

MATH20901 Multivariable Calculus: Problems 4 ¹

1. The length of a curve C is given by $\int_C |d\mathbf{r}|$. A cycloid is the path traced out in space by, say, a nail in a tyre on a wheel of radius a . The path is given by $\mathbf{p}(t) = a(t - \sin t, 1 - \cos t, 0)$ with $0 < t < 2\pi$ for one revolution. Find the distance the nail travels during one revolution of the wheel.
2. Let $\mathbf{v}(\mathbf{r}) = (x, xy, xyz)$ and let the curve C be parametrised by the path

$$\mathbf{p}(t) = \left(\sin\left(\frac{\pi}{2}t\right), \cos\left(\frac{\pi}{2}t\right), t \right), \quad 0 \leq t \leq 1.$$

Evaluate the line integral $\int_C \mathbf{v} \cdot d\mathbf{r}$.

3. The surface area of an object with surface S is given by $\int_S dS$. Use a transformation of coordinates $x = ra \cos \theta$, $y = rb \sin \theta$ to show that the area of an ellipse with semi-major and minor axes a and b is πab .
4. Calculate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ when $\mathbf{F} = (-x^2y, xy^2, 0)$ and C is a square in the (x, y) -plane with vertices at $(0, 0)$, $(l, 0)$, (l, l) , $(0, l)$ which is oriented anticlockwise.
5. Let $\mathbf{F}(\mathbf{r}) = (x^2z, xy^2, z^2)$ and define a closed curve C in $\mathbb{R}^3$ comprised of three straight line segments C_1 , C_2 and C_3 formed by the intersection of the plane $x + y + z = 1$ with the planes $y = 0$, $z = 0$ and $x = 0$ (respectively).
 - (a) First you should sketch the curve $C = C_1 \cup C_2 \cup C_3$ in the (x, y, z) -space.
 - (b) Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, by considering each line segment separately. [*Hint: be careful to make sure the orientation of each segment of C is aligned with the other segments.*]
 - (c) Calculate $\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$ over an appropriately-defined surface with boundary $\partial S = C$.
 - (d) If, instead, $\mathbf{F} = (yz, xz, xy)$ explain how you could have calculated parts (b) and (c) instantly.
6. Let $\mathbf{F}(\mathbf{r}) = (-y^2, x, z^2)$ and define a curve C in $\mathbb{R}^3$ to be the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.

- (a) Compute

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

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(b) Compute

$$\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$$

where S is a surface of your choice with boundary ∂S coinciding with the curve C . Hence show that Stokes' theorem is satisfied.

7. In this question you are asked to prove Green's theorem on a rectangle directly. Let $\mathbf{v}(\mathbf{r}) = f(x, y)\hat{\mathbf{x}} + g(x, y)\hat{\mathbf{y}}$ be a vector field that lies in the (x, y) -plane and depends only on x and y . Without using Stokes' theorem, show that

$$\int_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx dy = \int_{\partial D} f dx + g dy,$$

where $D = \{(x, y) \mid a < x < b, c < y < d\}$ and ∂D is its boundary.

8. Show that for any two scalar fields $f(\mathbf{r}), g(\mathbf{r})$ around a closed curve C

$$\int_C (f \nabla g + g \nabla f) \cdot d\mathbf{r} = 0$$