Multivariable Calculus MATH20901, Exam Solutions, January 2018

1. (a) (*First part of course. Bookwork definition & basic example.*)
Definition of linearity:
$$\mathbf{F}(\alpha\mathbf{x}+\beta\mathbf{y})=\alpha\mathbf{F}(\mathbf{x})+\beta\mathbf{F}(\mathbf{y})$$
for all $\mathbf{x},\mathbf{y}$ and scalars α,β.
(Can add: as a *consequence* of linearity $\mathbf{F}(\mathbf{x})=A\mathbf{x}$ for a constant matrix A.)
(i) Is non-linear
$$\mathbf{G}'(\mathbf{x})=\begin{pmatrix} 1/x_2 & -x_1/x_2^2 \\ 2x_1 & 2x_2 \\ \mathrm{e}^{x_1} & 0 \end{pmatrix}$$
(ii) Is linear
$$\mathbf{H}'(\mathbf{x})=\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

(b) *(Following notes, examples in class & examples on worksheets. Also similar Q on 2015 exam.)*
First, normalise direction as $\mathbf{u}=(1,-2)/\sqrt{5}$, then method 1 is:
$$D_{\mathbf{u}}\mathbf{G}=\mathbf{G}'(\mathbf{x})\begin{pmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{pmatrix}=\frac{1}{\sqrt{5}}\begin{pmatrix} 1/x_2+2x_1/x_2^2 \\ 2x_1-4x_2 \\ \mathrm{e}^{x_1} \end{pmatrix}$$
and method 2 is:
$$D_{\mathbf{u}}\mathbf{G}=\frac{d}{dt}\left(\frac{x_1+t/\sqrt{5}}{x_2-2t/\sqrt{5}},(x_1+t/\sqrt{5})^2+(x_2-2t/\sqrt{5})^2,\mathrm{e}^{x_1+t/\sqrt{5}}\right)_{t=0}$$
which results in same answer.

(c) (i) (*Bookwork & same as in 2016 exam*)
Jacobian is
$$J_{\mathbf{r}}=\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$
Jacobian determinant is $|J_{\mathbf{r}}|=r$, so invertible if $r\neq 0$.

(ii) (*Bookwork & same as in 2016 exam*)
First,
$$\hat{\mathbf{r}}=\frac{\partial\mathbf{r}}{\partial r}\Big/\left|\frac{\partial\mathbf{r}}{\partial r}\right|=(\cos\theta,\sin\theta)$$
and
$$\hat{\boldsymbol{\theta}}=\frac{\partial\mathbf{r}}{\partial\theta}\Big/\left|\frac{\partial\mathbf{r}}{\partial\theta}\right|=(-\sin\theta,\cos\theta)$$
with scale factors $h_r=1$, $h_\theta=r$. The gradient is (from the formula in lectures)
$$\boldsymbol{\nabla}=\frac{\hat{\mathbf{r}}}{h_r}\frac{\partial}{\partial r}+\frac{\hat{\boldsymbol{\theta}}}{h_\theta}\frac{\partial}{\partial\theta}=\hat{\mathbf{r}}\frac{\partial}{\partial r}+\frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial}{\partial\theta}$$

(iii) (*New, though done something similar on a problem sheet for spherical polars*)

$$\left(\hat{\mathbf{r}}\frac{\partial}{\partial r}+\frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial}{\partial \theta}\right)\cdot F_r\hat{\mathbf{r}}=\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}\frac{\partial F_r}{\partial r}+F_r\frac{\partial \hat{\mathbf{r}}}{\partial r}\cdot\hat{\mathbf{r}}+\frac{\hat{\boldsymbol{\theta}}}{r}\cdot\hat{\mathbf{r}}\frac{\partial F_r}{\partial \theta}+F_r\frac{\hat{\boldsymbol{\theta}}}{r}\cdot\frac{\partial \hat{\mathbf{r}}}{\partial \theta}$$

Since $\hat{\mathbf{r}}$ does not depend on r, $\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}=1$, $\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\theta}}=0$ and $\partial\hat{\mathbf{r}}/\partial\theta=\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}}=1$ we end up with the answer given.

