MATH20901 Multivariable Calculus
Problems Classes Week 6

1. Let $\mathbf{v}(x,y) = (y^2, -xy)$ be a vector field on $\mathbb{R}^2$, and let C be the part of the circle $x^2+y^2=1$ that starts at $(1,0)$ and ends at $(0,1)$, oriented clockwise. Compute $\int_C \mathbf{v}\cdot d\mathbf{r}$.

2. Calculate
$$\int_C \mathbf{r}\cdot d\mathbf{r}$$
where C is any curve connecting the point $\mathbf{r}_1$ to $\mathbf{r}_2$.

3. (i) A vector field $\mathbf{F}$ is given by $\mathbf{F}(x,y,z) = (-y, x, z)$. Calculate $\nabla\times\mathbf{F}$.

   (ii) The surface, S, of an open cone of unit height is defined by $z^2 = x^2+y^2$ for $0 < z < 1$. Design a mapping $\mathbf{s}(u,v)$ which parametrises the surface S. Hence calculate the vector $\mathbf{N}(u,v) = \mathbf{s}_u \times \mathbf{s}_v$. What geometric property does this vector have ?

   (iii) Calculate the surface integral
$$\int_S \nabla\times\mathbf{F}\cdot d\mathbf{S}$$
   where the surface is defined to be directed away from the z-axis.

   (iv) (8 marks)
   State Stokes' theorem and use it to calculate the integral in (iii) by an independent method.

4. Let ϕ be a scalar field. Use the divergence theorem to show that
$$\int_V \boldsymbol{\nabla}\phi dV = \int_{\partial V} \phi\hat{\mathbf{n}} dS.$$

5. (i) A vector field $\mathbf{F}$ is given by $\mathbf{F}(x,y,z) = (xy, yz, xz)$. Calculate $\nabla\cdot\mathbf{F}$.

   (ii) The volume V of a tetrahedron is bounded by four triangular surfaces formed by the intersection of the planes $x=0$, $y=0$, $z=0$ and $x+y+z=1$. Sketch V.

   (iii) Calculate the volume integral
$$\int_V \nabla\times\mathbf{F}\cdot dV$$

   (iv) State the divergence theorem and use it to calculate the value of the integral in (iii) by an independent method.

6. If $\mathbf{f} = (0,x,0)$ and $\mathbf{g} = (-y,0,0)$ show that, for any closed curve $C \in \mathbb{R}^3$
$$\int_C \mathbf{f}\cdot d\mathbf{r} = \int_C \mathbf{g}\cdot d\mathbf{r}$$

7. A surface S is define as the intersection of the hemisphere of radius 2 given by the equation $(x+1)^2+y^2+z^2=4$ in $z\geq 0$ and the cylinder $x^2+y^2\leq 1$.

   Using cylindrical polar coordinates centred on $z=0$, write down a parametrisation $\mathbf{s}(r,\theta)$ of the surface S.