MATH20901 Multivariable Calculus
Problems Class Week 4

1. [Exam 2012/13]

   (b) (iii) (7 marks)
   Calculate $(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u}$ in cylindrical coordinates, where $\mathbf{u} = u_r\hat{\mathbf{r}} + u_\theta\hat{\boldsymbol{\theta}} + u_z\hat{\mathbf{z}}$.

   (c) (i) (6 marks)
   Let $f(r)$ be a smooth scalar-valued function of $r = |\mathbf{r}|$, and let $\mathbf{a} \in \mathbb{R}^3$ be a constant vector. Calculate
   $$\boldsymbol{\nabla} \times (\mathbf{r} \times \mathbf{a}f(r))$$

   (ii) (4 marks)
   Let $\mathbf{u}$ be a vector field in $C(\mathbb{R}^3, \mathbb{R}^3)$. Show that
   $$\mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{u}) = \frac{1}{2}\boldsymbol{\nabla}(\mathbf{u} \cdot \mathbf{u}) - (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u}$$

2. [Exam 2016/17]

   (b) (i) (8 marks)
   Define $\nabla(\mathbf{u} \cdot \mathbf{v})$ in terms of $\mathbf{u}'(\mathbf{x})$ and $\mathbf{v}'(\mathbf{x})$.

   (ii) (10 marks)
   By considering $\mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{v})$ and $\mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{u})$ show that
   $$\nabla(\mathbf{u} \cdot \mathbf{v}) = (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{v} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{u} + \mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{v}) + \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{u}).$$

   (c) Let $\mathbf{r} : \mathbb{R}^2 \to \mathbb{R}^2$ be a map defining the transformation $x = r\cos\theta$, $y = r\sin\theta$.

   (i) (6 marks)
   Compute the Jacobian determinant of the transformation, $\dfrac{\partial(x, y)}{\partial(r, \theta)}$. When is the map invertible ?

   (ii) (6 marks)
   Construct a local basis $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ in the new coordinate system.

   (iii) (8 marks)
   For a scalar function $f(x, y)$ derive the expression for $\boldsymbol{\nabla} f$ in the local basis $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$. Hence, or otherwise show that
   $$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2$$