(iv) (*Unseen*)
The Laplacian is $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\phi$ and $\boldsymbol{\nabla}\phi=\partial\phi/\partial r\hat{\mathbf{r}}$ so use $F_r=\partial\phi/\partial r$ and $F_\theta=0$ in result above to give

$$\Delta=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}.$$

The equation given (now just an ODE[1]) can be written

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)=r$$

and integrating up twice gives

$$\phi(r)=\tfrac{1}{9}r^3+A\log r+B$$

2. (a) (*On set homework, used in lectures*)
Written in component form

$$\boldsymbol{\nabla}\cdot(\mathbf{u}\times\mathbf{v})=\frac{\partial}{\partial x_i}\epsilon_{ijk}u_jv_k=v_k\epsilon_{kij}\frac{\partial u_j}{\partial x_i}-u_j\epsilon_{jik}\frac{\partial v_k}{\partial x_i}=\mathbf{v}\cdot(\boldsymbol{\nabla}\times\mathbf{u})-\mathbf{u}\cdot(\boldsymbol{\nabla}\times\mathbf{v})$$

where product rule and properties of ϵ_{ijk} have been used.

(b) (i) (*Unseen but basic calculation*)
First $\boldsymbol{\nabla}\cdot\mathbf{F}=\frac{\partial}{\partial x}(z^2-y^2)+\frac{\partial}{\partial y}(x^2-z^2)+\frac{\partial}{\partial z}(y^2-x^2)=0.$
Second, $\boldsymbol{\nabla}\times\mathbf{F}=(2y+2z)\hat{\mathbf{x}}+(2z+2x)\hat{\mathbf{y}}+(2x+2y)\hat{\mathbf{z}}.$

(ii) (*Unseen example, but integrals along triangular contours like this done on problems sheets and in problems classes. The bit on cyclic rotation is unseen.*)
Do the integral along segment C_1 say *from* $(0,0,1)$ *to* $(1,0,0)$ first.
Parametrise C_1 by $\mathbf{p}(t)=(t,0,1-t)$ for $0<t<1$. Then $\mathbf{p}'(t)=(1,0,-1)$ and $\mathbf{F}(\mathbf{p}(t))=((1-t)^2,t^2-(1-t)^2,-t^2)$ so

$$\int_{C_1}\mathbf{F}\cdot d\mathbf{r}=\int_0^1((1-t)^2,t^2-(1-t)^2,-t^2)\cdot(1,0,-1)dt=\frac{2}{3}$$

The other two line segments come to the same value because they just involve rotations about the origin in which $x\to z$, $z\to y$, $y\to x$ and the components of the vectors follow the same cyclic rotation $F_1\to F_2$ ...
So the answer is $\int_C\mathbf{F}\cdot d\mathbf{r}=2$. The direction of C is $(0,0,1)$ to $(1,0,0)$ to $(0,1,0)$ to $(0,0,1)$.

[1]Students may recall the definition of the Laplacian in polars and use it here without having derived it

(iii) *(Bookwork)*
Stokes' theorem is

$$\int_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$$

Given that the surface points towards the origin, need to make sure the orientation of C is aligned with direction of surface via RH thumb rule. Here it isn't, so answer from (ii) needs to be negated. Thus

$$\int_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\mathbf{S} = -2$$

(iv) *(As per comments made in (ii))*
Surface is parametrised by, say, $\mathbf{s}(u, v) = (u, v, 1-u-v)$ where $(u, v) \in D = \{0 < u < 1,\ 0 < v < 1-u\}$. Now normal to surface is

$$\mathbf{N} = \mathbf{s}_u \times \mathbf{s}_v = (1, 0, -1) \times (0, 1, -1) = (1, 1, 1)$$

However, need to negate because of definition of surface towards origin, so $\mathbf{N} = -(1, 1, 1)$.
Then, using result of part (b)(i)

$$\begin{aligned} \int_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\mathbf{S} &= -2 \int_0^1 du \int_0^{1-u} dv (v + 1 - u - v, u + 1 - u - v, u + v) \cdot (1, 1, 1) \\ &= -2 \int_0^1 \int_0^{1-u} 2 dv\, du = -2 \end{aligned}$$

(since area of D is a half – i.e. don't even need to do double integral).

(c) (*Unseen*)
The key is spotting that the LHS is

$$\int_V \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} + \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} dV = \int_V \mathbf{E} \cdot (\boldsymbol{\nabla} \times \mathbf{B}) - \mathbf{B} \cdot (\boldsymbol{\nabla} \times \mathbf{E}) = -\int_V \boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{B}) dV$$

by part (a). Then use the divergence theorem to get the RHS.