametrised by $\underline{r} = \underline{f}(t)$ where $\underline{f}(t) = (2\cos t, 2\sin t, 0)$. Then

$$\oint_C \underline{F}\cdot d\underline{r} = \int_0^{2\pi} (0, 2\cos t, 4\sin^2 t)\cdot\underbrace{(-2\sin t, 2\cos t, 0)}_{\underline{f}'(t)}\,dt$$

$$= 4\int_0^{2\pi} \cos^2 t\,dt = 4\cdot\int_0^{2\pi} \tfrac{1}{2} - \tfrac{1}{2}\cos 2t\,dt$$

$$= \underline{4\pi}$$