MATH20901 Multivariable Calculus: Problems 2 [1]

1. (a) Let δ_{ij} be Kronecker delta symbol and ϵ_{ijk} the Levi-Civita tensor. Compute $\delta_{ij}\delta_{jk}\delta_{ki}$ and ϵ_{ijk}^2.
   (b) Define three matrices $A \in \mathbb{R}^{n\times p}$, $B \in \mathbb{R}^{q\times p}$, and $C \in \mathbb{R}^{q\times s}$. Express the i, jth component of the product AB^TC in terms of the components of A, B, C using Einstein summation convention.

2. For $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$, show that:

$$\text{(i) } \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}); \qquad \text{(ii) } (\mathbf{a}\times\mathbf{b})\times(\mathbf{a}\times\mathbf{c}) = (\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}))\,\mathbf{a}.$$

3. Compute
   (a) the gradient of $f(\mathbf{r}) = \cos(xy) + \cos(yz)$, and verify that $\boldsymbol{\nabla}\times\boldsymbol{\nabla} f = 0$ in this case.
   (b) the divergence of $\mathbf{u}(\mathbf{r}) = (x\sin z, yz, \cos z)$,
   (c) the curl of $\mathbf{v}(\mathbf{r}) = (ayz, bzx, cxy)$, and verify that $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{v}) = 0$ in this case.

4. In each case of the following cases, use suffix notation. Compute:
   (a) the gradient of $f(\mathbf{r}) = \mathbf{a}\cdot\mathbf{r}$, where $\mathbf{a}\in\mathbb{R}^3$ is a fixed vector;
   (b) the divergence of $\mathbf{v}(\mathbf{r}) = \boldsymbol{\nabla} r^n$ where $r = |\mathbf{r}|$; For which value of n does the divergence vanish ?
   (c) The curl of $\mathbf{v}(\mathbf{r}) = \boldsymbol{\omega}\times\mathbf{r}$, where $\boldsymbol{\omega}\in\mathbb{R}^3$ is a fixed vector.

5. (a) Let $f(r)$ be a smooth scalar-valued function of $r = |\mathbf{r}|$, and let $\mathbf{a}\in\mathbb{R}^3$ be a constant vector. Use suffix notation to calculate:

$$\text{(i) } \boldsymbol{\nabla}\times(\mathbf{r}\times\mathbf{a}f(r)); \qquad \text{and (ii) } \boldsymbol{\nabla}\cdot(\mathbf{a}f(r)).$$

   (b) Let $\mathbf{u}$ be a vector field. Show, using suffix notation, that

$$\mathbf{u}\times(\boldsymbol{\nabla}\times\mathbf{u}) = \frac{1}{2}\boldsymbol{\nabla}(\mathbf{u}\cdot\mathbf{u}) - (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}.$$

6. Prove the following identities for scalar functions f, g and the vector function $\mathbf{v}$:
   (a) $\boldsymbol{\nabla}\cdot(f\mathbf{v}) = f\boldsymbol{\nabla}\cdot\mathbf{v} + \boldsymbol{\nabla} f\cdot\mathbf{v}$;
   (b) $\boldsymbol{\nabla}\cdot(f\boldsymbol{\nabla} g) = f\Delta g + \boldsymbol{\nabla} f\cdot\boldsymbol{\nabla} g$;
   (c) $\boldsymbol{\nabla}\times(f\mathbf{v}) = f\boldsymbol{\nabla}\times\mathbf{v} + \boldsymbol{\nabla} f\times\mathbf{v}$.

7. Without doing any calculations, how do you know the following result is false ?

$$\boldsymbol{\nabla}\cdot(\mathbf{u}\times\mathbf{v}) = \mathbf{u}\cdot(\boldsymbol{\nabla}\times\mathbf{v}) + \mathbf{v}\cdot(\boldsymbol{\nabla}\times\mathbf{u}).$$

   Find a corrected version of this result.

8. Show that, for vector fields $\mathbf{u}(\mathbf{r})$, $\mathbf{v}(\mathbf{r})$ both in $\mathbb{R}^3$

$$\boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{v}) = (\boldsymbol{\nabla} \cdot \mathbf{v})\mathbf{u} - (\boldsymbol{\nabla} \cdot \mathbf{u})\mathbf{v} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{u} - (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{v}.$$

9. (a) The force per unit volume of gravity on a particle of density ρ is given by

$$\mathbf{F} = -\rho g \hat{\mathbf{z}}$$

where g is gravitational acceleration. Find a ϕ such that $\mathbf{F} = \boldsymbol{\nabla}\phi$.

(b) Euler's equations governing the motion of a so-called 'ideal fluid' are given by

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p, \qquad \text{with} \qquad \boldsymbol{\nabla} \cdot \mathbf{u} = 0$$

where p is a scalar (pressure) function, t is time and $\mathbf{u}$ is a vector function (the fluid velocity).

Take the curl of Euler's equation, using the results of Q5(b) and Q8 to derive the following equation for the *vorticity* $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{u}$

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\mathbf{u}.$$

(c) Deduce that if $\mathbf{u}$ is two-dimensional (e.g. no component in the $\hat{\mathbf{z}}$ direction) then $\boldsymbol{\omega}$ is constant.

10. Navier's equation governs the motion of a linear isotropic elastic solid and is given by

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + \mu)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}) + \mu \Delta \mathbf{u}$$

where ρ is the density of the elastic solid, λ and μ are material constants, t is time and $\mathbf{u}$ is the (small) displacement of the solid as a function of space and time.

Assuming space and time derivatives are interchangeable, apply the divergence and curl operators to Navier's equation to show that $\phi = \boldsymbol{\nabla} \cdot \mathbf{u}$ and $\mathbf{H} = \boldsymbol{\nabla} \times \mathbf{u}$ satisfy

$$\frac{\partial^2 \phi}{\partial t^2} = c_1^2 \Delta \phi, \qquad \text{and} \qquad \frac{\partial^2 \mathbf{H}}{\partial t^2} = c_2^2 \Delta \mathbf{H},$$

where c_1 and c_2 are to be found.

[Hint: You will need to use results from lectures, namely: (i) $\Delta \mathbf{u} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}) - \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{u})$; (ii) $\boldsymbol{\nabla} \times \boldsymbol{\nabla} f = 0$; (iii) $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{v}) = 0$ for any f, $\mathbf{v}$.]