MATH20901 Multivariable Calculus: Problems 3 [1]

1. Let $(x, y, z) = \mathbf{r}(\mathbf{q}) = \mathbf{r}(q_1, q_2, q_3)$ define an *orthogonal* coordinate system.

   (a) Write down the Jacobian matrix $\mathbf{r}'(\mathbf{q})$ and express the result in terms of the orthogonal basis vectors $\hat{\mathbf{q}}_\alpha$.

   (b) Show that the inverse of the Jacobian matrix is

$$(\mathbf{r}'(\mathbf{q}))^{-1} = \begin{pmatrix} \frac{1}{h_1^2}\frac{\partial x}{\partial q_1} & \frac{1}{h_1^2}\frac{\partial y}{\partial q_1} & \frac{1}{h_1^2}\frac{\partial z}{\partial q_1} \\ \frac{1}{h_2^2}\frac{\partial x}{\partial q_2} & \frac{1}{h_2^2}\frac{\partial y}{\partial q_2} & \frac{1}{h_2^2}\frac{\partial z}{\partial q_2} \\ \frac{1}{h_3^2}\frac{\partial x}{\partial q_3} & \frac{1}{h_3^2}\frac{\partial y}{\partial q_3} & \frac{1}{h_3^2}\frac{\partial z}{\partial q_3} \end{pmatrix}.$$

   [*HINT: an orthogonal matrix R has inverse R^T*].

   (c) Use the result from (b) to calculate $\dfrac{\partial \phi}{\partial y}$, where ϕ is the latitudinal angle in spherical coordinates.

2. Spherical coordinates are defined by

$$\mathbf{r}(r, \phi, \theta) = (r \sin\phi \cos\theta, r \sin\phi \sin\theta, r \cos\phi).$$

   (a) Calculate the basis vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\boldsymbol{\theta}}$, as well as the scale factors h_r, h_ϕ, and h_θ.

   (b) Show that

$$\frac{\partial \hat{\mathbf{r}}}{\partial \phi} = \hat{\boldsymbol{\phi}}, \quad \frac{\partial \hat{\boldsymbol{\phi}}}{\partial \phi} = -\hat{\mathbf{r}}, \quad \frac{\partial \hat{\mathbf{r}}}{\partial \theta} = \sin\phi\, \hat{\boldsymbol{\theta}}, \quad \frac{\partial \hat{\boldsymbol{\phi}}}{\partial \theta} = \cos\phi\, \hat{\boldsymbol{\theta}}, \quad \frac{\partial \hat{\boldsymbol{\theta}}}{\partial \theta} = -\sin\phi\, \hat{\mathbf{r}} - \cos\phi\, \hat{\boldsymbol{\phi}}.$$

   (c) Calculate $\boldsymbol{\nabla} \cdot \mathbf{u}$ in spherical coordinates, where $\mathbf{u} = u_r \hat{\mathbf{r}} + u_\phi \hat{\boldsymbol{\phi}} + u_\theta \hat{\boldsymbol{\theta}}$.

   (d) Treat yourself. How about $\boldsymbol{\nabla} \times \mathbf{u}$?

   (e) Finally, write down Δf in spherical polars.

3. (a) Let $\phi(\mathbf{r}) = f(r)$, where f is a function of a single variable, $r = |\mathbf{r}|$. Compute $\Delta\phi$.

   (b) Let $\boldsymbol{\mu} \in \mathbb{R}^3$ be a nonzero vector. Compute $\Delta \dfrac{\boldsymbol{\mu} \cdot \mathbf{r}}{r^3}$.

4. This question concerns the transformation to elliptical coordinates (μ, ν) given by the relation

$$x = a \cosh\mu \cos\nu, \qquad y = a \sinh\mu \sin\nu$$

   where $\mu \in [0, \infty)$, $\nu \in [0, 2\pi)$.

(a) Show that curves of constant μ correspond to ellipses in the (x, y)-plane and that curves of constant ν are hyperbolae.

(b) Derive a basis $\hat{\boldsymbol{\mu}}$, $\hat{\boldsymbol{\nu}}$ for elliptical coordinates and, in doing so, show that the scale factors are

$$h_\mu = h_\nu = a\sqrt{\sinh^2\mu + \sin^2\nu}.$$

Also confirm that $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\nu}}$ are orthogonal.

(c) Calculate the Jacobian determinant. Is the mapping of coordinates always invertible ? If not, when is it non-invertible ?

(d) Express $\boldsymbol{\nabla} f$ in elliptical coordinates.

(e) Find Δf in elliptical coordinates.

5. (a) For the two scalar function f, g, derive the relation $\Delta(fg) = f\Delta g + g\Delta f + 2\boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g$

(b) Assuming that we are working in 2D so that $\mathbf{r} = (x, y, 0)$ show that $\Delta(r^2 \log r) = 4 + 4\log r$.

(c) Hence show that $\Delta^2(r^2 \log r) = 0$ where $\Delta^2 \equiv \Delta\Delta$ is called the biharmonic operator.